# Diagrammatic Approach to the Two-Body Dynamics & PostNewtonian Corrections

Pierpaolo Mastrolia

*Amplitudes 2018*
*SLAC National Lab, 18.6.2018*

*based on:*
**Foffa, PM, Sturani, Sturm, PRD95 (2017) 10, 104009**

*in collaboration with:*
- **Foffa, Sturani, Sturm, Cristofoli, Torres-Bobadilla**

Physics and Astronomy Department
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# Outline

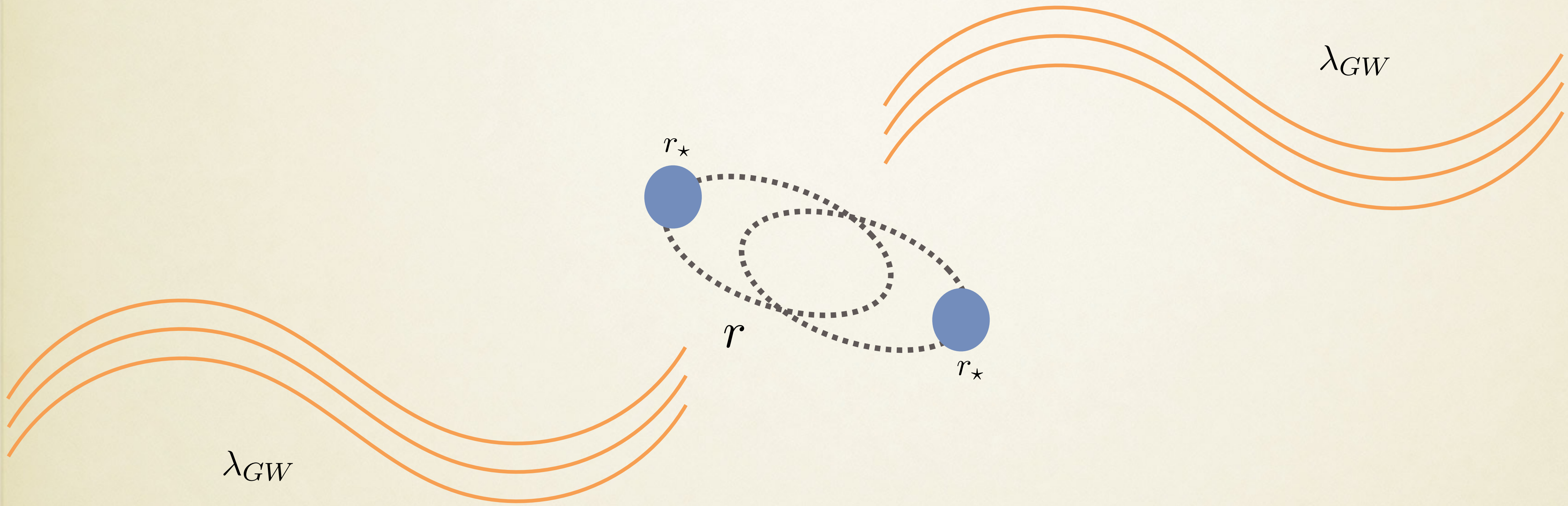
- Effective Field Theory - General Relativity Feynman Rules
- Post-Newtonian expansion
- Amplitudes @ 4PN - O(G^5)
- Feynman Integrals in Dimensional Regularisation
  - Integration-by-parts identities
  - Master Integrals
  - Dimensional Recurrence Relations
  - Differential Equations
- Results
- Toward the Potential @ 5PN - O(G^6)
- Conclusions/Outlook

# Motivations

- Describing the Two-body dynamics at high precision
- Providing an independent calculation of the 4PN-O(G^5) Lagrangian
- Providing a new calculation of the 5PN-O(G^6) Lagrangian
- Broadening the application range
  of Multi-loop Amplitudes/High-Energy Computational Tools

>> A. Buonanno
>> C. Cheung
>> D. O'Connell
>> S. Caron-Huot

# Length scales in a Binary System

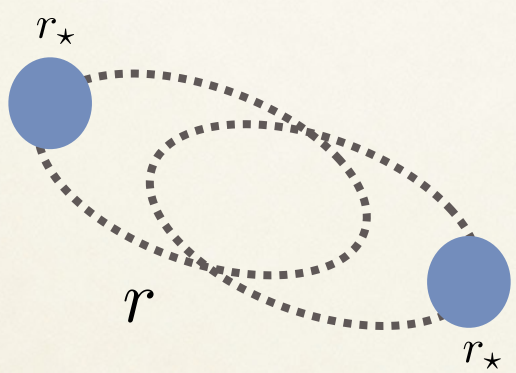



- **Double Hierarchy**

$$r_\star << r << \lambda_{GW}$$

Dissipative system ::
GW emission

# Length scales in a Binary System




- **Double Hierarchy**

$$r_\star << r << \lambda_{GW}$$

Conservative system ::
~~GW emission~~

$$G_N \frac{m}{r} \sim v^2 << 1$$

expansion parameter

# Effective Field Theory

Goldberger, Ross, Rothstein

## Einstein-Hilbert

$$S_{tot}[x,h] = S_g[h] + S_m[x,h] \qquad e^{iS_{eff}[x]} = \int \mathcal{D}[h] e^{iS_{tot}[x,h]}$$

$$S_g = \frac{1}{32\pi G}\int d^{(d+1)}x\sqrt{-g}\left[R - g_{\mu\nu}g^{\alpha\beta}g^{\gamma\delta}\Gamma^{\mu}_{\alpha\beta}\Gamma^{\nu}_{\gamma\delta}\right]$$

gauge fixing term

$$S_{m_a} \simeq S^a_{pp} = -m_a\int\sqrt{-g_{\mu\nu}dx^\mu_a dx^\nu_a} = -m_a\int dt\sqrt{-g_{\mu\nu}\dot{x}^\mu_a\dot{x}^\nu_a} \qquad a = 1,2$$

## Kaluza-Klein parametrisation

Kol, Smolkin; Blanchet, Damour;

$$g_{\mu\nu} = e^{\frac{2\phi}{\Lambda}}\begin{pmatrix} -1 & \frac{A_i}{\Lambda} \\ \frac{A_i}{\Lambda} & e^{-c_d\frac{\phi}{\Lambda}}\gamma_{ij} - \frac{A_iA_j}{\Lambda^2}\end{pmatrix} \qquad \gamma_{ij} = \delta_{ij} + \frac{\sigma_{ij}}{\Lambda}$$

$$c_d = 2\left(\frac{d-1}{d-2}\right) \qquad \Lambda^{-1} = (32\pi G_N)^{\frac{1}{2}}.$$

$$\mathcal{S}_g = \int d^{d+1}x\sqrt{\gamma}\left[\frac{1}{4}\left((\nabla\sigma)^2 - 2(\nabla\sigma_{ij})^2 - (\dot{\sigma}^2 - 2\dot{\sigma}^{ij}\dot{\sigma}_{ij})\right)\right] - c_d\left[(\nabla\phi)^2 - \dot{\phi}^2\right] + \left[\frac{1}{2}F_{ij}F^{ij} + (\vec{\nabla}\cdot\vec{A})^2 - \dot{\vec{A}}\cdot\dot{\vec{A}}\right] + \ldots$$

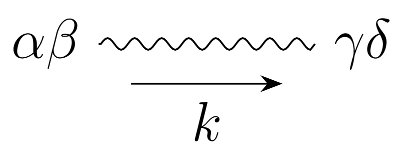
$$\mathcal{S}_{pp} = -m_a\int d\lambda_a\, e^{\frac{\phi}{\Lambda}}\sqrt{1 - e^{-(2+c_d)\frac{\phi}{\Lambda}}v_iv_j\gamma^{ij}}$$

$$e^{i\mathcal{S}_{eff}(x_a)} = \int D\phi DA_i D\sigma_{nm}\, e^{i\mathcal{S}_g + i\mathcal{S}_{pp}}$$

# Feynman Rules

## Propagators

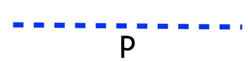
### Graviton




$$D^{\alpha\beta\gamma\delta} = \frac{-i}{2k^2}\left(\frac{2}{2-D}\,\eta^{\alpha\beta}\eta^{\gamma\delta} + \eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma}\right)$$

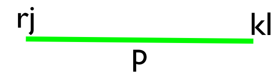
### EFT Non-Rel.Gravity fields

$\phi$-propagator,




$$\rightarrow -\frac{\mathrm{i}}{2c_d p^2}$$

$\sigma$-propagator,




$$\rightarrow \frac{\mathrm{i}P^{\sigma_{rj}\sigma_{kl}}}{2p^2}\,.$$

$$P^{\sigma_{ij}\sigma_{kl}} = -\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + (2-c_d)\delta_{ij}\delta_{kl}\right)$$

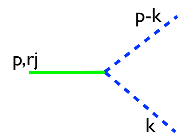
$A_i$-propagator


$$\rightarrow \mathrm{i}\,\frac{\delta^{rt}}{2k^2}$$

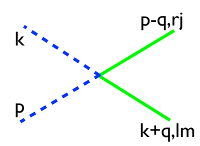
# Feynman Rules

## Interactions

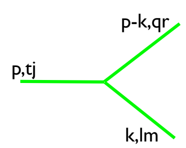
### Self-interactions



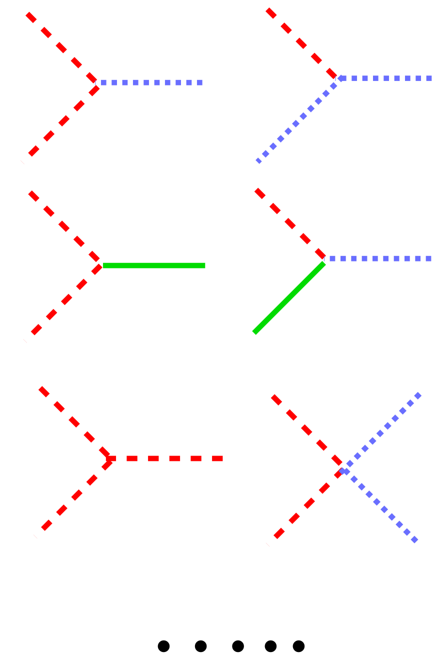

$$\rightarrow \mathrm{i}\frac{2c_d}{\Lambda}\left[\frac{1}{2}(p-k)\cdot k\,\delta^{rj} - k^r(p-k)^j + \Big(r\leftrightarrow j\Big)\right],$$



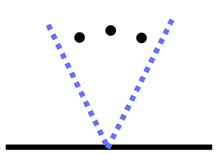

$$\rightarrow \mathrm{i}\frac{4c_d}{\Lambda^2}\left[k^r p^l \delta^{jm} - \frac{1}{2}k^l p^m \delta^{rj} - \frac{1}{8}p\cdot k\,\mathcal{Q}^{rjlm} + \Big(r\leftrightarrow j\,,l\leftrightarrow m\Big)\right],$$



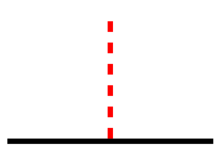

$$\rightarrow \mathrm{i}\frac{1}{8\Lambda}\Bigg\{(p-k)\cdot k\left(\frac{1}{2}\delta^{tr}\mathcal{I}^{lmjq} - \frac{1}{4}\delta^{qr}\mathcal{I}^{tjlm} - \frac{1}{8}\delta^{tj}\mathcal{Q}^{qrlm}\right) +$$
$$+\ \frac{1}{4}(p-k)^t k^j \mathcal{Q}^{qrlm} + \left[\left(\frac{1}{2}\delta^{tj}\delta^{mr} - \delta^{tr}\delta^{jm}\right)(p-k)^q k^l - \Big(l\leftrightarrow q\Big)\right] +$$
$$+\ \delta^{lm}\delta^{tr}(p-k)^q k^j - \delta^{tm}\delta^{qr}(p-k)^l k^j + \Big(t\leftrightarrow j\,,l\leftrightarrow m\,,q\leftrightarrow r\Big)\Bigg\},$$



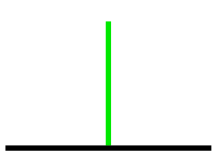
### matter-interactions



$$= -\frac{im_a}{\Lambda^n n!} + \mathcal{O}(v^2)$$



$$= -i\frac{m_a v_i}{\Lambda} + \mathcal{O}(v^3)$$

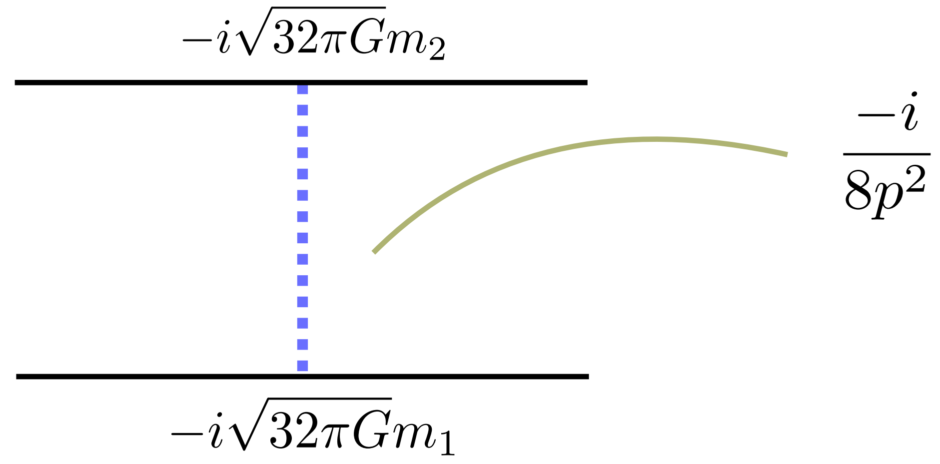


$$= \frac{im_a}{2\Lambda}v^i v^j + \mathcal{O}(v^4)$$

$$\Lambda^{-1} = (32\pi G_N)^{\frac{1}{2}}.$$

# Newton Potential

- **d=3** space-dimension




- **Fourier tfm**

$$V = i\int \frac{\mathrm{d}^3 p}{(2\pi)^3}(-i\sqrt{32\pi G}m_1)(-i\sqrt{32\pi G}m_2)\frac{-i}{8p^2}\mathrm{e}^{ip.(x_1-x_2)}$$

