Cosmic Galois Theory in Planar $\mathcal{N} = 4$ supersymmetric Yang-Mills Theory

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Amplitudes
June 18, 2018

arXiv:1609.00669 with S. Caron-Huot, L. Dixon, and M. von Hippel

arXiv:1806.01361 with S. Caron-Huot, L. Dixon, M. von Hippel, and G. Papathanasiou

and ongoing work with these authors

Outline

- The surprising simplicity of scattering amplitudes
- Planar $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory
  - Kinematics, symmetries, and analytic properties
  - The coaction on multiple polylogarithms
  - The space of Steinmann hexagon functions
- Cosmic Galois Theory
  - The coaction principle
  - Through six loops on the six-point amplitude
  - To all loops on the Ω space

Scattering Amplitudes

Scattering amplitudes exhibit a great deal of structure that isn't made manifest by Feynman diagrams

- Parke-Taylor amplitude
- BCFW recursion

$\vdots$

- generalized unitarity
- symbology

$\vdots$

This hidden structure is perhaps easiest to uncover in $\mathcal{N} = 4$ SYM

- SUSY Ward identities $\Rightarrow$ many relations among amplitudes with different helicity structure
- Conformal theory $\Rightarrow$ no running of the coupling or UV divergences
- $AdS_5 \times S^5$ dual theory $\Rightarrow$ multiple ways to calculate quantities of interest
- Supersymmetric version of QCD $\Rightarrow$ the types of functions that show up here also appear in QCD

Planar Limit and Dual Conformal Symmetry

We work with in the $N_c \to \infty$ limit with fixed $g^2 = g^2_{\text{YM}} N_c/(16\pi^2)$

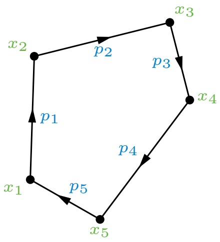
- All non-planar graphs are suppressed in this limit, giving rise to a natural ordering of external particles
- This ordering can be used to define a set of dual coordinates

$$p_i^\mu \sigma_\mu^{\alpha\dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} = x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}}$$

- The coordinates x_i^μ label the cusps of a light-like polygonal Wilson loop in the dual theory, which respects a superconformal symmetry in this dual space

Alday, Maldacena, arXiv:0705.0303 [hep-th];

Drummond, Korchemsky, Sokatchev, arXiv:0707.0243 [hep-th]



Helicity and Infrared Structure

Since we are working with all massless particles, our amplitudes $\mathcal{A}_n$ have universal infrared divergences

- These are accounted for by the 'BDS ansatz' Bern, Dixon, Smirnov, arXiv:hep-th/0505205
- In the dual theory, the BDS ansatz solves an anomalous conformal Ward identity that determines the Wilson loop up to a function of dual conformal invariants Drummond, Henn, Korchemsky, Sokatchev, arXiv:0712.1223 [hep-th]
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- Dual conformal invariants can first be formed in six-particle kinematics, so the four- and five-particle amplitudes are entirely described by the BDS ansatz

$$\mathcal{A}_n = \underbrace{\mathcal{A}_n^{\text{BDS}}}_{\text{IR structure}} \times \underbrace{\exp(R_n) \times \overbrace{\left(1 + \mathcal{P}_n^{\text{NMHV}} + \mathcal{P}_n^{\text{N}^2\text{MHV}} + \cdots + \mathcal{P}_n^{\overline{\text{MHV}}}\right)}^{\text{helicity structure}}}_{\text{finite function of dual conformal invariants}}$$

$$= \overbrace{\mathcal{A}_n^{\text{BDS-like}}}^{\text{IR structure}} \times \overbrace{\rho \times \underbrace{\left(\mathcal{E}_n^{\text{MHV}} + \mathcal{E}_n^{\text{NMHV}} + \mathcal{E}_n^{\text{N}^2\text{MHV}} + \cdots + \mathcal{E}_n^{\overline{\text{MHV}}}\right)}_{\text{helicity structure}}}^{\text{finite function of dual conformal invariants}}$$

