Form factors in $\mathcal{N} = 4$ SYM and beyond

Amplitudes 2018

- 18/06/18 - SLAC -

hep/th

1707.09897
1804.05703
1804.05828

based on work with:

A. Brandhuber, M. Kostacinska, G. Travaglini

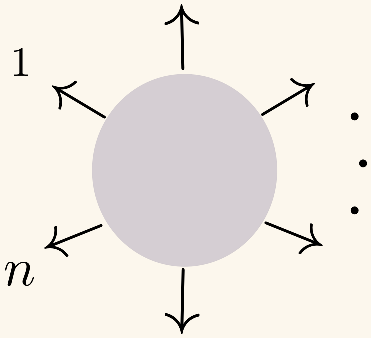
Brenda Penante

Amplitudes in planar $\mathcal{N} = 4$

- Dual conformal symmetry
- Wilson loop duality
- Conformal symmetry
- Bootstrap
- Grassmannian
- Amplituhedron
- Colour-kinematics duality
- Recursion relations (tree and loop integrand)

Amplitudes

$$A_n = \langle 1 \cdots n | 0 \rangle$$



$$p_i^2 = 0$$

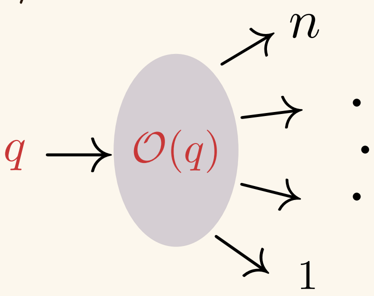
Form factors

gauge invariant
operator

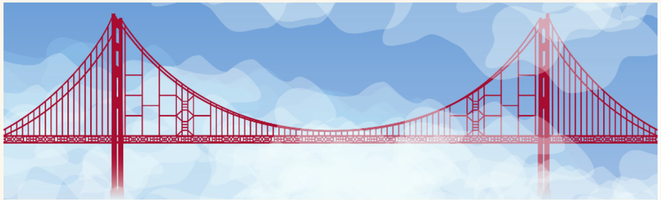
$$F_{\mathcal{O},n}(q) := \int d^4x \, e^{-iqx} \, \langle 1 \cdots n | \mathcal{O}(x) | 0 \rangle$$

$$= \ \delta^{(4)}\Big(q - \sum_{i=1}^{n} p_i\Big) \langle 1 \cdots n | \mathcal{O}(0) | 0 \rangle$$

$q^2 \neq 0$



Form Factors



Amplitudes

$A_n = \langle 1 \cdots n | 0 \rangle$

Corr. functions

$C_n = \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$

on-shell

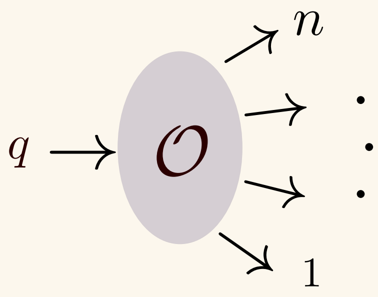
off-shell

$$F_{\mathcal{O},n} = \langle 1 \cdots n | \mathcal{O} | 0 \rangle$$

Can be used to study properties of operators
(eg renormalisation, anomalous dimension)

Form Factors as effective interactions

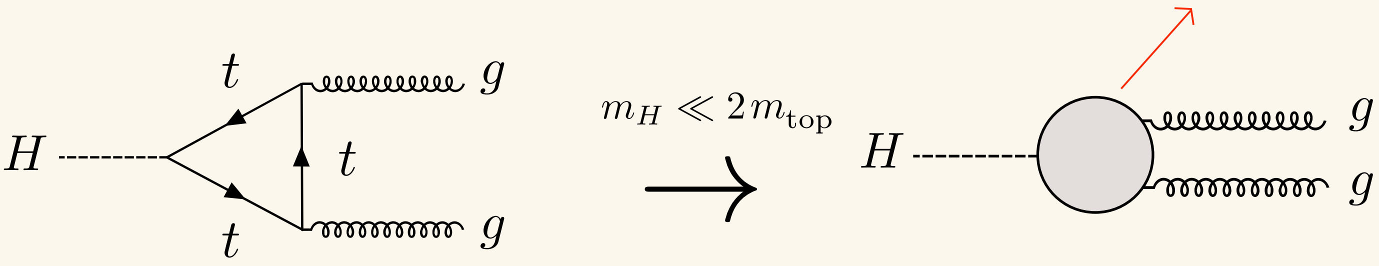
$$\mathcal{L}_{\text{EFT}} = \mathcal{L} + \frac{c_i}{\Lambda^{\#}}\mathcal{O}_i$$