# Post-Newtonian Corrections

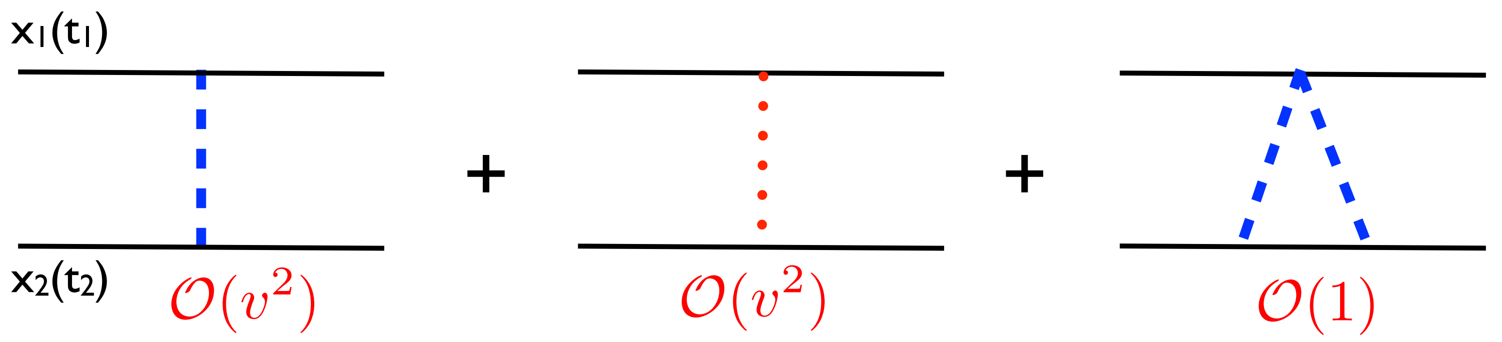
- **Post-Newtonian expansion**
  - **n-**th order correction

$$\mathcal{O}\left(G_N^k \ , \ v^{2m}\right) \ , \qquad n = k + 2m - 1$$

$$G_N \frac{m}{r} \sim v^2$$

Virial

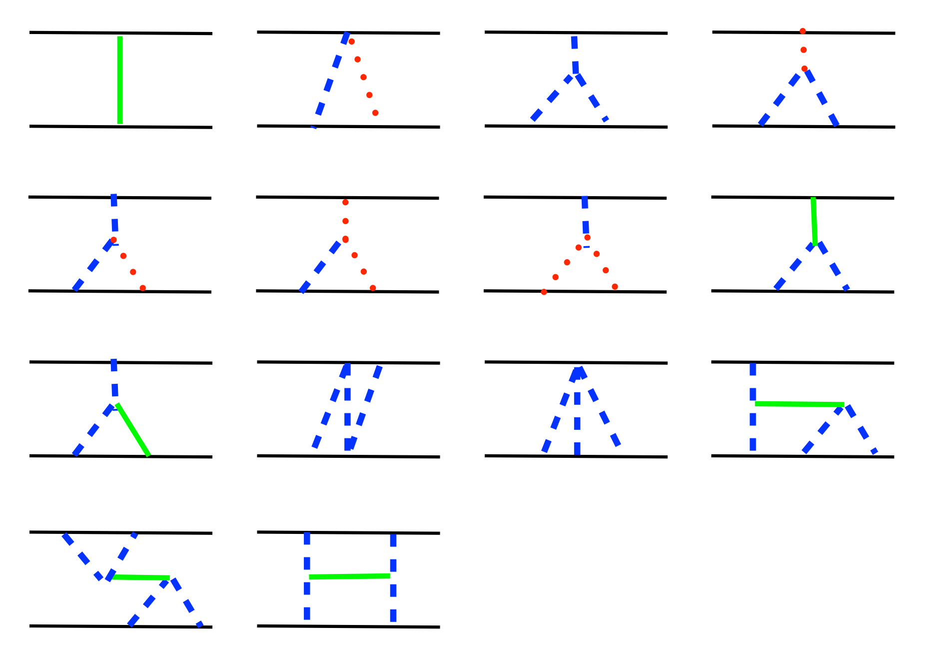
- **1-PN**



# Post-Newtonian Corrections

courtesy of Foffa & Sturani

- **2-PN**

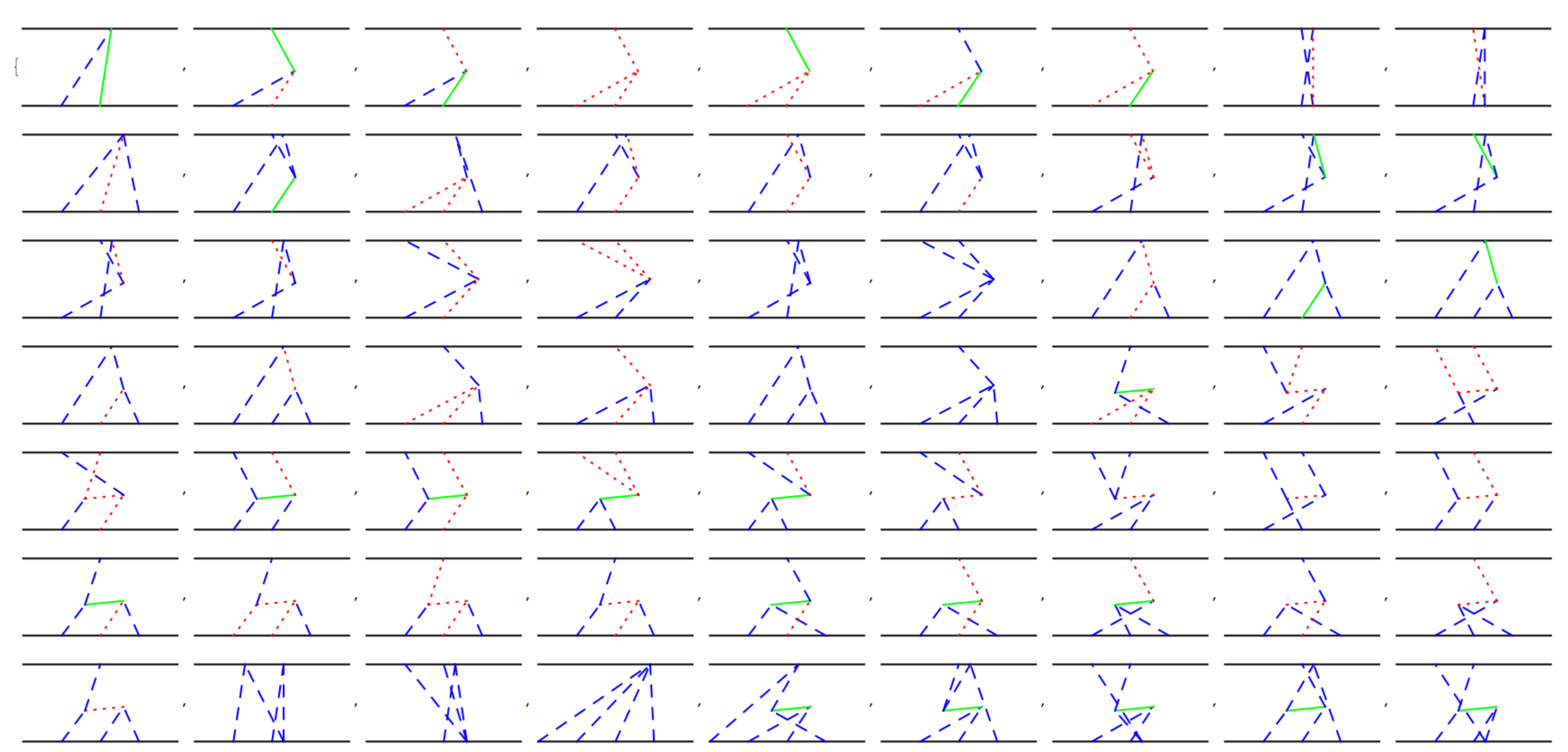


School on Gravitational Waves: from data to theory and back
UNESP Sao Paulo, August 3-11, 2015
Effective field theory methods
to model astrophysical binaries
Stefano Foffa

additional material

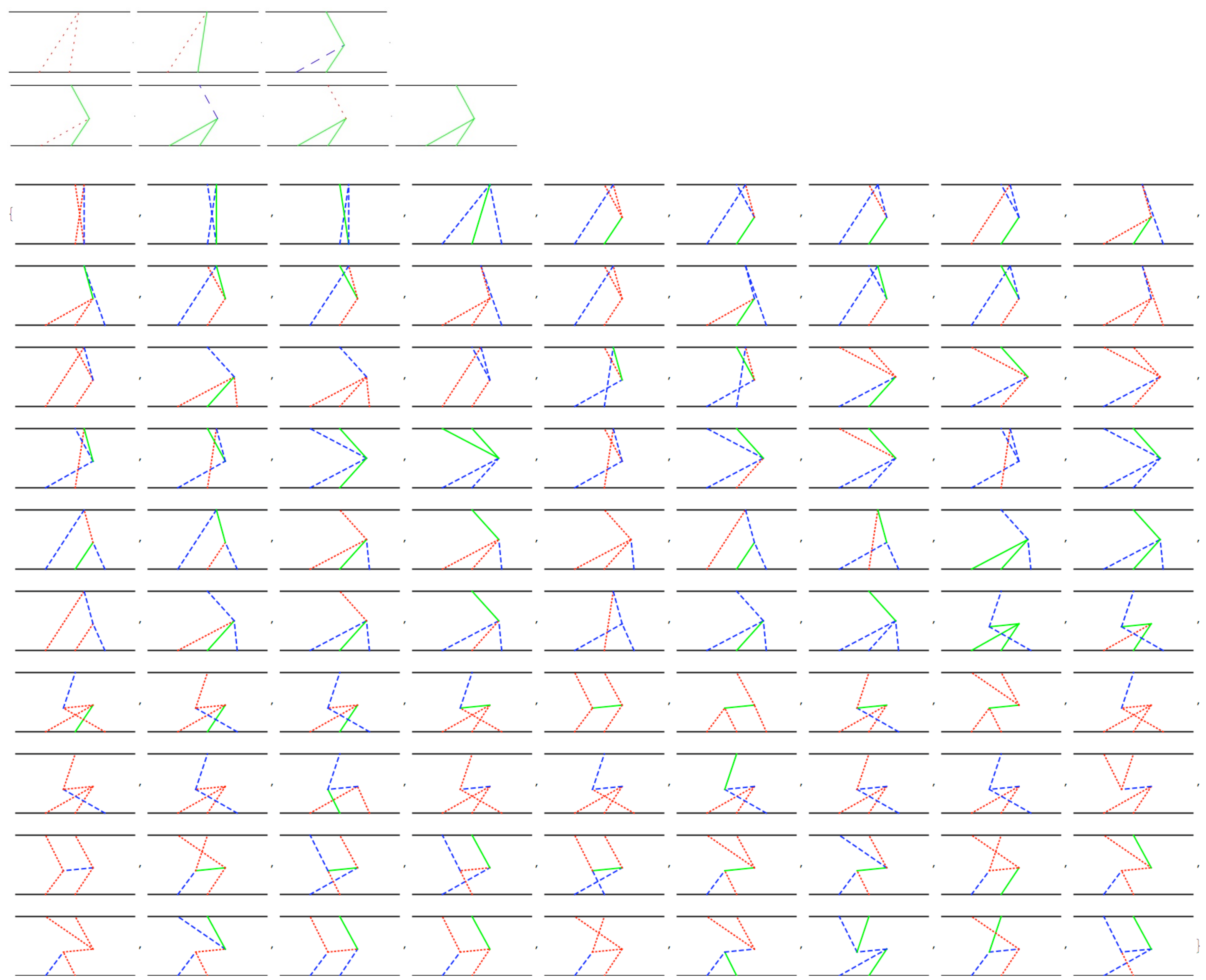
Goldberger
Rothstein
Porto

- **3-PN**



# 4-PN (595 diagrams)

courtesy of Foffa & Sturani



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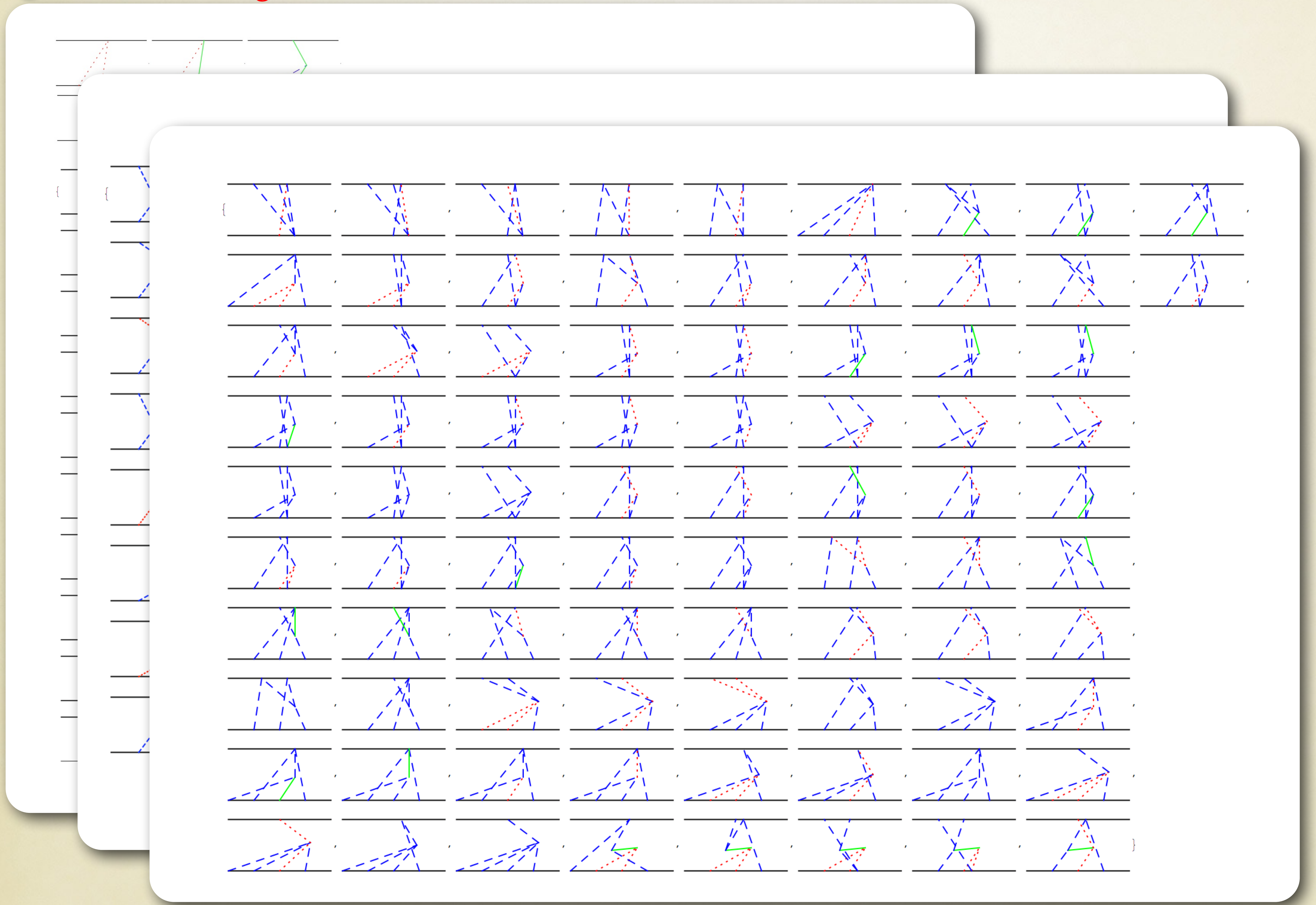

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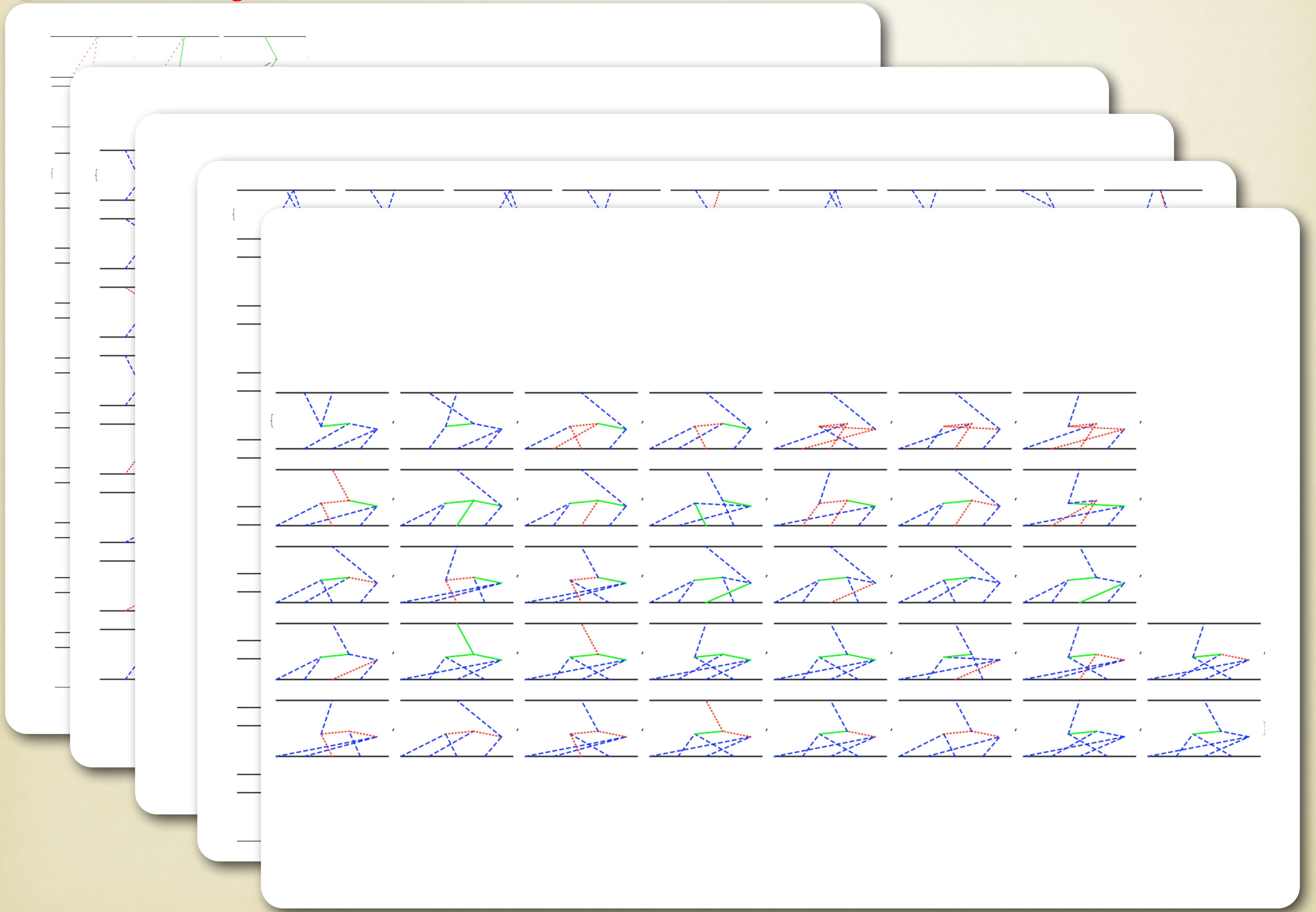