Polylogarithms

- Loop-level contributions to MHV and NMHV amplitudes are expected to be multiple polylogarithms of uniform transcendental weight $2L$, meaning that the derivatives of these functions satisfy

$$dF = \sum_i F^{s_i} d \log s_i$$

  for some set of 'symbol letters' $\{s_i\}$, where F^{s_i} is a multiple polylogarithm of weight $2L - 1$
- The symbol letters $\{s_i\}$ will be algebraic functions of kinematic invariants
- Examples of such functions (and their special values) include $\log(z)$, $i\pi$, $\mathrm{Li}_m(z)$, and ζ_m. The classical polylogarithms $\mathrm{Li}_m(z)$ involve only the symbol letters $\{z, 1-z\}$

$$\mathrm{Li}_1(z) = -\log(1-z), \quad \mathrm{Li}_m(z) = \int_0^z \frac{\mathrm{Li}_{m-1}(t)}{t} dt$$

The Coaction on Polylogarithms

- Multiple polylogarithms are endowed with a coaction that maps functions to a tensor space of lower-weight functions

$$\mathcal{H}_w \xrightarrow{\Delta} \bigoplus_{p+q=w} \mathcal{H}_p \otimes \mathcal{H}_q^{\mathfrak{dr}}$$

- The location of branch cuts is encoded in the first component of the coaction
- The derivatives of a function are encoded in the second component of the coaction
- If we iterate this map $w-1$ times we arrive at a function's 'symbol', in terms of which all identities reduce to familiar logarithmic identities

$$\Delta_{1,\ldots,1}\mathrm{Li}_m(z) = -\log(1-z) \otimes \log z \otimes \cdots \otimes \log z$$

Symbol Alphabets and Branch Cuts

The symbol alphabet for six-particle kinematics is believed to be

$$\mathcal{S}_6 = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$

$$s_{i\ldots k} = (p_i + \cdots + p_k)^2$$

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}}, \quad v = \frac{s_{23}s_{56}}{s_{234}s_{456}}, \quad w = \frac{s_{34}s_{61}}{s_{345}s_{561}}$$

$$y_u = \frac{u - z_+}{u - z_-}, \quad y_v = \frac{v - z_+}{v - z_-}, \quad y_w = \frac{w - z_+}{w - z_-}$$

$$z_\pm = \frac{1}{2}(-1 + u + v + w \pm \sqrt{\Delta}), \quad \Delta = (1 - u - v - w)^2 - 4uvw$$

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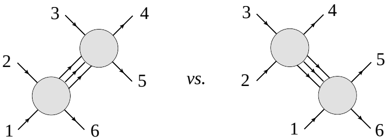
- Massless scattering amplitudes in the Euclidean region only have branch cuts where one of the Mandelstam invariants $s_{i\dots k}$ vanishes

$$\Delta_{1,w-1}F = \log u \otimes {}^{u}F + \log v \otimes {}^{v}F + \log w \otimes {}^{w}F$$

The Steinmann Conditions

- The Steinmann relations additionally tell us that amplitudes cannot have double discontinuities in partially overlapping channels Steinmann, Helv. Physica Acta 33 257, 349 (1960); Cahill, Stapp, Annals Phys. 90, 438 (1975)



$$\text{Disc}_{s_{234}}(\text{Disc}_{s_{345}}(\mathcal{A}_n)) = 0$$

~~$\log\left(\frac{u}{vw}\right) \otimes \log\left(\frac{w}{uv}\right)$~~ $\otimes \ldots$

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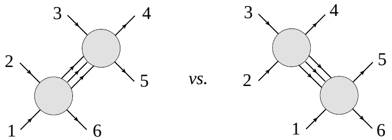

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$$\cdots \otimes \cancel{\log\left(\tfrac{u}{vw}\right) \otimes \log\left(\tfrac{w}{uv}\right)} \otimes \ldots \qquad \cdots \otimes \cancel{\log\left(\tfrac{u}{vw}\right) \otimes \log\left(\tfrac{v}{uw}\right)} \otimes \ldots$$

