The Hepp bound for Feynman periods

Erik Panzer

All Souls College (Oxford)

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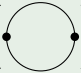
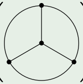
Amplitudes (SLAC)

Feynman periods

$$\text{graph } G \quad \longrightarrow \quad \overbrace{\mathcal{U} = \sum_{T \in \mathrm{ST}(G)} \prod_{e \notin T} x_e}^{\text{graph polynomial}} \quad \longrightarrow \quad \overbrace{\mathcal{P}(G) = \int_{\mathbb{R}^E_{>0}} \frac{\delta(1 - x_i)}{\mathcal{U}^2} \mathrm{d}^E \vec{x}}^{\text{period}}$$

Examples

$\mathcal{P}\Big($  $\Big) = \int_0^\infty \frac{\mathrm{d}x_1}{(x_1 + 1)^2} = 1$

$\mathcal{P}\Big($  $\Big) = 6\zeta(3) \approx 7.2$

Assumptions:

- logarithmic divergence: (no kinematics)

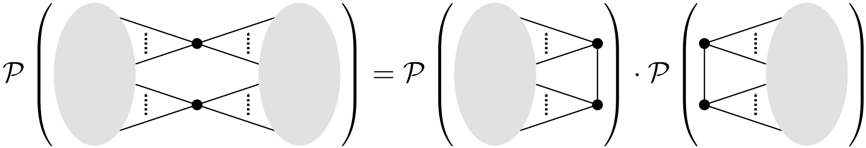
$$\omega(G) := |E(G)| - 2 \cdot \mathrm{loops}(G) \overset{!}{=} 0$$

- no subdivergences:

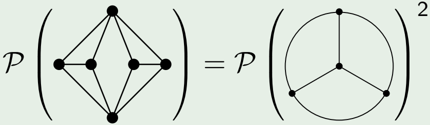
$$\omega(\gamma) > 0 \quad \forall \quad \emptyset \neq \gamma \subsetneq G$$

Symmetries

1. Product:



Example



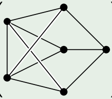
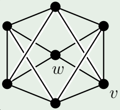
$$= (6\zeta(3))^2$$

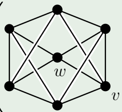
2. Planar duality: $\mathcal{P}(G) = \mathcal{P}(G^{\text{dual}})$

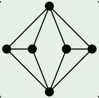
3. Completion: If $\widehat{G}$ is 4-regular, then

$$\mathcal{P}(\widehat{G} \setminus v) = \mathcal{P}(\widehat{G} \setminus w) \quad \text{for all} \quad v, w \in V(\widehat{G})$$

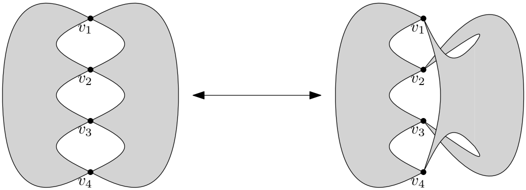
Example

$\mathcal{P}\Bigg($  $\Bigg) = \mathcal{P}\Bigg($ 

$\setminus v \Bigg) = \mathcal{P}\Bigg($ 

$\setminus w \Bigg) = \mathcal{P}\Bigg($  $\Bigg)$

4. Twist:



[drawing by Crump]

Invariants

	product	duality	completion	twist	
c_2 [Schnetz]	yes	yes [Doryn]	for $p = 2$ [Yeats, Doryn]	open	few values, sees number theory
permanent [Crump]	yes	yes	yes	yes	almost faithful
$\mathcal{H}$	yes	yes	yes	yes	faithful (conj.), sees magnitude

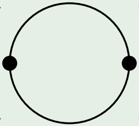
$P_{7,11}$

p	2	3	5	7	11	13	17	19	23
$c_2(p)$	1	0	1	-1	1	-1	1	-1	1
$\text{Perm}(p)$		0	1	1	1	11	5	0	22

Hepp bound

$$\mathcal{H}(G) := \int \frac{\delta(1 - x_i)\mathrm{d}^E \vec{x}}{\mathcal{U}_{\max}^2} \quad \text{where} \quad \mathcal{U}_{\max} := \max_T \prod_{e \notin T} x_e$$

Example

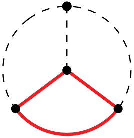
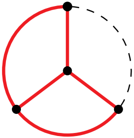
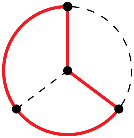
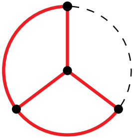
$\mathcal{H}\Bigg($  $\Bigg)$

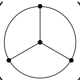
$$= \int_0^\infty \frac{\mathrm{d}x_1}{(\max\{x_1, 1\})^2} = \int_0^1 \mathrm{d}x_1 + \int_1^\infty \frac{\mathrm{d}x_1}{x_1^2} = 2$$