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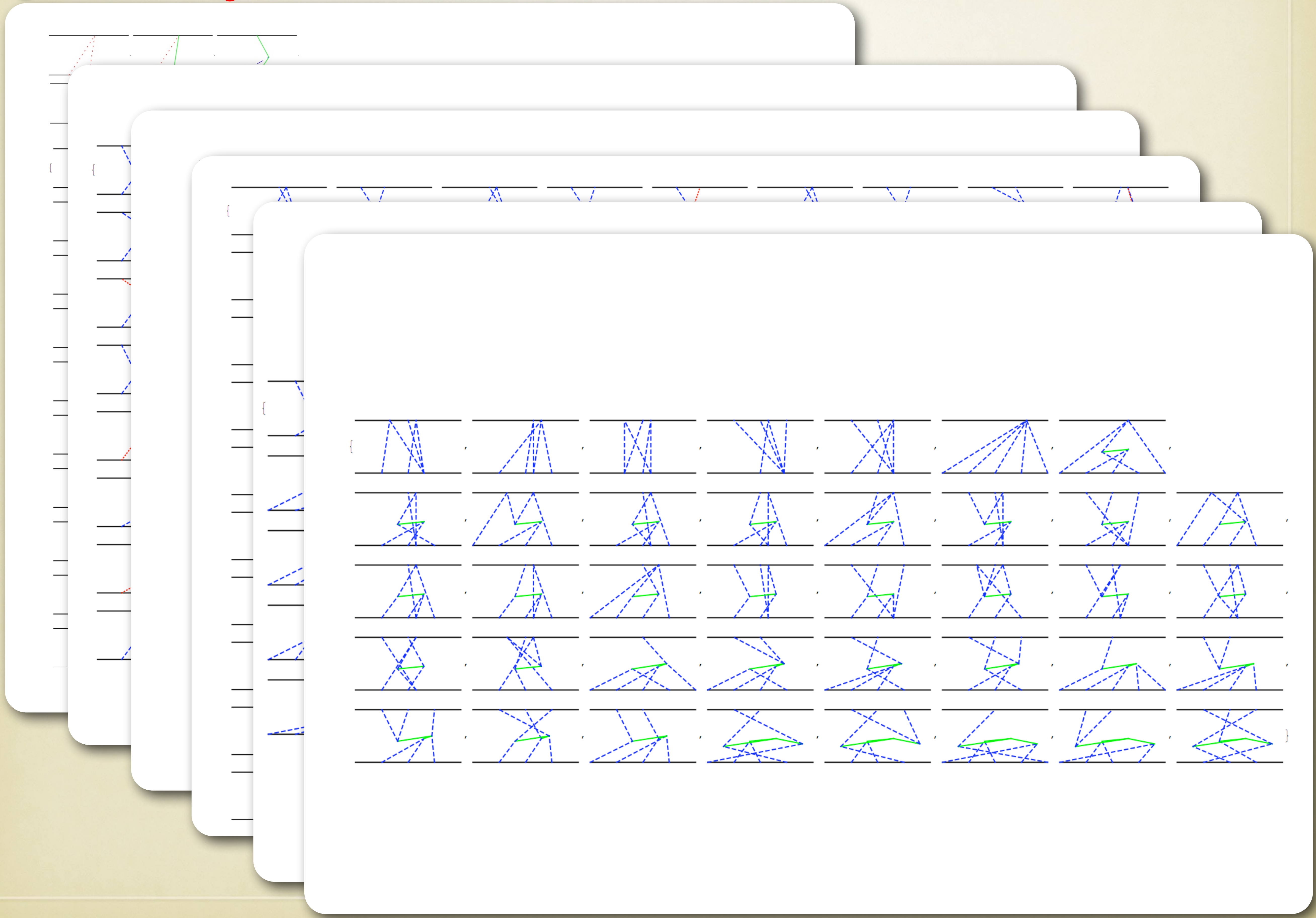
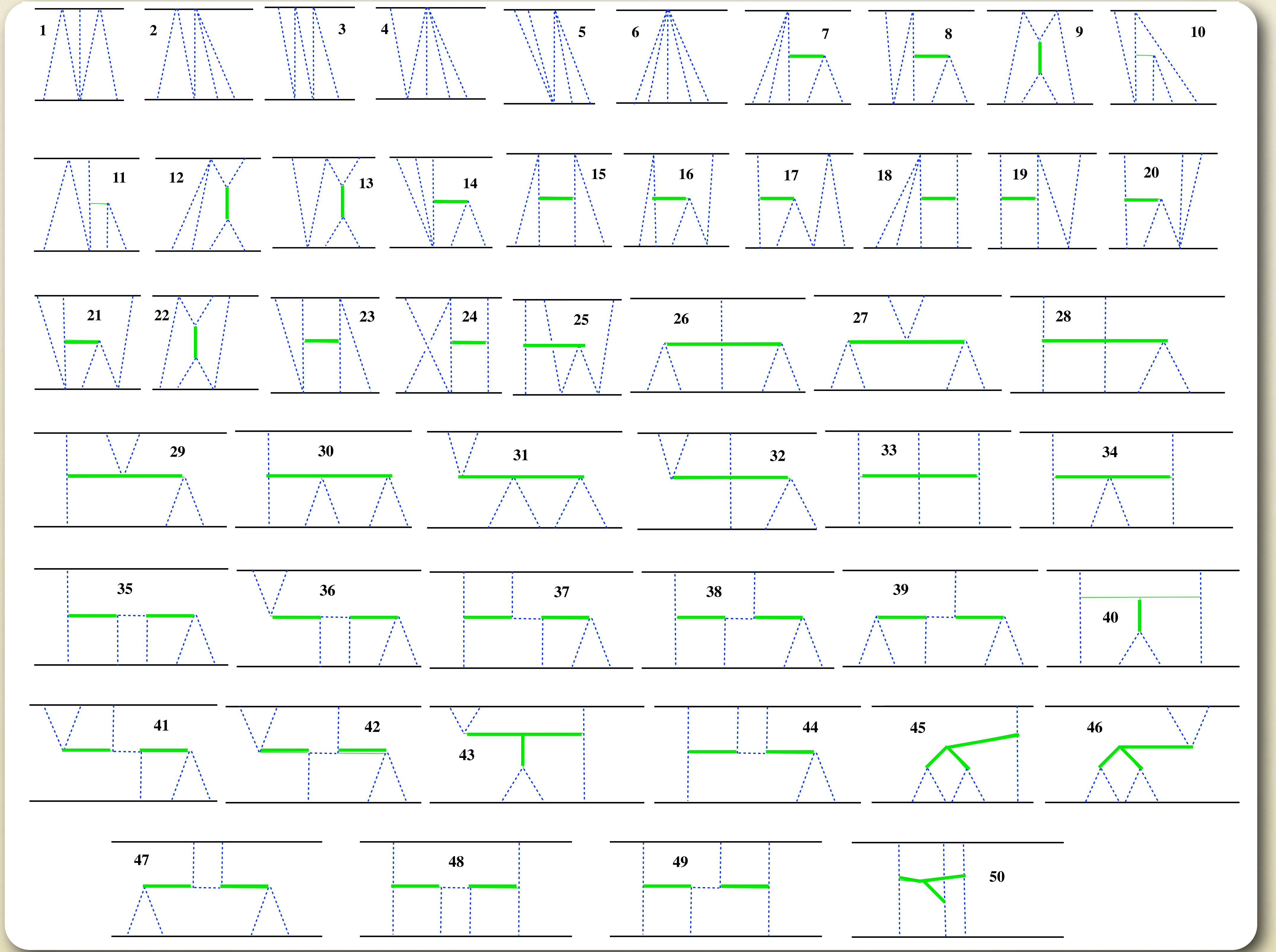


# Amplitudes @ 4PN - O(G^5)

Foffa, Sturani, Sturm, & **P.M.**

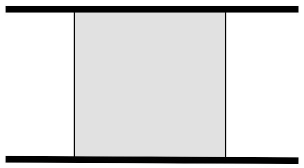
- **50 Amplitudes @ 4-loops**



# From Amplitudes to Lagrangian

- **Fourier Tfm**

$$\mathcal{L}_a = -\mathrm{i} \lim_{d \to 3} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \, e^{\mathrm{i} p \cdot r}$$



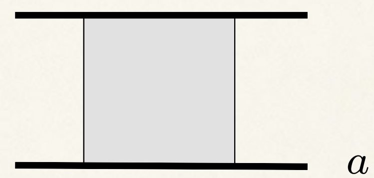
$a$

$$a = 1, \ldots, 50$$

# From Amplitudes to Lagrangian

- **Fourier Tfm**

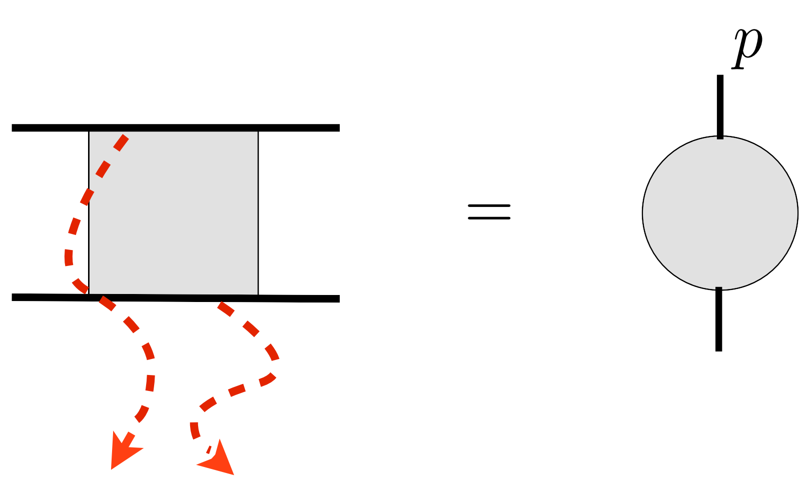
$$\mathcal{L}_a = -\mathrm{i} \lim_{d \to 3} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \, e^{\mathrm{i} p \cdot r}$$



$$a = 1, \ldots, 50$$

## EFT-GR Diagrams *vs* 2-point QFT Diagrams

- **key observation**



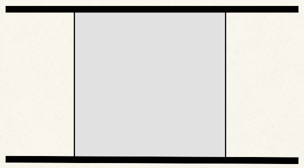

$$p^2 \equiv s \neq 0$$

static sources <==> non propagating <==> pinching lines

# From Amplitudes to Lagrangian

- **Fourier Tfm**

$$\mathcal{L}_a = -\mathrm{i} \lim_{d\to 3} \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, e^{\mathrm{i}p\cdot r}$$



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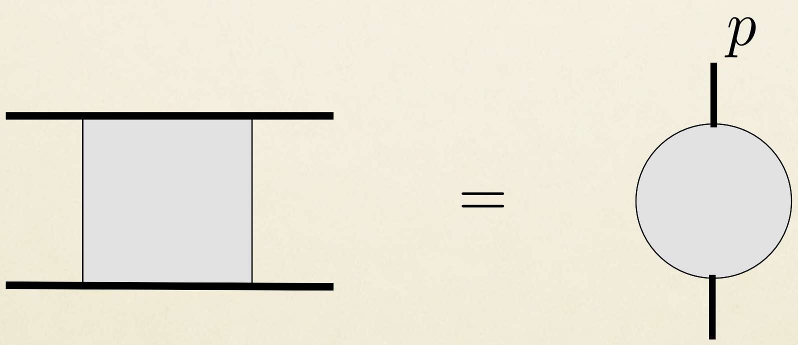
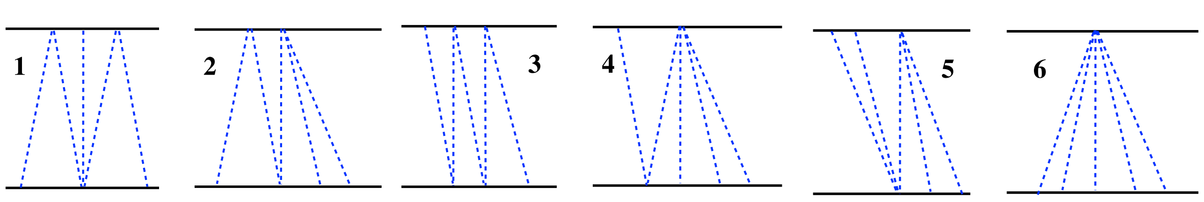
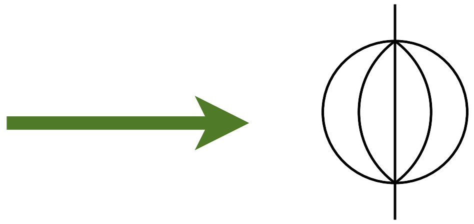