- ...in fact, the Steinmann relations constrain not just double discontinuities, but all iterated discontinuities
Caron-Huot, Dixon, von Hippel, AJM, Papathanasiou, to appear

$$\ldots \otimes \log s_{i-1} \otimes \log\left(\frac{u}{vw}\right) \otimes \log s_{i+1} \otimes \ldots$$

$$s_{i-1}, s_{i+1} \in \left\{ \frac{u}{vw}, \frac{v}{1-v}, \frac{w}{1-w}, y_u, y_v y_w \right\}$$

- This appears to be equivalent to requiring 'cluster adjacency'
Drummond, Foster, Gurdogan, arXiv:1710.10953 [hep-th]

Normalization Choices

- Steinmann functions don't form a ring—products of functions with incompatible branch cuts are not Steinmann
  - The BDS ansatz leads to products of amplitudes starting at two loops, which scrambles the Steinmann relations
- Therefore, we instead normalize by a 'BDS-like' ansatz that depends on only two-particle Mandelstam invariants

$$\mathcal{A}_n^{\text{BDS}} \times \exp(R_n) \to \rho \times \mathcal{A}_n^{\text{BDS-like}} \times \mathcal{E}_n^{\text{MHV}}$$
$$\mathcal{A}_n^{\text{BDS}} \times \exp(R_n) \times \mathcal{P}_n^{\text{N}^k\text{MHV}} \to \rho \times \mathcal{A}_n^{\text{BDS-like}} \times \mathcal{E}_n^{\text{N}^k\text{MHV}}$$

  where we can also shift the amplitudes by a zeta-valued constant ρ
- This only scrambles the Steinmann relations involving two-particle invariants, which are obfuscated in massless kinematics anyways

Steinmann Hexagon Functions

- Putting this all together, the functions $\mathcal{E}_6^{\mathrm{MHV}}$ and $\mathcal{E}_6^{\mathrm{NMHV}}$ are believed to exist in the space of Steinmann hexagon functions $\mathcal{H}^{\text{hex}}$, which is the space of all multiple polylogarithms that
  - draw on the hexagon symbol alphabet
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  - satisfy the extended Steinmann relations

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  - draw on the hexagon symbol alphabet
  - satisfy the branch cut condition
  - satisfy the extended Steinmann relations
- We only include transcendental constants in $\mathcal{H}^{\text{hex}}$ when forced to
  - satisfying the branch-cut condition beyond the first entry of the symbol requires introducing transcendental constants
  - the only constants we are required to include as independent elements of our basis are even zeta values

$$\zeta_4, \ \zeta_6, \ \zeta_8, \ \zeta_{10}, \ \zeta_{12}, \ \ldots$$

Steinmann Hexagon Functions

- To get the amplitudes to fit into this minimal space, we choose

$$\rho(g^2) = 1 + 8(\zeta_3)^2 g^6 - 160\zeta_3\zeta_5\, g^8 + \left[1680\zeta_3\zeta_7 + 912(\zeta_5)^2 - 32\zeta_4(\zeta_3)^2\right] g^{10} - \left[18816\zeta_3\zeta_9 + 20832\zeta_5\zeta_7 - 448\zeta_4\zeta_3\zeta_5 - 400\zeta_6(\zeta_3)^2\right] g^{12} + \mathcal{O}\left(g^{14}\right)$$

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- The independent $\{n, 1, 1, \ldots, 1\}$ coproduct entries in the MHV amplitudes then saturate this space weight at weight L

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12
$L=1$	1	3	1										
$L=2$	1	3	6	4	1								
$L=3$	1	3	6	13	14	6	1						
$L=4$	1	3	6	13	27	35	20	6	1				
$L=5$	1	3	6	13	27	54	78	51	21	6	1		
$L=6$	1	3	6	13	27	54	105	170	128	58	21	6	1

- The coproduct entries of the NMHV amplitudes also saturate this space through weight L

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Thus, $\mathcal{H}^{\text{hex}}$ (as constructed per the last slide) seems to be the minimal space needed to describe six-particle scattering in this theory

Cosmic Galois Theory

Extends classical Galois theory to the study of periods—namely, integrals of rational functions over rational domains