- $\mathcal{H}(G) > \mathcal{P}(G) > \mathcal{H}(G) / |\mathrm{ST}(G)|^2$
- fulfils the four symmetries
- $\mathcal{H}(G) \in \mathbb{Q}_{>0}$
- can be computed very efficiently

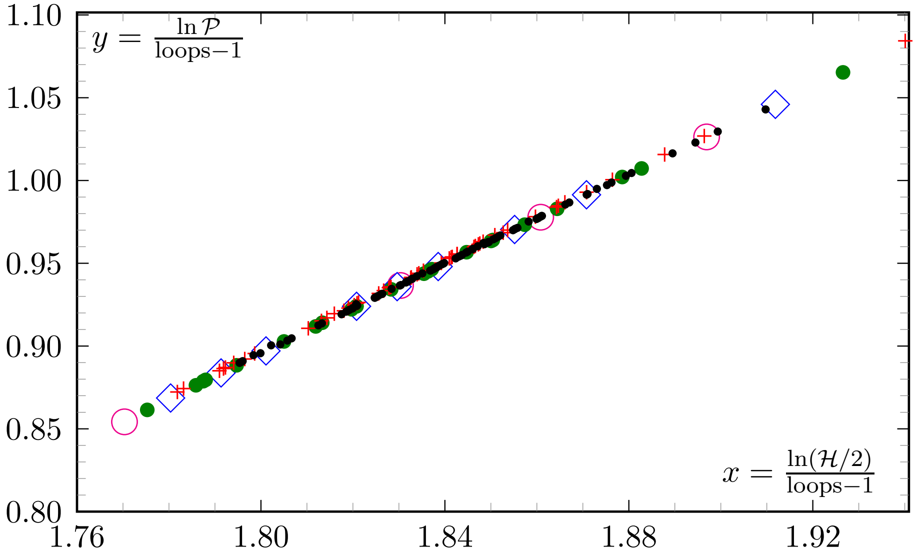
Theorem (Flag formula)

$$\mathcal{H}(G) = \sum_{\substack{\gamma_1 \subset \gamma_2 \subset \cdots \subset \gamma_{\mathrm{loops}(G)} = G \\ \textit{each } \gamma_i \textit{ is 1PI} \\ \mathrm{loops}(\gamma_i) = i}} \frac{|\gamma_1| \cdot |\gamma_2 \setminus \gamma_1| \cdots \left| G \setminus \gamma_{\mathrm{loops}(G)-1} \right|}{\omega(\gamma_1) \cdots \omega(\gamma_{\mathrm{loops}(G)-1})}$$

γ_1	$\subset$	γ_2	summand	#	$\sum$
	$\subset$		$\frac{3\cdot2\cdot1}{1\cdot1} = 6$	12	72
	$\subset$		$\frac{4\cdot1\cdot1}{2\cdot1} = 2$	6	12

$\Rightarrow \mathcal{H}\Big($  $\Big) = 84$

Period correlation



Conjecture

$$\mathcal{H}(G_1) = \mathcal{H}(G_2) \quad \Leftrightarrow \quad \mathcal{P}(G_1) = \mathcal{P}(G_2)$$

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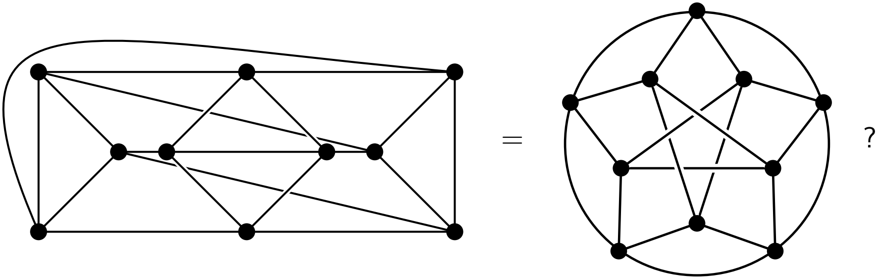
For example, we find a pair of unknown 8 loop periods with:

- $\mathcal{H}(P_{8,30}) = \frac{1724488}{3} = \mathcal{H}(P_{8,36})$
- $\mathcal{P}(P_{8,30}) \approx 505.5 \approx \mathcal{P}(P_{8,36})$



The Hepp bound is the volume of a polytope

Spanning tree polytope:

$$\mathcal{N}_G := \operatorname{conv}\left\{\vec{T} - \vec{T^c} :\ T \text{ spanning tree of } G\right\} \subset \left\{\vec{G}\cdot\vec{a} = 0\right\} \subset \mathbb{R}^{E_G}$$