$$p^2 \equiv s \neq 0$$

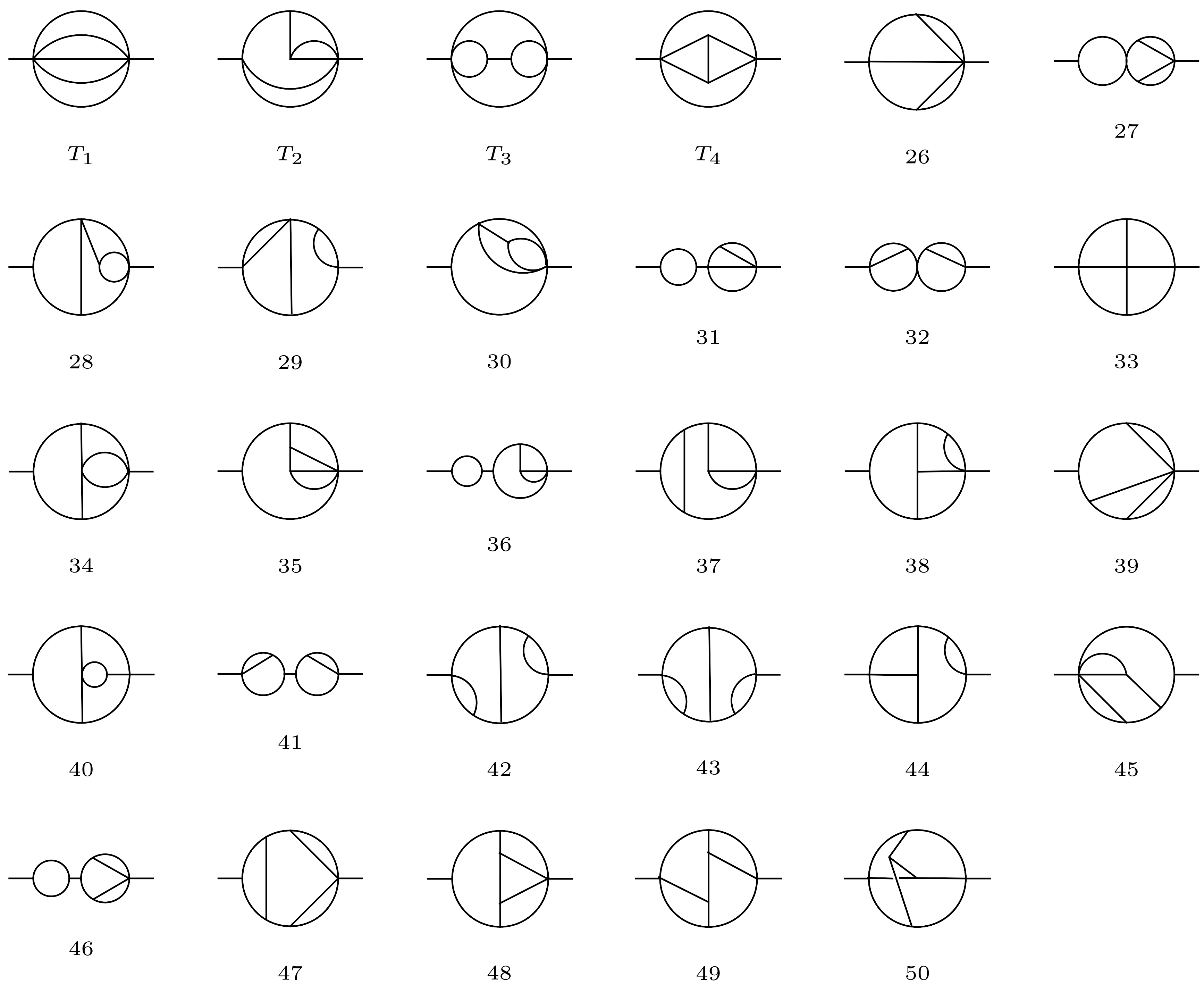
- **example**



# Four-Loop QFT-like Graphs

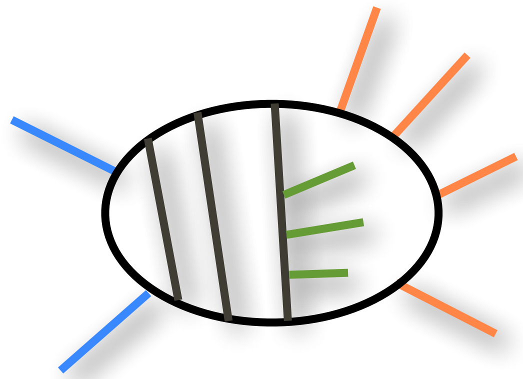
- **29 Topologies @ 4-loop**




$T_1 = \{1,2,3,4,5,6\},\ T_2 = \{7,8,10,11,14,16,17,20,21,25\},\ T_3 = \{9,12,13,22\},\ T_4 = \{15,18,19,23,24\}$

# Dimensionally Regulated Integrals

# Graph Topology & Integrals



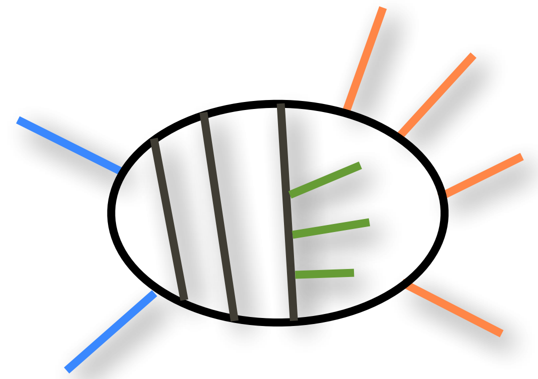
$e = \#$ legs :: $p_i$, $\quad (i = 1, \ldots, e)$;
$\ell = \#$ loops :: $q_i$ $\quad (i = 1, \ldots, \ell)$;
$n = \#$ denominators :: $D_i$ $\quad (i = 1, \ldots, n)$;

$N = \#$ scalar products (of types $q_i \cdot p_j$ and $q_i \cdot q_j$ ) $\qquad N = \ell(e-1) + \frac{\ell(\ell+1)}{2}$

$n = \#$ **reducible** scalar products (expressed in terms of denominators);

$m = \#$ **irreducible** scalar products $= N - n$ :: $S_i$ $\quad (i = 1, \ldots, m)$

# Graph Topology & Integrals



$e = \#$ legs :: $p_i$, $(i = 1, \ldots, e)$;
$\ell = \#$ loops :: $q_i$ $(i = 1, \ldots, \ell)$;
$n = \#$ denominators :: $D_i$ $(i = 1, \ldots, n)$;

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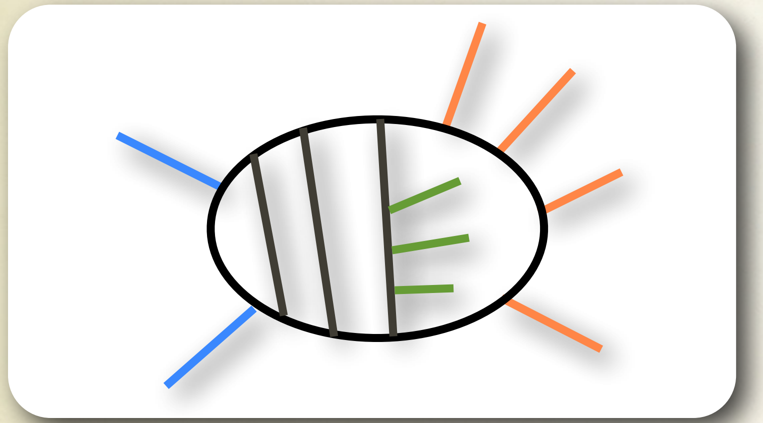
# Graph Topology & Integrals



$e = \#$ legs :: $p_i, \quad (i = 1, \dots, e);$

$\ell = \#$ loops :: $q_i \quad (i = 1, \dots, \ell);$

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$n = \#$ **reducible** scalar products (expressed in terms of denominators);

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**Associated Integrals ::**

$$F^{[d]}_{n,m}(\mathbf{x}, \mathbf{y}) \equiv \int_{q_1 \dots q_\ell} f_{n,m}(\mathbf{x}, \mathbf{y}) \ , \qquad \int_{q_1 \dots q_\ell} \equiv \int \frac{\mathrm{d}^d q_1}{(2\pi)^d} \cdots \frac{\mathrm{d}^d q_\ell}{(2\pi)^d}$$

$$f_{n,m}(\mathbf{x}, \mathbf{y}) = \frac{S_1^{y_1} \cdots S_m^{y_m}}{D_1^{x_1} \cdots D_n^{x_n}}$$

# Integration-by-parts Identities (IBPs)

Tkachov; Chetyrkin, Tkachov; Laporta;

$$\int_{q_1 \ldots q_\ell} \frac{\partial}{\partial q_i^\mu} \Big( v^\mu \; f_{n,m}(\mathbf{x}, \mathbf{y}) \Big) = 0 \; , \qquad v = q_1, \ldots, q_\ell, \; p_1, \ldots, p_{e-1}.$$

$\forall (n,m)$, $N_{\rm IBP}$ = # of IBP relations $= \ell(\ell + e - 1)$

Relations between integrals associated to the same topology (or subtopologies)

$$c_0 \; F^{[d]}_{n,m}(\mathbf{x}, \mathbf{y}) + \sum_{i,j} c_{i,j} \; F^{[d]}_{n,m}(\mathbf{x_i}, \mathbf{y_j}) = 0 \; ,$$

$$\mathbf{x_i} = \{x_1, \ldots, x_i \pm 1, \ldots, x_n\}$$

$$\mathbf{y_j} = \{y_1, \ldots, y_j \pm 1, \ldots y_n\}$$

**public codes :: AIR; Reduze2; FIRE; LiteRed;**
**private codes :: ... many authors ... Sturm ...**

# Master Integrals (MIs)

Independent set of integrals $M_i^{[d]}$,

$$M_i^{[d]} \equiv \int_{q_1 \dots q_\ell} m_i(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \ ,$$

with a definite set of powers $\bar{\mathbf{x}}, \bar{\mathbf{y}}$ such that

$$F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \stackrel{\text{IBP}}{=} \sum_k c_k M_k^{[d]} \ , \qquad \forall (n,m)$$

They form a *basis* for the integrals of the corresponding topology.

## Two special cases

Two types of integrals generated from the **master integrands**

- Polynomial insertion: $$\int_{q_1 \dots q_\ell} P(q_i \cdot p_j, q_i \cdot q_j)\ m_i(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = \sum_{n,m} \alpha_{n,m}\ F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \stackrel{\text{IBP}}{=} \sum_i c_i\ M_i^{[d]}$$
- External-leg derivatives: $$p_i^\mu \frac{\partial}{\partial p_j^\mu} M_k^{[d]} = \int_{q_1 \dots q_\ell} p_i^\mu \frac{\partial}{\partial p_j^\mu}\ m_k(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = \sum_{n,m} \beta_{n,m}\ F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \stackrel{\text{IBP}}{=} \sum_i c_i\ M_i^{[d]}$$

# Dimensional Recurrence Relations for MIs

Bern, Dixon, Kosower
Tarasov; Baikov; Lee;
Gluza, Kajda, Kosower

**Gram determinant** $$P(q_i \cdot p_j, q_i \cdot q_j) = \mathbf{G}(q_i, p_j) = \begin{vmatrix} q_1^2 & \dots & (q_1 \cdot p_{e-1}) \\ \vdots & \ddots & \vdots \\ (p_{e-1} \cdot q_1) & \dots & p_{e-1}^2 \end{vmatrix}$$

**Dimension-shifted integrals**

$$F^{[d]}_{n,m}(\mathbf{x}, \mathbf{y}) \equiv \int_{q_1 \dots q_\ell} f_{n,m}(\mathbf{x}, \mathbf{y})$$

# Dimensional Recurrence Relations for MIs

Bern, Dixon, Kosower
Tarasov; Baikov; Lee;
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**Gram determinant**

$$P(q_i \cdot p_j, q_i \cdot q_j) = \mathbf{G}(q_i, p_j) = \begin{vmatrix} q_1^2 & \dots & (q_1 \cdot p_{e-1}) \\ \vdots & \ddots & \vdots \\ (p_{e-1} \cdot q_1) & \dots & p_{e-1}^2 \end{vmatrix}$$

**Dimension-shifted integrals**

$$F^{[d]}_{n,m}(\mathbf{x}, \mathbf{y}) \equiv \int_{q_1 \dots q_\ell} f_{n,m}(\mathbf{x}, \mathbf{y}) \qquad \Rightarrow \int_{q_1 \dots q_\ell} \mathbf{G} \; f_{n,m}(\mathbf{x}, \mathbf{y}) = \Omega(d, p_i) \; F^{[d+2]}_{n,m}(\mathbf{x}, \mathbf{y})$$

# Dimensional Recurrence Relations for MIs

Bern, Dixon, Kosower
Tarasov; Baikov; Lee;
Gluza, Kajda, Kosower

**Gram determinant** $$P(q_i \cdot p_j, q_i \cdot q_j) = \mathbf{G}(q_i, p_j) = \begin{vmatrix} q_1^2 & \ldots & (q_1 \cdot p_{e-1}) \\ \vdots & \ddots & \vdots \\ (p_{e-1} \cdot q_1) & \ldots & p_{e-1}^2 \end{vmatrix}$$

**Dimension-shifted integrals**

$$F^{[d]}_{n,m}(\mathbf{x}, \mathbf{y}) \equiv \int_{q_1 \ldots q_\ell} f_{n,m}(\mathbf{x}, \mathbf{y}) \qquad \Rightarrow \int_{q_1 \ldots q_\ell} \mathbf{G} \; f_{n,m}(\mathbf{x}, \mathbf{y}) = \Omega(d, p_i) \, F^{[d+2]}_{n,m}(\mathbf{x}, \mathbf{y})$$

**G-insertion** generates shifted dim. integrals: **d --> d+2**

# Dimensional Recurrence Relations for MIs

Bern, Dixon, Kosower
Tarasov; Baikov; Lee;
Gluza, Kajda, Kosower

**Gram determinant**
$$P(q_i \cdot p_j, q_i \cdot q_j) = \mathbf{G}(q_i, p_j) = \begin{vmatrix} q_1^2 & \dots & (q_1 \cdot p_{e-1}) \\ \vdots & \ddots & \vdots \\ (p_{e-1} \cdot q_1) & \dots & p_{e-1}^2 \end{vmatrix}$$

**Dimension-shifted integrals**

$$F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \equiv \int_{q_1 \dots q_\ell} f_{n,m}(\mathbf{x}, \mathbf{y}) \qquad \Rightarrow \int_{q_1 \dots q_\ell} \mathbf{G} \; f_{n,m}(\mathbf{x}, \mathbf{y}) = \Omega(d, p_i) \; F_{n,m}^{[d+2]}(\mathbf{x}, \mathbf{y})$$

In the case of **Master integrals**

$$M_k^{[d+2]} = \Omega(d, p_i)^{-1} \int_{q_1 \dots q_\ell} \mathbf{G} \; m_k(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \stackrel{\text{IBP}}{=} \sum_i c_{k,i} \; M_i^{[d]}$$

# Dimensional Recurrence Relations for MIs

Bern, Dixon, Kosower
Tarasov; Baikov; Lee;
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**Gram determinant** $P(q_i \cdot p_j, q_i \cdot q_j) = \mathbf{G}(q_i, p_j) = \begin{vmatrix} q_1^2 & \dots & (q_1 \cdot p_{e-1}) \\ \vdots & \ddots & \vdots \\ (p_{e-1} \cdot q_1) & \dots & p_{e-1}^2 \end{vmatrix}$

**Dimension-shifted integrals**

$$F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \equiv \int_{q_1 \dots q_\ell} f_{n,m}(\mathbf{x}, \mathbf{y}) \qquad \Rightarrow \int_{q_1 \dots q_\ell} \mathbf{G}\ f_{n,m}(\mathbf{x}, \mathbf{y}) = \Omega(d, p_i)\ F_{n,m}^{[d+2]}(\mathbf{x}, \mathbf{y})$$

In the case of **Master integrals**

$$M_k^{[d+2]} = \Omega(d, p_i)^{-1} \int_{q_1 \dots q_\ell} \mathbf{G}\ m_k(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \stackrel{\text{IBP}}{=} \sum_i c_{k,i}\ M_i^{[d]}$$

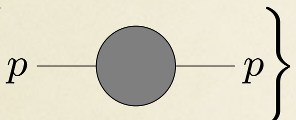
which can be seen as a **Dimensional recurrence relation**

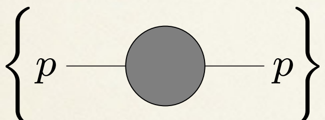
In general, $n$ MIs obey a **system of Dimensional recurrence relations**

$$\mathbf{M}^{[d]} \equiv \begin{pmatrix} M_1^{[d]} \\ \vdots \\ M_n^{[d]} \end{pmatrix} \qquad \mathbf{M}^{[d+2]} = \mathbb{C}(d)\ \mathbf{M}^{[d]}$$

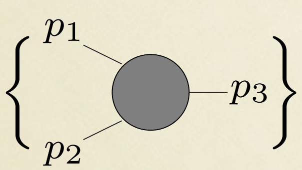
# Differential Equations for MIs

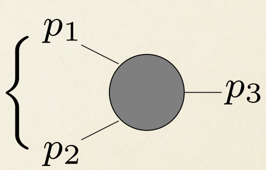
Bern, Dixon, Kosower
Kotikov; Remiddi;
Gehrmann, Remiddi
Argeri, Bonciani, Ferroglia, Remiddi, **P.M.**
...
Henn; Henn, Smirnov;
Lee; Papadopoulos;
Argeri, diVita, Mirabella, Schlenk, Schubert, Tancredi, **P.M.**
diVita, Schubert, Yundin, **P.M.**
Zeng
Primo, Tancredi
...