- Provides a setting in which we can explore the stability of amplitudes and integrals under the coaction
- Namely, we can ask whether the space of Steinmann hexagon functions $\mathcal{H}^{\text{hex}}$ satisfies a 'coaction principle'

$$\Delta \mathcal{H}^{\text{hex}} \subset \mathcal{H}^{\text{hex}} \otimes \mathcal{H},$$

  where $\mathcal{H}$ can be a more general space of functions

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  where $\mathcal{H}$ can be a more general space of functions
- Coaction principles of this type have been observed in other settings Schnetz, arXiv:1302.6445 [math.NT]; Brown, arXiv:1512.06409 [math-ph]
  - Tree-level string theory amplitudes
    Schlotterer, Stieberger, arXiv:1205.1516 [hep-th]
  - Feynman graphs in ϕ^4 theory
    Panzer, Schnetz, arXiv:1603.04289 [hep-th]
  - The electron anomalous magnetic moment
    Schnetz, arXiv:1711.05118 [math-ph]

The Coaction Principle

$$\Delta \mathcal{H}^{\text{hex}} \subset \mathcal{H}^{\text{hex}} \otimes \mathcal{H},$$

- The coproduct preserves the branch-cut conditions, so the coaction principle is satisfied at symbol level by construction
- Despite only including a restricted set of constants in our basis, the full basis will evaluate to more general spaces of constants in various kinematic limits
  - multiple zeta values (MZVs)
  - alternating sums
  - higher roots of unity
- Thus, we can investigate the coaction principle by looking at well-behaved kinematic points in our space

The coaction on MZVs

- For instance, we can consider our function space at $(u, v, w) = (1, 1, 1)$, where everything evaluates to MZVs
- To study the behavior of the MZVs under the coaction, it's convenient to map to an f-alphabet Brown, arXiv:1102.1310 [math.NT]
- In this setting one has natural derivations ∂_{2m+1} that act on the (f-alphabet representation of motivic) zeta values as

$$\partial_{2m+1}\zeta_{2n+1} = \delta_{m,n}$$

  and that satisfy the Leibniz rule—for example,

$$\partial_3(\zeta_7\zeta_3^2) = 2\zeta_7\zeta_3$$

- These operators act nontrivially on multiple zeta values, in a way that is easy to calculate using the f-alphabet
- There is no ∂_2

The coaction principle at (1,1,1)

Weight	Multiple Zeta Values	Appear in $\mathcal{H}^{\text{hex}}\big|_{u,v,w\to 1}$
0	1	1
1		
2	ζ_2	ζ_2
3	ζ_3	
4	ζ_4	ζ_4
5	$\zeta_5,\ \zeta_3\zeta_2$	$5\zeta_5 - 2\zeta_3\zeta_2$
6	$\zeta_3^2,\ \zeta_6$	ζ_6
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✓ 5	$\zeta_5,\ \zeta_3\zeta_2$	$5\zeta_5 - 2\zeta_3\zeta_2$
✓ 6	∂_3 $\zeta_3^2,\ \zeta_6$ $\frac{1}{2}\partial_3$	ζ_6
✓ ✓ 7	$\zeta_7,\ \zeta_5\zeta_2,\ \zeta_3\zeta_4$	$\zeta_5\zeta_2 - 7\zeta_7 + 3\zeta_3\zeta_4$
8	$\zeta_5\zeta_3,\ \zeta_{5,3},\ \zeta_8,\ \zeta_3^2\zeta_2$	$\zeta_{5,3} + 5\zeta_5\zeta_3 - \zeta_3^2\zeta_2,\ \zeta_8$

$$\partial_3(\zeta_{5,3}) = 0, \qquad \partial_5(\zeta_{5,3}) = -5\zeta_3$$

$$\Downarrow$$

$$\partial_3(\zeta_{5,3} + 5\zeta_5\zeta_3 - \zeta_3^2\zeta_2) = 5\zeta_5 - 2\zeta_3\zeta_2$$

$$\partial_5(\zeta_{5,3} + 5\zeta_5\zeta_3 - \zeta_3^2\zeta_2) = 0$$

The coaction principle at (1,1,1)