Facets of $\mathcal{N}_G$ are indexed by certain subgraphs:

$$A_G := \{\gamma \subset G :\ \gamma \text{ and } G/\gamma \text{ are 2-vertex connected}\}$$

Factorisation of the facets:

$$F_\gamma := \mathcal{N}_G \cap \{\vec{\gamma}\cdot\vec{a} = \omega_\gamma\} \cong \mathcal{N}_\gamma \times \mathcal{N}_{G/\gamma}$$

Lemma

$$\mathcal{H}(G) = (E_G - 1)! \cdot \operatorname{Vol}\left(\mathcal{N}_G^\circ\right)$$

$$\mathcal{N}_G^\circ = \bigcap_{T \text{ spanning tree of } G} \left\{\vec{a} :\ \vec{a}\cdot(\vec{T} - \vec{T^c}) \leq 1\right\}$$

Multivariate version & canonical form

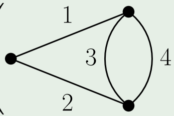
Now consider arbitrary indices:

$$\mathcal{H}(G;\vec{\nu}) := \int \frac{\delta(1-x_i)\mathrm{d}^E\vec{x}}{\mathcal{U}_{\max}^{D/2}} \prod_e x_e^{\nu_e-1}$$

The dimension is fixed by $\omega(G) = \sum_e \nu_e - (D/2)\cdot \mathrm{loops}(G) \stackrel{!}{=} 0$.

Example

The flag formula generalizes to this case, e.g.

$\mathcal{H}\Big($ 

$;\vec{\nu}\Big)$

$$= \frac{1}{\nu_1\nu_2\nu_3\nu_4} \times \left\{ \frac{(\nu_1+\nu_2+\nu_3)\nu_4}{\nu_1+\nu_2+\nu_3-D/2} + \frac{(\nu_1+\nu_2+\nu_4)\nu_3}{\nu_1+\nu_2+\nu_4-D/2} + \frac{(\nu_3+\nu_4)(\nu_1+\nu_2)}{\nu_3+\nu_4-D/2} \right\}$$

Consider the Hepp bound $\mathcal{H}(G;\vec{\nu})$:

- it is a rational function in $\vec{\nu}$
- it has simple poles
- at hyperplanes $\omega(\gamma) = 0$ for 1PI subgraphs γ

Lemma (factorization of residues)

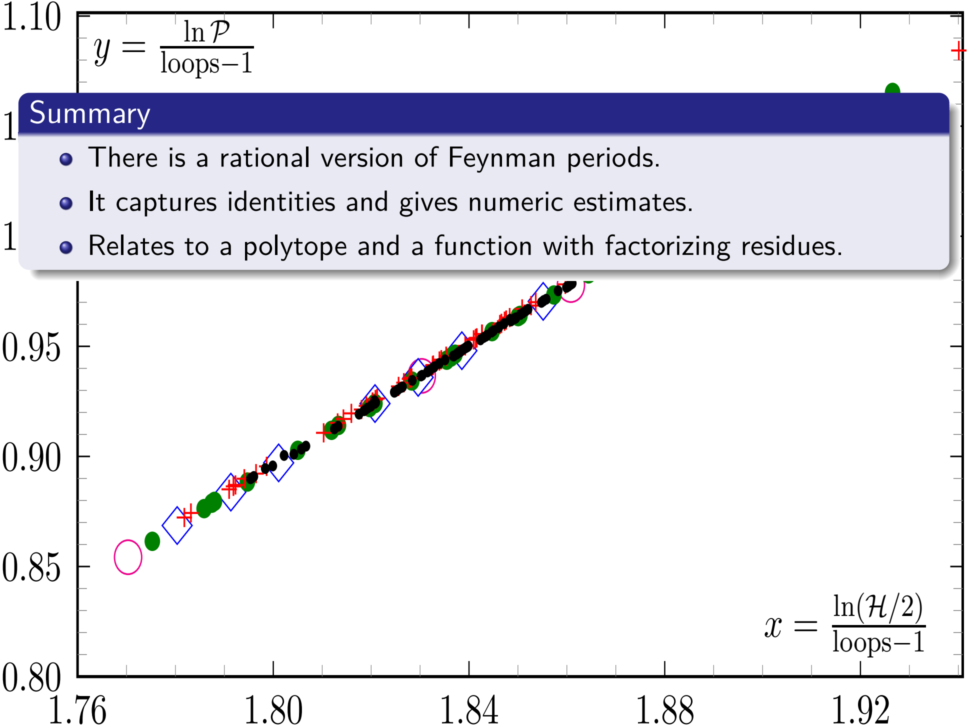
$$\operatorname{Res}_{\omega(\gamma)=0} \mathcal{H}(G;\vec{\nu}) = \mathcal{H}(\gamma;\vec{\nu}_\gamma)\Big|_{\omega(\gamma)=0} \cdot \mathcal{H}(G/\gamma;\vec{\nu}_{G/\gamma})\Big|_{\omega(G/\gamma)=0}$$