$$p^2 \frac{\partial}{\partial p^2} \Big\{$$ 

$$\Big\} = \frac{1}{2} p_\mu \frac{\partial}{\partial p_\mu} \Big\{$$ 

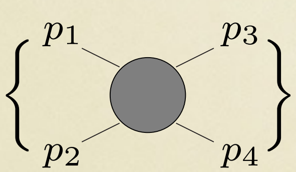
$$\Big\}$$

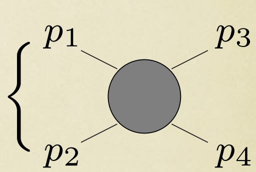
$$P^2 \frac{\partial}{\partial P^2} \Big\{$$ 

$$\Big\} = \left[ A \left( p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} + p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} \right) + B \left( p_{1,\mu} \frac{\partial}{\partial p_{2,\mu}} + p_{2,\mu} \frac{\partial}{\partial p_{1,\mu}} \right) \right] \Big\{$$ 

$$\Big\}$$

$$P = p_1 + p_2,$$

$$P^2 \frac{\partial}{\partial P^2} \Big\{$$ 

$$\Big\} = \left[ C \left( p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} - p_{3,\mu} \frac{\partial}{\partial p_{3,\mu}} \right) + D p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} + E (p_{1,\mu} + p_{3,\mu}) \left( \frac{\partial}{\partial p_{3,\mu}} - \frac{\partial}{\partial p_{1,\mu}} + \frac{\partial}{\partial p_{2,\mu}} \right) \right] \Big\{$$ 

$$\Big\}$$

In general, $n$ MIs obey a **system of 1st ODE**

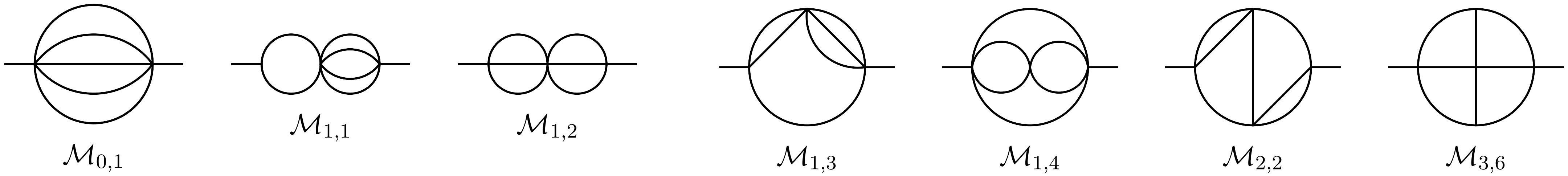
$$\partial_z \mathbf{M}^{[d]} = \mathbb{A}(d, z) \; \mathbf{M}^{[d]}$$

# back to EFT-GR @ 4PN - $O(G^5)$

# 7 Master Integrals

- IBP Reduction (i. *in-house* code + ii. Reduze2)
- 50 EFT Integrals ==> 29 Topologies ==> 7 MIs

$$\bar{\mathbf{x}} = (1, \ldots, 1)\ , \quad \bar{\mathbf{y}} = (0, \ldots, 0)$$




$$\mathcal{M}_{0,1} = \int_{k_{1\ldots4}} \frac{1}{D_{1\ldots4} D_{14}}, \qquad \mathcal{M}_{1,1} = \int_{k_{1\ldots4}} \frac{1}{D_{1\ldots4} D_9 D_{12}},$$

$$\mathcal{M}_{1,2} = \int_{k_{1\ldots4}} \frac{1}{D_{1\ldots4} D_{10} D_{11}}, \qquad \mathcal{M}_{1,3} = \int_{k_{1\ldots4}} \frac{1}{D_{1\ldots4} D_8 D_{10}},$$

$$\mathcal{M}_{1,4} = \int_{k_{1\ldots4}} \frac{1}{D_{1\ldots4} D_7 D_{13}}, \qquad \mathcal{M}_{2,2} = \int_{k_{1\ldots4}} \frac{1}{D_{1\ldots4} D_{10} D_{15} D_{16}},$$

$$\mathcal{M}_{3,6} = \int_{k_{1\ldots4}} \frac{1}{D_{1\ldots4} D_5 D_6 D_{10} D_{14}},$$

$$D_{1\ldots4} = k_1^2\, k_2^2\, k_3^2\, k_4^2, \qquad D_5 = (k_2 - k_3)^2, \qquad D_6 = (k_1 - k_4)^2,$$

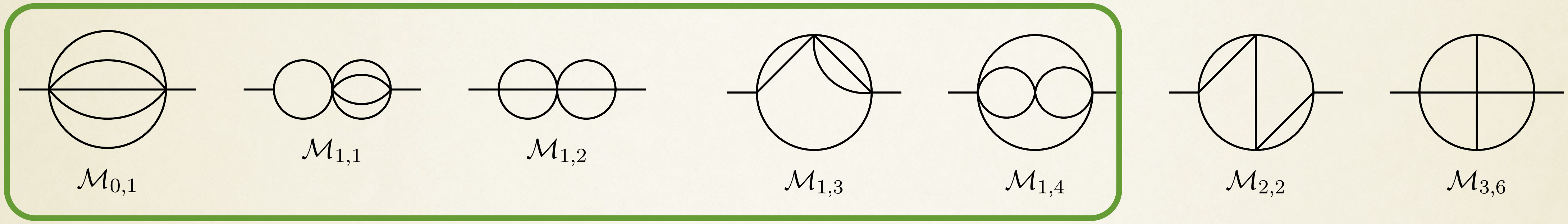
$$D_7 = (k_2 + k_3 - k_4)^2, \qquad D_8 = (k_1 + k_2 + k_3 - k_4)^2, \qquad D_9 = (k_1 - p)^2,$$

$$D_{10} = (k_1 + k_2 - p)^2, \qquad D_{11} = (k_3 + k_4 + p)^2, \qquad D_{12} = (k_2 - k_3 - k_4 + p)^2,$$

$$D_{13} = (k_1 - k_2 - k_3 + p)^2, \qquad D_{14} = (k_1 + k_2 - k_3 - k_4 - p)^2,$$

$$D_{15} = (k_1 + k_4 - p)^2, \qquad D_{16} = (k_2 + k_3 - p)^2.$$

# 7 Master Integrals




- 5 easy MIs

$$\mathcal{M}_{0,1} = (4\pi)^{-2d} s^{2d-5} \frac{\Gamma(5-2d)\Gamma(\frac{d}{2}-1)^5}{\Gamma\left(\frac{5}{2}d-5\right)}$$

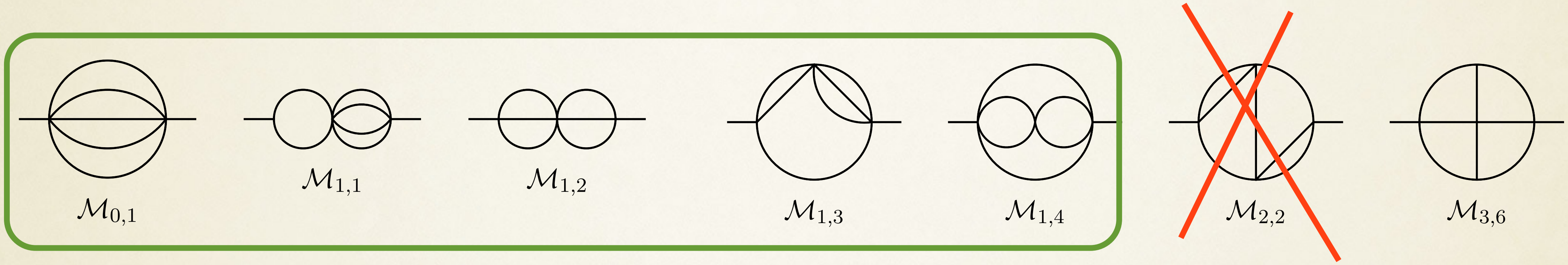
$$\mathcal{M}_{1,1} = (4\pi)^{-2d} s^{2d-6} \frac{\Gamma\left(4-\frac{3}{2}d\right)\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)^6}{\Gamma(d-2)\Gamma(2d-4)}$$

$$\mathcal{M}_{1,2} = (4\pi)^{-2d} s^{2d-6} \frac{\Gamma(3-d)^2\Gamma\left(\frac{d}{2}-1\right)^6}{\Gamma\left(\frac{3}{2}d-3\right)^2}$$

$$\mathcal{M}_{1,3} = (4\pi)^{-2d} s^{2d-6} \frac{\Gamma(6-2d)\Gamma(3-d)\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)^6\Gamma(2d-5)}{\Gamma\left(5-\frac{3}{2}d\right)\Gamma(d-2)\Gamma\left(\frac{3}{2}d-3\right)\Gamma\left(\frac{5}{2}d-6\right)}$$

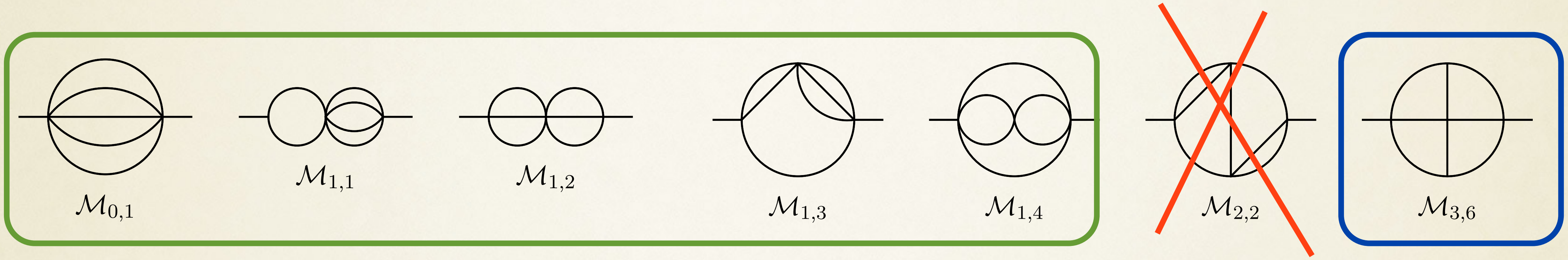
$$\mathcal{M}_{1,4} = (4\pi)^{-2d} s^{2d-6} \frac{\Gamma(6-2d)\Gamma\left(2-\frac{d}{2}\right)^2\Gamma\left(\frac{d}{2}-1\right)^6\Gamma\left(\frac{3}{2}d-4\right)}{\Gamma(4-d)\Gamma(d-2)^2\Gamma\left(\frac{5}{2}d-6\right)}$$

# 7 Master Integrals



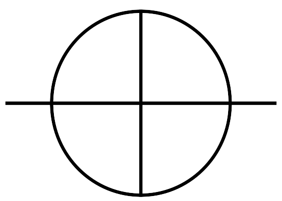
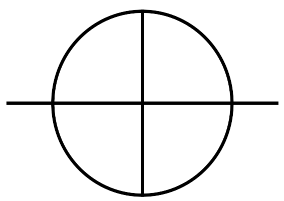
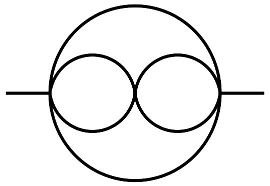
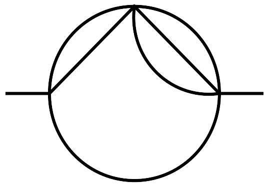
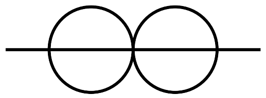
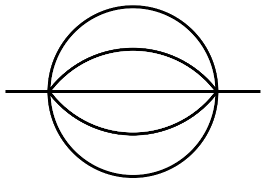

- 5 easy MIs
- $M_{2,2}$ drops out in the d --> 3 limit

# 7 Master Integrals



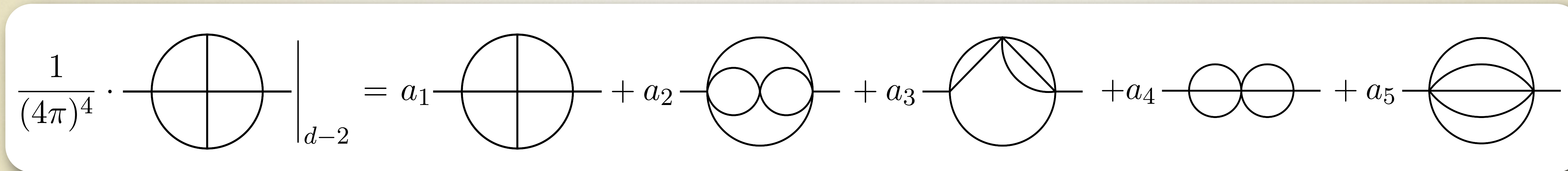

- 5 easy MIs
- $M_{2,2}$ drops out in the d --> 3 limit
- $M_{3,6}$ non-trivial

# Dimensional Recurrence for $M_{3,6}$

$\frac{1}{(4\pi)^4} \cdot$  $\Big|_{d-2} = a_1$  $+ a_2$  $+ a_3$  $+a_4$  $+ a_5$ 

- ☑ d=3 from d=5

# Dimensional Recurrence for M3,6




☑ **d=3 from d=5**

**Numerical Solution of Dim. Rec. Rel.** SUMMERTIME Lee, Mingulov

$$
\begin{aligned}
\mathcal{M}_{3,6} = s^{2\varepsilon-2}\,[\quad & 0.0000200507465911803421663140298185911994957574254953872318\,7/\varepsilon^2 \\
& -0.000098401388124608332497837406855433503738550841537985141\,38/\varepsilon \\
& +0.000087517902709298124514308005959307153063894547697305056\,64 \\
& +0.000836408964802425654539965887062813673417581308375565484\,95\,\varepsilon \\
& +\mathcal{O}\left(\varepsilon^2\right)]
\end{aligned}
$$

# Experimental Mathematics for $M_{3,6}$

- **Numerical Reconstruction**
- **from SUMMERTIME**

In[1]:=
```
nM36 = (
    0.0000200507465911803421663140298185911994957574254953872318 7 / ϵ^2
      - 0.0000984013881246083324978374068554335037385508415379851413 8 / ϵ
     + 0.0000875179027092981245143080059593071530638945476973050566 4
     + 0.0008364089648024256545399658870628136734175813083755654849 5 * ϵ
     + ϵ^2 * Help[ϵ, 2]
   );
```

In[2]:=
```
pref = (4 * Pi) ^ (- 4 - 2 * ϵ) * Exp[2 * ϵ * EulerGamma]
```

Out[2]= $e^{2\gamma\epsilon}\,(4\,\pi)^{-2\,\epsilon-4}$

In[3]:=
```
npref = N[Series[pref, {ϵ, 0, 2}], 50] // Chop
```

Out[3]= $0.000040101493182360684332628059637182398991514850990774 - 0.00015670128306685598066304675407368460848558683208520\,\epsilon + 0.00030616431167705224971803922217880567378514178260532\,\epsilon^2 + O(\epsilon^3)$

In[4]:=
```
nBexp = nM36 / npref
```

Out[4]= $\frac{0.5000000000000000000000000000000000000000000000}{\epsilon^2} - \frac{0.5000000000000000000000000000000000000000000000}{\epsilon} - 3.5887664832879433908818962083384937026952625247 0 + O(\epsilon^1)$

# Experimental Mathematics for $M_{3,6}$

- **Numerical Reconstruction**
- **from SUMMERTIME**

In[1]:=
```
nM36 = (
    0.000020050746591180342166314029818591199495757425495387231 87 / ϵ^2
      - 0.0000984013881246083324978374068554335037385508415379851 4138 / ϵ
     + 0.00008751790270929812451430800595930715306389454769730505664
     + 0.00083640896480242565453996588706281367341758130837556548495 * ϵ
     + ϵ^2 * Help[ϵ, 2]
  );
```

In[2]:=
```
pref = (4 * Pi) ^ (-4 - 2 * ϵ) * Exp[2 * ϵ * EulerGamma]
```

Out[2]= $e^{2\gamma\epsilon}(4\pi)^{-2\epsilon-4}$

In[3]:=
```
npref = N[Series[pref, {ϵ, 0, 2}], 50] // Chop
```

Out[3]= $0.000040101493182360684332628059637182398991514850990774 - 0.00015670128306685598066304675407368460848558683208520\,\epsilon + 0.00030616431167705224971803922217880567378514178260532\,\epsilon^2 + O(\epsilon^3)$