Weight	Multiple Zeta Values	Appear in $\mathcal{H}^{\text{hex}}\big\vert_{u,v,w\to 1}$
0	1	1
1		
2	ζ_2	ζ_2
✓ 3	ζ_3	
4	ζ_4	ζ_4
✓ 5	$\zeta_5,\ \zeta_3\zeta_2$	$5\zeta_5 - 2\zeta_3\zeta_2$
✓ 6	$\zeta_3^2,\ \zeta_6$	ζ_6
✓✓ 7	$\zeta_7,\ \zeta_5\zeta_2,\ \zeta_3\zeta_4$	$\zeta_5\zeta_2 - 7\zeta_7 + 3\zeta_3\zeta_4$
✓✓ 8	$\zeta_5\zeta_3,\ \zeta_{5,3},\ \zeta_8,\ \zeta_3^2\zeta_2$	$\zeta_{5,3} + 5\zeta_5\zeta_3 - \zeta_3^2\zeta_2,\ \zeta_8$

- Unexplained dropouts were required at low weights for the coaction principle to be nontrivial
- Each zeta value that drops out seeds an infinite tower of constraints at higher loop orders, which we find are satisfied

Consistency Across Kinematics

- If true, the coaction principle is expected to hold in general, not just at one point
- Different constants can drop out at different kinematic points—for instance, we can look at limits where hexagon functions evaluate to alternating sums
  - at $(½,1,½)$ the space of alternating sums is saturated
  - at $(½,v \to 0,½)$ dropouts are observed starting at weight 6
  - at $(u, v \to 0, u)|_{u\to\infty}$ dropouts are observed starting at weight 1 ($\log 2$ doesn't appear)

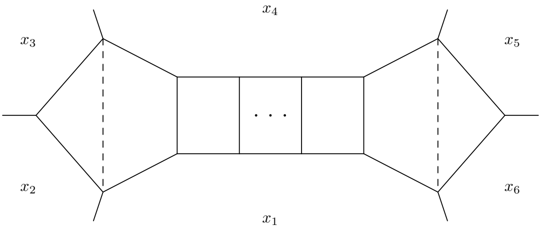
Consistency Across Kinematics

- If true, the coaction principle is expected to hold in general, not just at one point
- Different constants can drop out at different kinematic points—for instance, we can look at limits where hexagon functions evaluate to alternating sums
  - at $(\frac{1}{2},1,\frac{1}{2})$ the space of alternating sums is saturated
  - at $(\frac{1}{2},v \to 0,\frac{1}{2})$ dropouts are observed starting at weight 6
  - at $(u, v \to 0, u)|_{u\to\infty}$ dropouts are observed starting at weight 1 ($\log 2$ doesn't appear)

Everywhere we have checked, the coaction principle is respected

The "Ω space"

$\Omega^{(L)} \equiv$



- $\Omega^{(L)}$ contributes to the six-point amplitude at all loops
- Related at adjacent loop orders by a second-order differential equation Drummond, Henn, Trnka, arXiv:1010.3679 [hep-th]
- These differential equations can be solved at finite coupling in terms of a Mellin integral over hypergeometric functions
  Caron-Huot, Dixon, von Hippel, AJM, Papathanasiou, arXiv:1806.01361 [hep-th]

$$\Omega = \sum_L (-g^2)^L \Omega^{(L)}$$

- From this representation, it is possible to show that the space of coproduct entries appearing in all $\Omega^{(L)}$ satisfy a coaction principle

Conclusions

- A large amount of information is encoded in the formal structure of amplitudes, much of which is still not well understood
- In particular, there exists surprising motivic structure in amplitudes that remains to be explained in terms of physical principles
  - a coaction principle seems to hold not only in planar $\mathcal{N} = 4$ SYM theory, but also string theory, ϕ^4, and QED
  - can these coaction principles be extended beyond the case of polylogarithms?
  - does the factor ρ admit a physical definition?
- What other mathematical properties are waiting to be found?
  - The $\Omega^{(L)}$ integrals give us some hints...