Example

$$\operatorname{Res}_{\nu_e=0} \mathcal{H}(G;\vec{\nu}) = \mathcal{H}(G/e;\vec{\nu}_{G/\gamma})$$

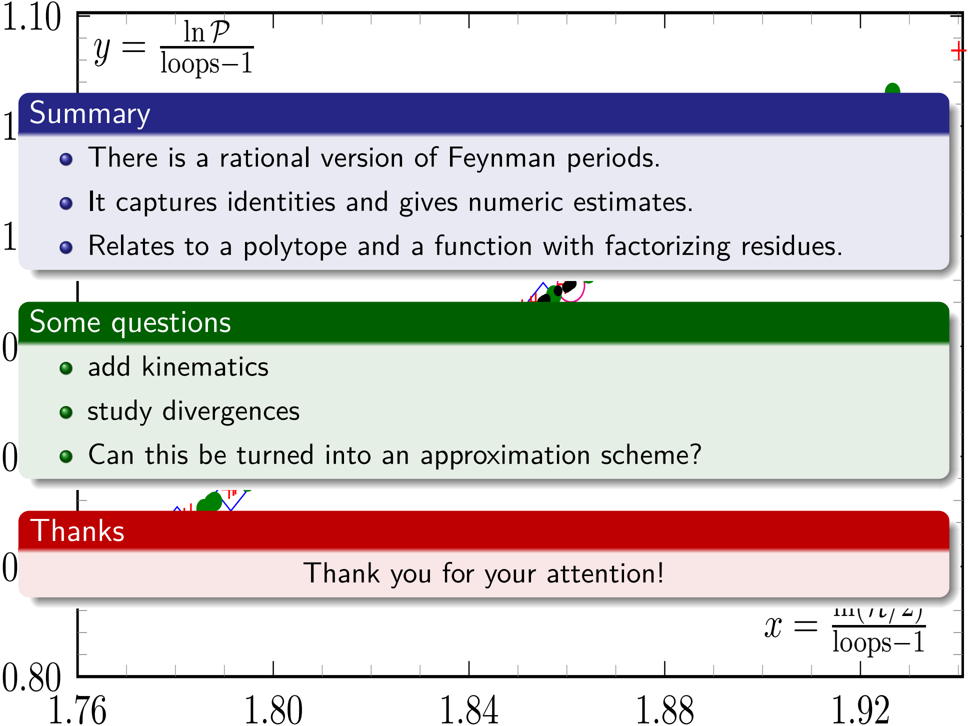
- it is the volume of a polytope:

$$\mathcal{H}(G;\vec{\nu}) = (E-1)! \cdot \operatorname{Vol}\left(\left(\mathcal{N}_G + (\vec{\nu} - \vec{1})\right)^{\circ}\right)$$



Summary

- There is a rational version of Feynman periods.
- It captures identities and gives numeric estimates.
- Relates to a polytope and a function with factorizing residues.



Summary

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- It captures identities and gives numeric estimates.
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Some questions

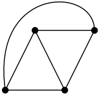
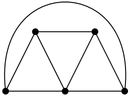
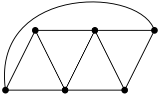
- add kinematics
- study divergences
- Can this be turned into an approximation scheme?

Thanks

Thank you for your attention!

Theorem (Brown & Schnetz)

$$\mathcal{P}(\mathrm{ZZ}_n) = 4\frac{(2n-2)!}{n!(n-1)!}\left(1 - \frac{1-(-1)^n}{2^{2n-3}}\right)\zeta(2n-3) \sim \frac{4^n}{n\sqrt{\pi n}}.$$

$\mathrm{ZZ}_3 =$  $\mathrm{ZZ}_4 =$  $\mathrm{ZZ}_5 =$ 

Theorem

The Hepp bound of ZZ_n *is the coefficient of* x^n *in the power series*

$$\frac{1}{(1-x)(5x+3)}\left[\frac{5x+28}{3} - \frac{2}{1+x} - \frac{1}{x}\sqrt{\frac{1-9x}{1-x}}\right] - 4x^2\log(1-x^2).$$

Corollary

$$\mathcal{H}(\mathrm{ZZ}_n) \sim \frac{3^7}{2^{10}\sqrt{2\pi}}\frac{9^n}{n^{3/2}} \approx 0.852\frac{9^n}{n^{3/2}}.$$