In[4]:=
```
nBexp = nM36 / npref
```

Out[4]= $\frac{0.5000000000000000000000000000000000000000000000000}{\epsilon^2} - \frac{0.500000000000000000000000000000000000000000000000}{\epsilon} - 3.58876648328794339088189620833849370269526252470 + O(\epsilon^1)$

(1/2) :: double pole

(-1/2) :: single pole

(?) :: finite term

# Experimental Mathematics for $M_{3,6}$

## Finite term O(1)

- **50 digits**

In[5]:=
```
test = N[Coefficient[nBexp, ϵ, 0], 50]
```

Out[5]= $-3.5887664832879433908818962083384937026952625247 0$

In[6]:=
```
vars = {1, Pi, Log[2], Zeta[2], Log[2]^2, Zeta[3], Log[2] * Zeta[2], Pi * Zeta[2]}
```

Out[6]= $\left\{1, \pi, \log(2), \frac{\pi^2}{6}, \log^2(2), \zeta(3), \frac{1}{6}\pi^2\log(2), \frac{\pi^3}{6}\right\}$ ← :: transcendental constants :: :: educated guess ::

In[7]:=
```
guess = {test, vars} // Flatten
```

Out[7]= $\left\{-3.5887664832879433908818962083384937026952625247 0, 1, \pi, \log(2), \frac{\pi^2}{6}, \log^2(2), \zeta(3), \frac{1}{6}\pi^2\log(2), \frac{\pi^3}{6}\right\}$

In[8]:=
```
fit = FindIntegerNullVector[guess]
```

Out[8]= $\{4, 16, 0, 0, -1, 0, 0, 0, 0\}$

In[9]:=
```
check = Sum[fit[[i]] * guess[[i]], {i, 1, Length[guess]}]
```

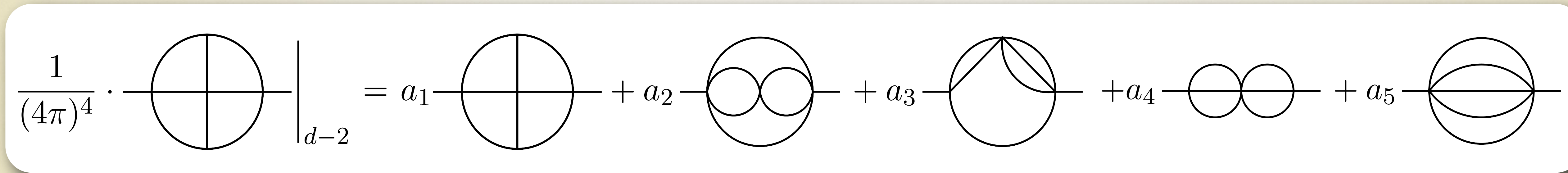
Out[9]= $0.\times 10^{-48}$

- **result**

In[10]:=
```
res = -1 / fit[[1]] * Sum[fit[[i]] * guess[[i]], {i, 2, Length[guess]}] // Expand
```

Out[10]= $\frac{\pi^2}{24} - 4$ ← :: finite term

# Dimensional Recurrence for $M_{3,6}$




☑ **d=3 from d=5**

## Numerical Solution of Dim. Rec. Rel.

**SUMMERTIME** Lee, Mingulov

$$\begin{aligned}\mathcal{M}_{3,6} = s^{2\varepsilon-2}\,[\quad & 0.00002005074659118034216631402981859119949575742549538723187/\varepsilon^2 \\ & -0.00009840138812460833249783740685543350373855084153798514138/\varepsilon \\ & +0.0000875179027092981245143080059593071530638945476973050564 \\ & +0.00083640896480242565453996588706281367341758130837556548495\,\varepsilon \\ & +\mathcal{O}\left(\varepsilon^2\right)]\end{aligned}$$

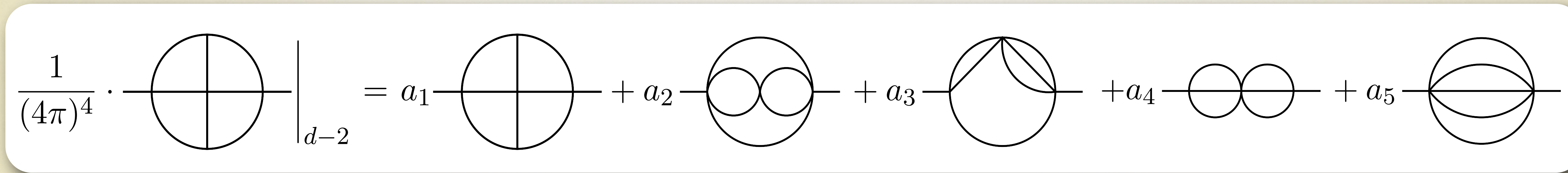
## Analytic *ansatz*

:: experimental mathematics ::

$$= s^{2\varepsilon-2}(4\pi)^{-4-2\varepsilon}e^{2\varepsilon\gamma_E}\frac{1}{2}\left[\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}-8+\frac{\pi^2}{12}-\varepsilon\left(18-\pi^2\left(\frac{13}{4}-2\log 2\right)-\frac{77}{3}\zeta_3\right)+\mathcal{O}\left(\varepsilon^2\right)\right]$$ ☑

confirmed by
Damour, Jaranowski (analytical)

# Dimensional Recurrence for $M_{3,6}$




☑ **d=3 from d=5**

## Numerical Solution of Dim. Rec. Rel.

**SUMMERTIME** Lee, Mingulov

$$
\begin{aligned}
\mathcal{M}_{3,6} = s^{2\varepsilon-2}\,[\;\; & 0.000020050746591180342166314029818591199495757425495387231 87/\varepsilon^2 \\
& -0.000098401388124608332497837406855433503738550841537985 14138/\varepsilon \\
& +0.00008751790270929812451430800595930715306389454769730505664 \\
& +0.00083640896480242565453996588706281367341758130837556548495\,\varepsilon \\
& +\mathcal{O}\left(\varepsilon^2\right)]
\end{aligned}
$$

## Analytic *ansatz*

:: experimental mathematics ::

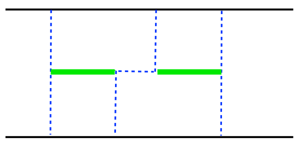
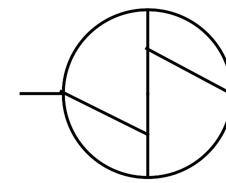
$$
= s^{2\varepsilon-2}(4\pi)^{-4-2\varepsilon}e^{2\varepsilon\gamma_E}\frac{1}{2}\left[\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}-8+\boxed{\frac{\pi^2}{12}}-\varepsilon\left(18-\pi^2\left(\frac{13}{4}-2\log 2\right)-\frac{77}{3}\zeta_3\right)+\mathcal{O}\left(\varepsilon^2\right)\right]
$$
☑

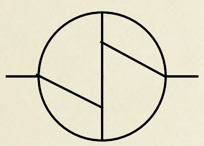
important impact

confirmed by
Damour, Jaranowski (analytical)

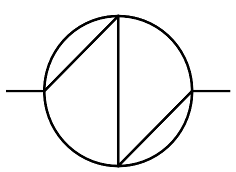
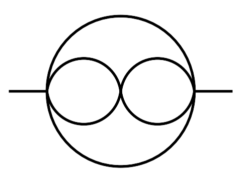
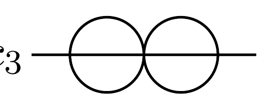
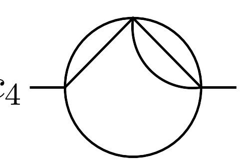
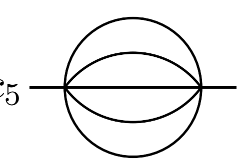
# Computational Algorithm

## Amplitudes

$$\mathcal{A}_{49} =$$  $$= -2\,\mathrm{i}\,(8\pi G_N)^5 \left(\frac{(d-2)}{(d-1)}\, m_1 m_2\right)^3$$  $$[N_{49}]$$

 $$[N_{49}] \equiv \int_{k_1,k_2,k_3,k_4} \frac{N_{49}}{k_1^2\; p_2^2\; k_3^2\; p_4^2\; k_{12}^2\; k_{13}^2\; k_{23}^2\; k_{24}^2\; k_{34}^2}\,,$$

$$N_{49} \equiv (k_1\cdot k_3\;k_{12}\cdot k_{23} - k_1\cdot k_{12}\;k_3\cdot k_{23} - k_1\cdot k_{23}\;k_3\cdot k_{12}) \times (p_2\cdot k_{23}\;p_4\cdot k_{34} + p_4\cdot k_{23}\;p_2\cdot k_{34} - p_2\cdot p_4\;k_{23}\cdot k_{34})\,,$$

$$= c_1$$  $$+ c_2$$  $$+ c_3$$  $$+ c_4$$  $$+ c_5$$ 

IBPs

$$\mathcal{A}_{49} = -\mathrm{i}(8\pi G_N)^5 (m_1 m_2)^3 2^{-4} (4\pi)^{-(4+2\varepsilon)} e^{2\varepsilon\gamma_E} s^{(1+2\varepsilon)} \left[\frac{1}{\varepsilon}\left(\frac{\pi^2}{16} - \frac{2}{3}\right) + \frac{29}{18} - \frac{13}{144}\pi^2 - \frac{\pi^2}{8}\log 2 + \mathcal{O}(\varepsilon^1)\right]$$

MIs

## Lagrangian

$$\mathcal{L}_{49} = -\mathrm{i}\lim_{d\to 3}\int_p \mathrm{e}^{\mathrm{i}p\cdot r}\mathcal{A}_{49} = (32 - 3\pi^2)\frac{G_N^5 m_1^3 m_2^3}{r^5}$$

Fourier Tfm

# (Impact on) **The 4PN O(G^5) Lagrangian**

Foffa, Sturani, Sturm, & **P.M.**
Damour, Jaranowski

## Individual terms

$$0 = \mathcal{L}_9 = \mathcal{L}_{12} = \mathcal{L}_{13} = \mathcal{L}_{22} = \mathcal{L}_{26} = \mathcal{L}_{27} = \mathcal{L}_{31} = \mathcal{L}_{36} = \mathcal{L}_{46} = \mathcal{L}_{47} \,,$$

$$\frac{1}{2}\frac{G_N^5 m_1^3 m_2^3}{r^5} = \mathcal{L}_1 = \mathcal{L}_3 = 4\mathcal{L}_5 = 3\mathcal{L}_{14} = \frac{\mathcal{L}_{19}}{8} = \frac{3\mathcal{L}_{20}}{2} = \frac{3\mathcal{L}_{21}}{4} = \frac{\mathcal{L}_{23}}{4} = \frac{\mathcal{L}_{24}}{4} = \frac{3\mathcal{L}_{25}}{2} \,,$$

$$\frac{1}{2}\frac{G_N^5 m_1^4 m_2^2}{r^5} = \mathcal{L}_2 = 3\mathcal{L}_4 = \frac{3\mathcal{L}_8}{2} = \frac{3\mathcal{L}_{10}}{2} = \frac{3\mathcal{L}_{11}}{2} = \frac{\mathcal{L}_{15}}{4} = \frac{3\mathcal{L}_{16}}{4} = \frac{3\mathcal{L}_{17}}{4} = \frac{\mathcal{L}_{18}}{4} \,,$$

$$\frac{1}{120}\frac{G_N^5 m_1^5 m_2}{r^5} = \mathcal{L}_6 = \frac{\mathcal{L}_7}{20} = \frac{3\mathcal{L}_{30}}{20} = -\frac{3\mathcal{L}_{35}}{56} = \frac{\mathcal{L}_{39}}{24} = \frac{\mathcal{L}_{45}}{12} \,,$$

$$\mathcal{L}_{28} = \frac{G_N^5 m_1^4 m_2^2}{r^5}\left[\frac{428}{75} + \frac{4}{15}\mathcal{P}\right] \,,$$

$$\mathcal{L}_{32} = \frac{G_N^5 m_1^3 m_2^3}{r^5}\left[-\frac{91}{450} + \frac{1}{15}\mathcal{P}\right] \,,$$

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$$\mathcal{L}_{41} = \frac{G_N^5 m_1^3 m_2^3}{r^5}\left[\frac{49}{18} + \frac{1}{3}\mathcal{P}\right] \,,$$

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$$\mathcal{L}_{50} = \frac{G_N^5 m_1^3 m_2^3}{r^5}\left(4\pi^2 - \frac{124}{3}\right)$$

$$\mathcal{P} \equiv \frac{1}{\varepsilon} - 5\log\frac{r}{L_0}$$

$$L = \sqrt{4\pi e^{\gamma_E}}\, L_0$$

$$\Lambda^{-2} \equiv 32\pi G_N L^{d-3}$$

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diverge @ d=3

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**finite @ d=3**

# (Impact on) **The 4PN O(G^5) Lagrangian**

Foffa, Sturani, Sturm, & **P.M.**
Damour, Jaranowski

**Total contribution**

$$\sum_{a=1}^{50} \mathcal{L}_a = \frac{3}{8}\frac{G_N^5 m_1^5 m_2}{r^5} + \frac{31}{3}\frac{G_N^5 m_1^4 m_2^2}{r^5} + \frac{141}{8}\frac{G_N^5 m_1^3 m_2^3}{r^5}\,.$$

finite @ d=3

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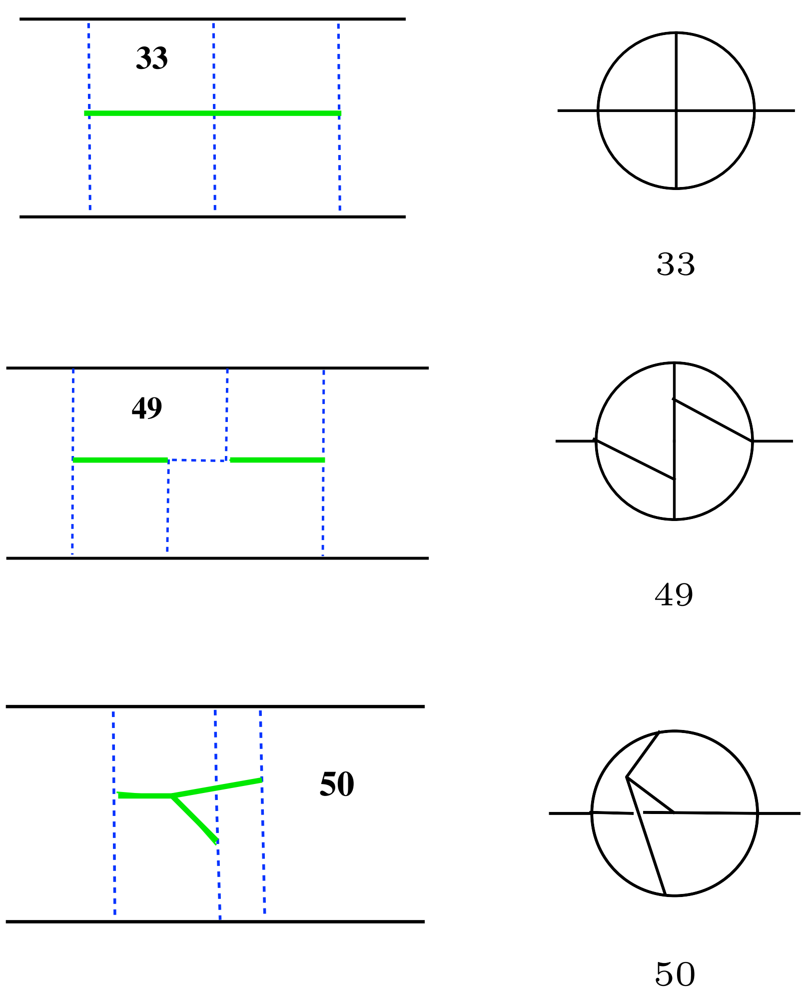
## Recap :: Pi^2-Terms

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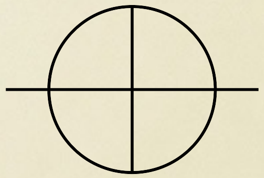
## $M_{3,6}$ contribution