Thanks!

Dual Conformal Invariants

- We can construct dual conformally invariant cross ratios out of combinations of Mandelstam invariants

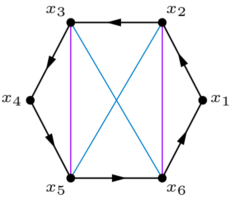
$$x_{ij}^2 = (x_i - x_j)^2 = (p_i + p_{i+1} + \cdots + p_{j-1})^2$$

that remain invariant under the dual inversion generator

$$I(x_i^{\alpha\dot{\alpha}}) = \frac{x_i^{\alpha\dot{\alpha}}}{x_i^2} \qquad \Rightarrow \qquad I(x_{ij}^2) = \frac{x_{ij}^2}{x_i^2 x_j^2}$$

- These can first be constructed for $n = 6$ since $x_{i,i+1}^2 = p_i^2 = 0$

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \quad v = \frac{x_{24}^2 x_{51}^2}{x_{25}^2 x_{41}^2}, \quad w = \frac{x_{35}^2 x_{62}^2}{x_{36}^2 x_{52}^2}$$



- In general, we can form $3n - 15$ independent ratios

The Steinmann Hexagon Space

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12
$L=1$	1	3	4										
$L=2$	1	3	6	10	6								
$L=3$	1	3	6	13	24	15	6						
$L=4$	1	3	6	13	27	53	50	24	6				
$L=5$	1	3	6	13	27	54	102	118	70	24	6		
$L=6$	1	3	6	13	27	54	105	199	269	181	78	24	6

The dimension of the space of coproduct weight-n first coproduct entries of the MHV and NMHV amplitudes at a given loop order L

weight n	0	1	2	3	4	5	6	7	8	9	10	11
total	1	3	6	13	27	54	105	200	372	679	1214	2136
P even, K	1	3	6	12	22	39	67	114	190	315	517	846
P even, non-K	0	0	0	0	3	9	25	56	123	244	474	872
P odd	0	0	0	1	2	6	13	30	59	120	223	418

The full space of Steinmann hexagon functions at weight n, graded by parity

Cosmic Galois Theory

In science you sometimes have to find a word that strikes, such as "catastrophe", "fractal", or "noncommutative geometry". They are words which do not express a precise definition but a program worthy of being developed.

- Pierre Cartier

The "Ω space"

- Ω and $\tilde{\Omega}$ naturally complete to a space involving two new integrals

$$\mathcal{O} = \frac{1}{g^2}\left(x\partial_x - y\partial_y\right)\Omega, \quad \mathcal{W} = \left(x\partial_x + y\partial_y\right)\Omega,$$

which perturbatively evaluate to polylogarithms of odd weight

Caron-Huot, Dixon, von Hippel, AJM, Papathanasiou, arXiv:1806.01361 [hep-th]

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- The space of all coproduct entries appearing in $\Omega^{(L)}$, $\tilde{\Omega}^{(L)}$, $\mathcal{O}^{(L)}$, and $\mathcal{W}^{(L)}$ can be constructed using

$$\text{Disc}_c\{\mathcal{W}, \Omega, \tilde{\Omega}_e, \mathcal{O}\} = \oint_{[g,g]} \frac{ds}{i\pi}\sqrt{s^2 - g^2}\; \tilde{\mathcal{V}}_i(s, g^2)$$

where $c = w/u/v$ corresponds to the channel carrying momentum along the ladder, and the vector $\tilde{\mathcal{V}}_i(s, g^2)$ is an expansion in s and g^2 that is closed under the coaction:

$$\Delta_{\bullet,1}\left(\tilde{\mathcal{V}}_i(s, g^2)\right) = \left(\tilde{\mathcal{V}}_i(s, g^2)\right) \otimes M_{ij}(s, g^2)$$

- Furthermore, this c discontinuity can then be undone, which gives us the full space of coproduct entries of these integrals

Caron-Huot, Dixon, von Hippel, AJM, Papathanasiou, arXiv:1806.01361 [hep-th]