Damour, Jaranowski
Foffa, Sturani, Sturm, & **P.M.**

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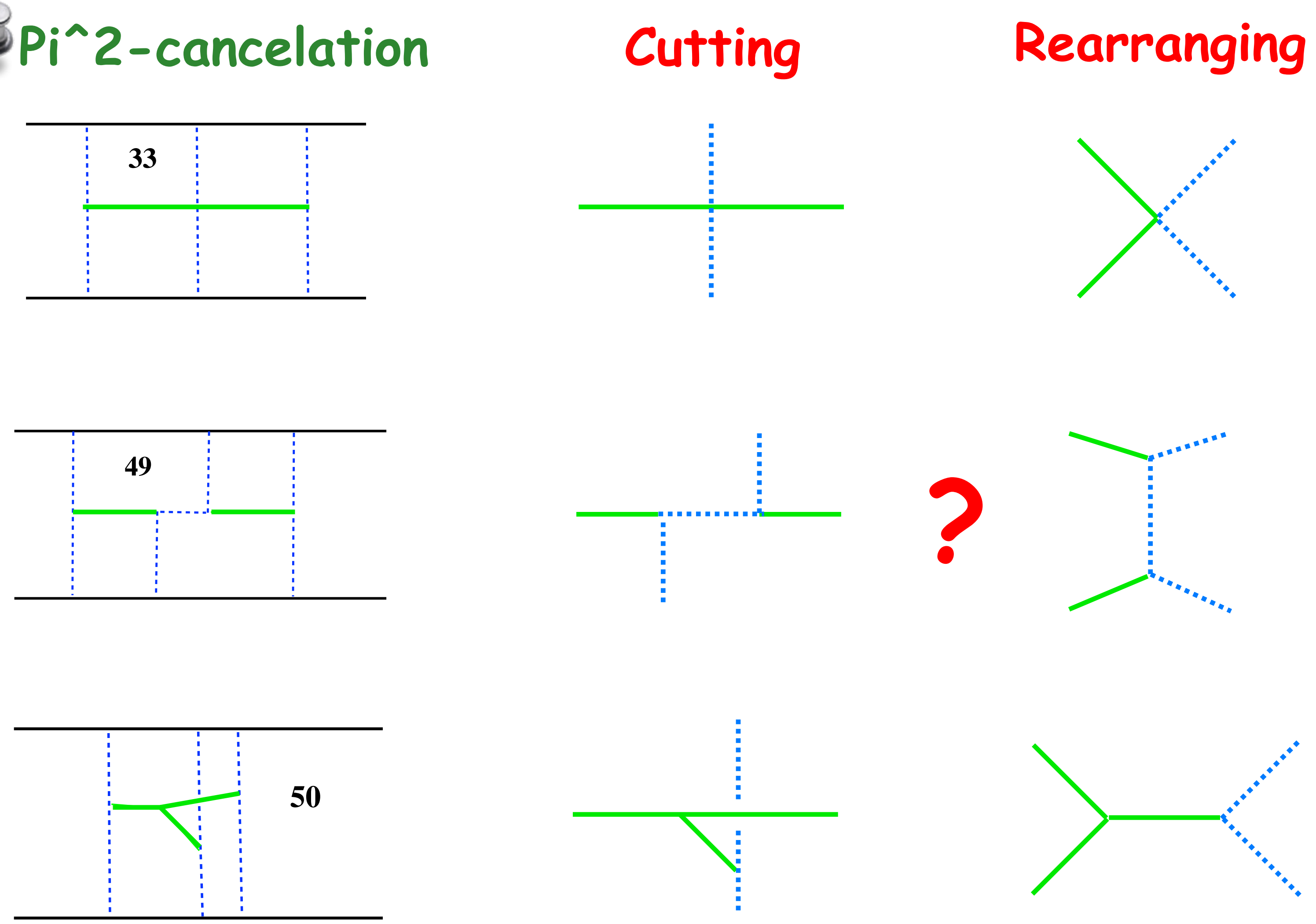
- **Pi^2-cancelation**

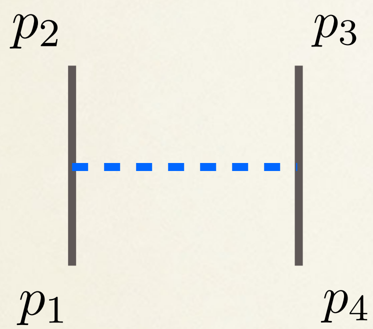


# Towards 5PN-O(G^6)

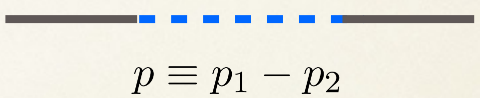
# Vacuum Diagrams for Newton Potential

Foffa, Sturani, Sturm,
Torres-Bobadilla & **P.M.**
(coming soon)

## Amplitudes

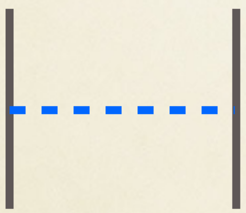



$$= \frac{1}{p^2} =$$




## Newton Potential

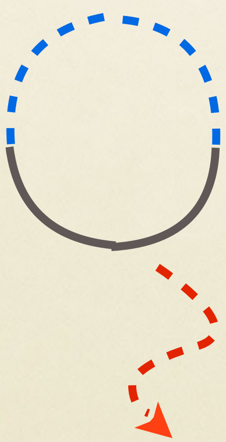
$$\int d^dp\ e^{ip\cdot r}$$



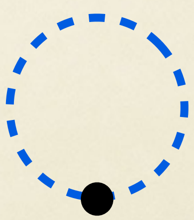
$$= \int d^dp\ \frac{e^{ip\cdot r}}{p^2} = \int d^dp\ e^{ip\cdot r}$$



$$=$$



$$=$$

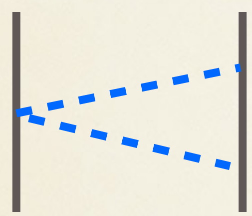
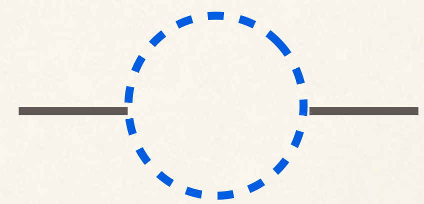


static sources <==> non propagating <==> pinching lines

# PostNewtonian Corrections

Foffa, Sturani, Sturm,
Torres-Bobadilla & **P.M.**
(coming soon)

## 1PN correction :: explicit calculation

$$\int d^d p \; e^{ip\cdot r}$$  $$= \int d^d p \; e^{ip\cdot r}$$ 

$$= \int d^d p \; e^{ip\cdot r} \int d^d k \, \frac{1}{(p-k)^2 \; k^2}$$

$$= \int d^d p \, \frac{e^{i(p-k)\cdot r}}{(p-k)^2} \int d^d k \, \frac{e^{ik\cdot r}}{k^2}$$

$$= \int d^d p \, \frac{e^{ip\cdot r}}{p^2} \int d^d k \, \frac{e^{ik\cdot r}}{k^2}$$

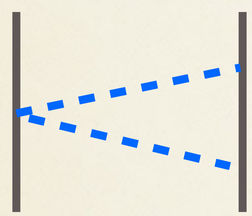
Static 1PN = Newton^2

# PostNewtonian Corrections

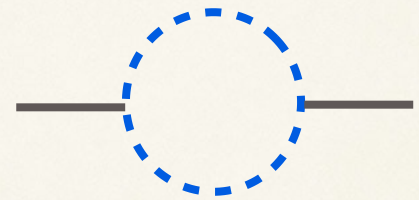
Foffa, Sturani, Sturm,
Torres-Bobadilla & **P.M.**
(coming soon)

## 1PN correction :: diagrammatic

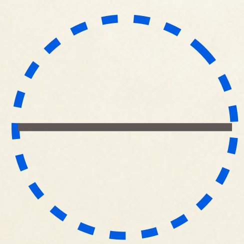
$$\int d^d p \; e^{ip\cdot r}$$



$$= \int d^d p \; e^{ip\cdot r}$$



$$=$$



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Foffa, Sturani, Sturm,
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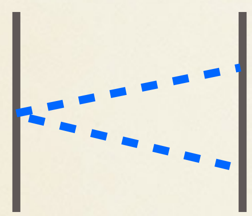
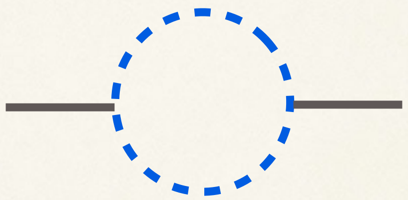
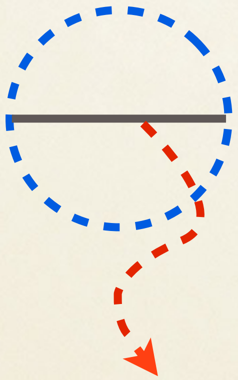
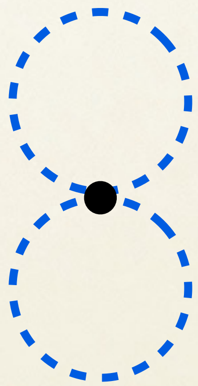
(coming soon)

## 1PN correction

$\int d^d p \, e^{ip\cdot r}$  $=$ $\int d^d p \, e^{ip\cdot r}$ 

$=$  $=$ 

Static 1PN = Newton^2

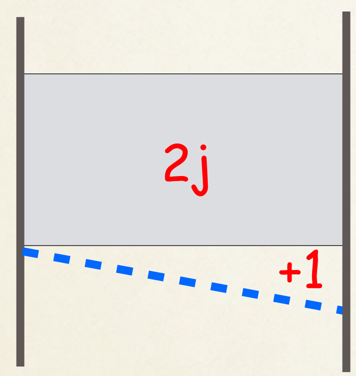
**static sources <==> non propagating <==> pinching lines**

# PostNewtonian Corrections

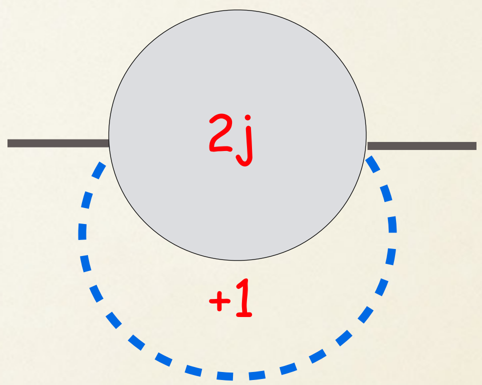
Foffa, Sturani, Sturm,
Torres-Bobadilla & **P.M.**
(coming soon)

- **(2j+1)PN correction**

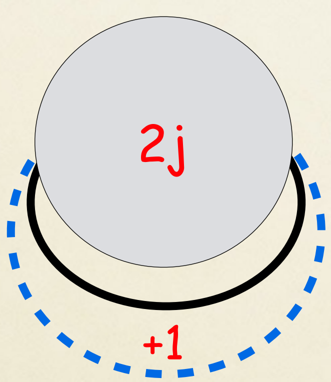
$$\int d^dp \; e^{ip\cdot r}$$




$$= \int d^dp \; e^{ip\cdot r}$$



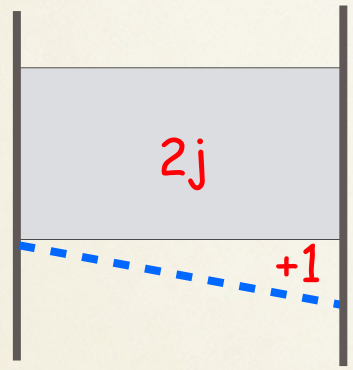

$$=$$

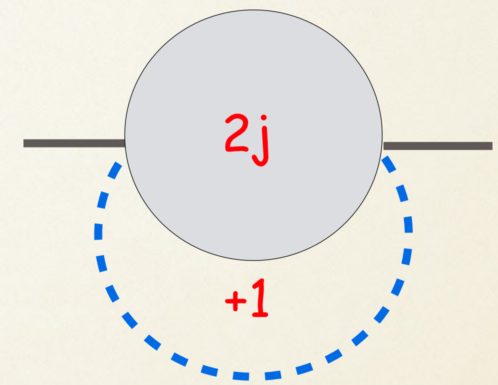


# PostNewtonian Corrections

Foffa, Sturani, Sturm, Torres-Bobadilla & **P.M.** (coming soon)

- (2j+1)PN correction

$$\int d^d p \; e^{ip\cdot r}$$


$$= \int d^d p \; e^{ip\cdot r}$$
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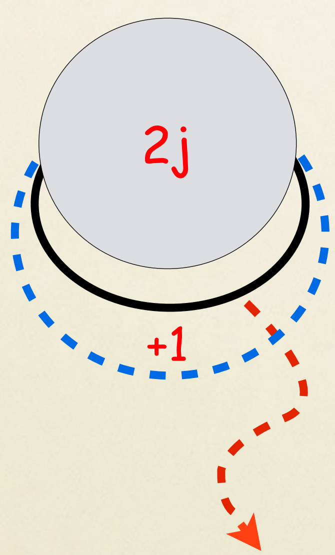

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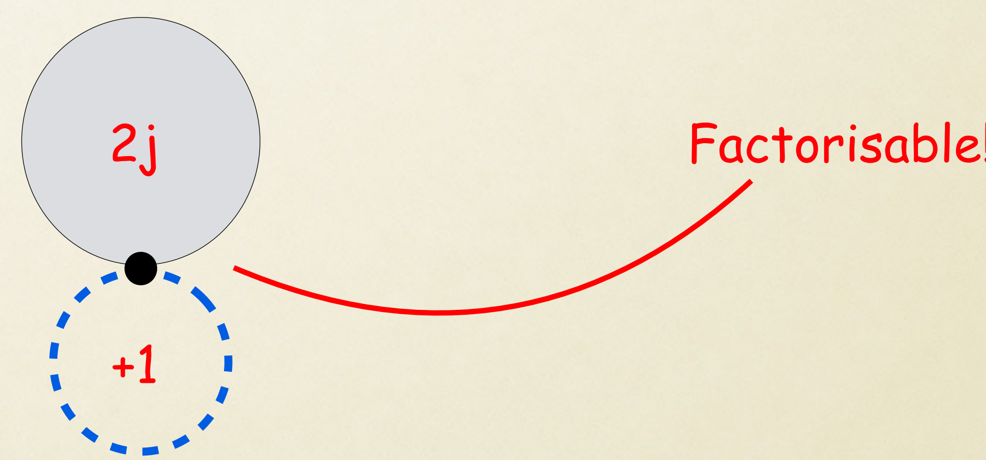


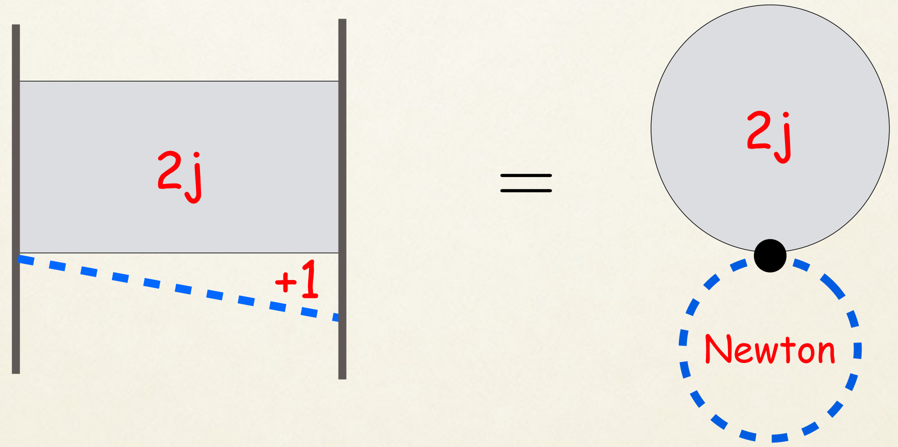

static sources <==> non propagating <==> pinching lines

# PostNewtonian Corrections

Foffa, Sturani, Sturm, Torres-Bobadilla & **P.M.** (coming soon)

- (2j+1)PN correction

$$\int d^d p \; e^{ip\cdot r}$$




**static (2j+1)-PN diagrams as product of (2j)-PN diagrams and the Newtonian term**

# Amplitudes @ 4PN - O(G^5)

Foffa, Sturani, Sturm, & **P.M.**

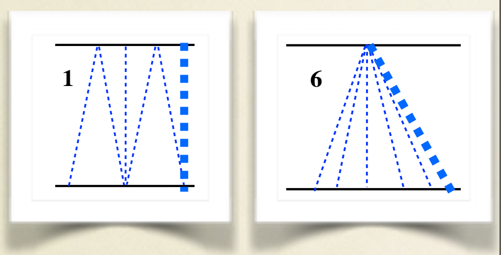
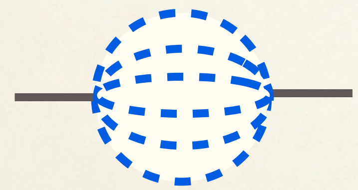
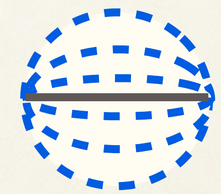
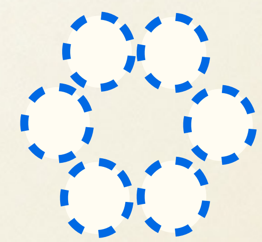
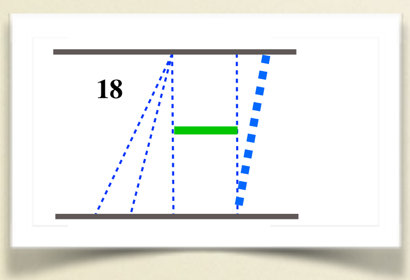
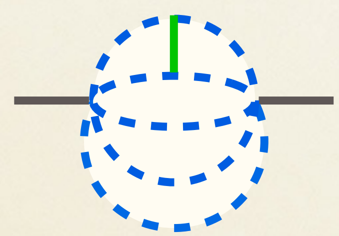
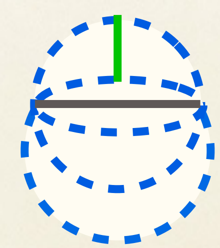
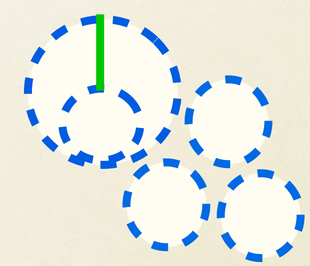
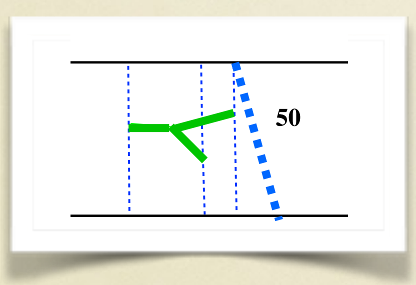
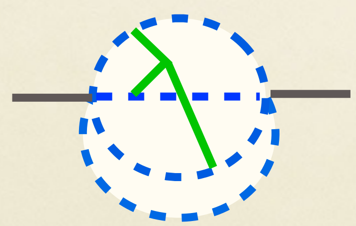
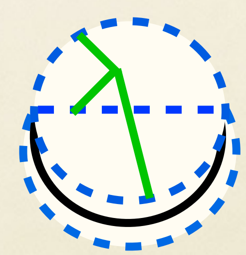
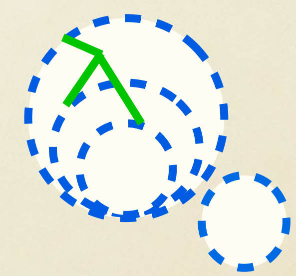
- **50 Amplitudes @ 4 loop: 25 of them…**

# Amplitudes @ 5PN - O(G^6)

Foffa, Sturani, Sturm,
Torres-Bobadilla & **P.M.**
(coming soon)

- **Amplitudes @ 5 loop: ALL give factorisable potential**

**Digram generation**: from our 4PN O(G^5) diagram + 1 Phi >······ < insertion

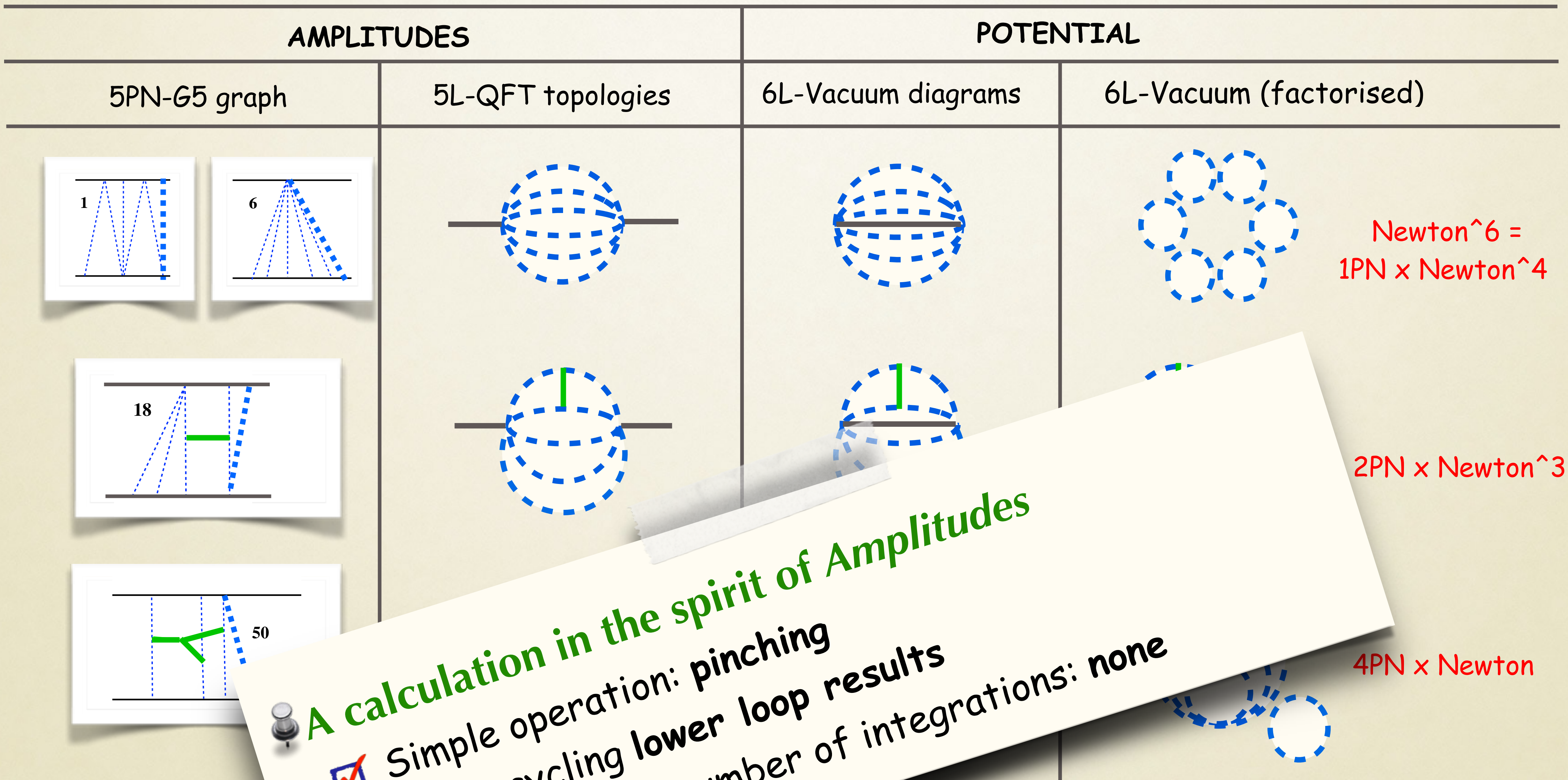
| AMPLITUDES | | POTENTIAL | |
|---|---|---|---|
| 5PN-G5 graph | 5L-QFT topologies | 6L-Vacuum diagrams | 6L-Vacuum (factorised) |
|  |  |  |  Newton^6 = 1PN x Newton^4 |
|  |  |  |  2PN x Newton^3 |
|  |  |  |  4PN x Newton |

# Amplitudes @ 5PN - O(G^6)

Foffa, Sturani, Sturm,
Torres-Bobadilla & **P.M.**
(coming soon)

- **Amplitudes @ 5 loop: ALL give factorisable potential**

Digram generation: from our 4PN O(G^5) diagram + 1 Phi >⋯⋯< insertion




- **A calculation in the spirit of Amplitudes**
  - ☑ Simple operation: **pinching**
  - ☑ ♻ Recycling **lower loop results**
  - ☑ Minimizing the number of integrations: **none**

# Summary ...

- **Multi-Loop Diagrammatic Techniques**
  - **Powerful tools for General Relativity**
  - **IBPs + Difference & Differential Equations**

- Application 1 **:: Coalescent Binaries Dynamics @ 4PN-O(G^5)**
  - **Basic idea @ *Amplitudes* :: EFT-GR Diagrams ~ 2-point QFT Diagrams**
  - **4-loop EFT-GR Diagrams mapped into 4-loop QFT Problem**

- Application 2 **:: Coalescent Binaries Dynamics @ 5PN-O(G^6)**
  - **Basic idea @ *Potential* :: EFT-GR Diagrams ~ Vacuum Diagrams + factorisation**
  - **5-loop EFT-GR Diagrams mapped into 5-loop factorised Vacuum Diagrams**

# ... and Outlook

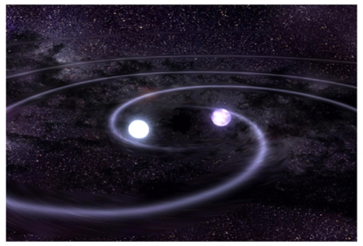
- How about more-legs (diagrams with radiation) ?
- GR & GW-physics EFT vs HEP & Amplitudes
- Unitarity-based methods, Multi-loop Integrals and Integrands decomposition, Amplitudes-inspired dualities, BCJ/Double-copy
- IBPs for Fourier Transform Integrals
- Post-Minkowskian approximation: v-expansion *vs* complete-v dependence

# Extra Slides

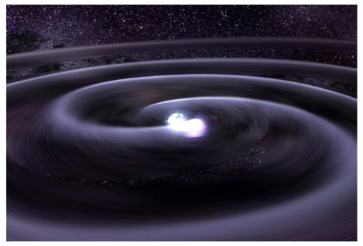
# Status of PN Corrections Porto

| | no spin | | spin |
|---|---|---|---|
| 4PN | Damour, Jaranowski<br>Foffa, Sturani, Sturm, & **PM**<br>Bernard, Blanchet, Bohe, Faye, Marsat<br>Damour, Jaranowski, Schaefer<br>Jaranowski, Schaefer<br>Le Tiec, Blanchet, Whiting<br>Jaranowski, Schaefer<br>Foffa, Sturani<br>Blanchet, Damour '88 | | |
| 3PN | De Andrade, Esposito-Farese, Itho, Futamase<br>Blanchet, Faye<br>Damour, Jaranowski, Schaefer | NNLO | Steinhoff<br>Levi |
| 2PN | Damour, Schaefer ’85<br>Damour, Deruelle ’81-'83 | NLO | Steinhoff, Hergt, Schaefer<br>Damour, Jaranowski, Schaefer<br>Buonanno, Faye, Blanchet<br>Porto, Rothstein |
| 1PN | Einstein-Infeld-Hoffman 1938 | LO | Barker, O’Connel '75 |
| 0PN | Newton 1687 | | |

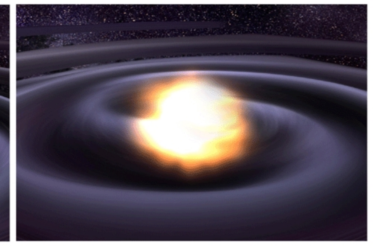
## Binary coalescence: a tale made of three stories



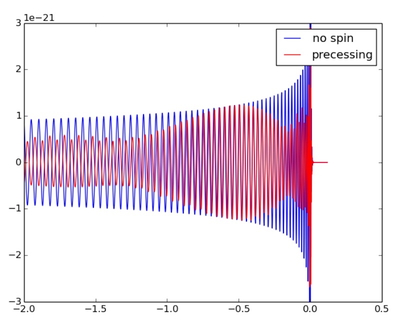
Inspiral phase post-Newtonian approximation: $v/c$



Merger: fully non-perturbative: Numerical Relativity



Ring-down: Perturbed Kerr Black Hole




Spin can induce precession and change the amplitude (and phase) of the waveform due to $\cos(\theta_{LN})$ factors in $h_{+,\times}$



## Modeling the inspiral

Inspiral $h = A\cos(\phi(t))$ $\quad \frac{\dot{A}}{A} \ll \dot{\phi}$

Virial relation:

$$v \equiv (G_N M \pi f_{GW})^{1/3} \qquad \nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

$$E(v) = -\frac{1}{2}\nu M v^2 \left(1 + \#(\nu)v^2 + \#(\nu)v^4 + \dots\right)$$

$$P(v) \equiv -\frac{dE}{dt} = \frac{32}{5G_N} v^{10} \left(1 + \#(\nu)v^2 + \#(\nu)v^3 + \dots\right)$$

E(v)(P(v)) known up to 3(3.5)PN

$$\frac{1}{2\pi}\phi(T) = \frac{1}{2\pi}\int^T \omega(t)dt = -\int^{v(T)} \frac{\omega(v)dE/dv}{P(v)} dv \sim \int \left(1 + \#(\nu)v^2 + \dots + \#(\nu)v^6 + \dots\right)\frac{dv}{v^6}$$

Post-Newtonian Coefficients



## Was it necessary to build a detector? The Hulse-Taylor binary pulsar

GW's first observed in the NS-NS binary system PSR B1913+16

Observation of orbital parameters ($a_p \sin\iota$, $e$, $P$, $\dot{\theta}$, $\gamma$, $\dot{P}$)

determination of $m_p$, $m_c$ (1PN physics, GR)

Energy dissipation in GW's $\to$ $\dot{P}^{(GR)}(m_p, m_c, P, e)$ vs. $\dot{P}^{(obs)}$

$$\frac{1}{2\pi}\phi = \int_0^T \frac{1}{P(t)}dt \simeq \frac{T}{P_0} - \frac{\dot{P}_0}{P_0^2}\frac{T^2}{2}$$

- Test of the 1PN conservative

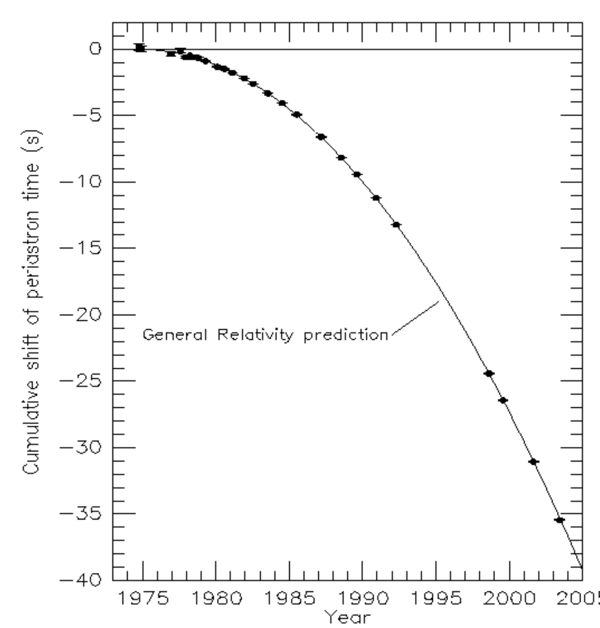
$$E(v) = -\frac{1}{2}\nu M v^2 \left(1 + \#(\nu)v^2 + \#(\nu)v^4 + \dots\right)$$

- leading order dissipative dynamics

$$F(v) \equiv -\frac{dE}{dt} = \frac{32}{5G_N} v^{10} \left(1 + \#(\nu)v^2 + \#(\nu)v^3 + \dots\right)$$



## Weisberg and Taylor (2004)




$$\frac{\dot{P}_{GR} - \dot{P}_{exp}}{\dot{P}} \sim 10^{-3}$$

10 pulsars in NS-NS, still $\sim$ 100Myr for coalescence