Ex: Higgs to gluons in $m_{\text{top}}^2 \to \infty$ limit

-- Wilczek / Shifman, Vainshtein, Voloshin, Zakharov / Dawson --

$$\mathcal{L}_{\text{EFT}} = H\,\text{Tr}(F^2)$$



Compute form factor $\langle gg\cdots g| \int d^4x\, e^{-iqx}\text{Tr}(F^2)(x)|0\rangle\Big|_{q^2=m_H^2}$

Protected operators in $\mathcal{N} = 4$:

(Stress tensor)

$$\mathrm{Tr}(\phi^2) \xrightarrow[\text{SUSY}]{} \mathcal{T}_2$$

$$\mathrm{Tr}(\phi^2) \longrightarrow \text{2 particles}$$

"Minimal"

Analytical		Numerical	Integrand	
2	3	4	5	→ L
van Neerven	Gehrmann, Henn, Huber	Boels, Kniehl, Yang Boels, Huber, Yang	Yang	

$$\mathrm{Tr}(\phi^2) \longrightarrow \text{3 particles}$$ -- Brandhuber, Travaglini, Yang --

scales: $u = \dfrac{s_{12}}{q^2},\ v = \dfrac{s_{23}}{q^2},\ w = \dfrac{s_{13}}{q^2}, \quad u + v + w = 1$

at 2 loops: uniform transcendentality 4 functions of $u,\ v,\ w$

$\mathrm{Tr}(\phi^2) \longrightarrow$ 3 particles -- Brandhuber, Travaglini, Yang --

scales: $u = \frac{s_{12}}{q^2}, v = \frac{s_{23}}{q^2}, w = \frac{s_{13}}{q^2}, \quad u + v + w = 1$

at 2 loops: uniform transcendentality 4 functions of u, v, w

$$\left[\text{Transc } n: \quad \mathrm{Li}_n(x) \quad \pi^n \quad \zeta_n \quad G(i_1, \ldots, i_n, x) \right]$$

Finite part coincides with max trasc part of helicity amplitudes $H \to ggg$ in $m^2_{\mathrm{top}} \to \infty$ limit. -- Gerhmann, Jaquier, Glover, Koukoutsakis --

Chiral part of stress tensor operator in $\mathcal{N} = 4$:

$$\mathcal{T}_2 = \mathrm{Tr}(\phi^2) + \cdots + (\theta)^4 \mathcal{L}_{\text{on-shell}}$$

$$\searrow \quad \mathrm{Tr}(F^2_{\mathrm{SD}}) + \ldots$$

Maximal transcendentality principle

"Certain quantities in planar $\mathcal{N}=4$ can be obtained from a truncation of a corresponding quantity in QCD"

Ex: $\mathrm{Li}_4(x)+\pi^4+\mathrm{Li}_3(x)+\zeta_3+\log^2(x)+\pi+1$

$$\longrightarrow \ \mathrm{Li}_4(x)+\pi^4$$

Cusp anomalous dimension Kotikov, Lipatov / Kotikov, Lipatov, Onishchenko, Velizhanin

Collinear anomalous dimension Dixon

Certain semi-infinite Wilson lines von Manteuffel, Schabinger, Zhu / Zhu, Li

DIS Wilson coeffs Bianchi. Forini, Kotikov

Certain form factors

~~Amplitudes~~

More protected operators:

$$\mathrm{Tr}(\phi^k) \xrightarrow{\text{SUSY}} \mathcal{T}_k$$

Brandhuber, Travaglini,
Penante, Wen

Two-loop finite part also uniform transc 4
and are identical for operators in several sectors:

$$\mathrm{Tr}(\phi^3)$$

SU(2)	/	SL(2)	/	SU(2|3)	/	Konishi
$\{X, Y\}$		$\{D_+, X\}$		$\{X, Y, Z, \psi\}$		$\mathcal{K} \to g\phi\phi$
Loebbert, Nandan, Sieg, Wilhelm, Yang		Loebbert , Sieg, Wilhelm, Yang		Brandhuber, Kostacinska, Penante, Travaglini, Young		Banerjee, Dhania, Mahakhud, Ravindran, Seth

Lower transc pieces for non-protected
operators also very similar!

Three wishes:

1. Understand what functions appear as form factor remainders, hidden structure, bootstrap.
2. Understand relations between QCD and $\mathcal{N}=4$. Are they confined to maximal transcendental weight? How far to they persist?
3. Use SUSY as an organisational tool

This talk: $\mathrm{Tr}(F^3_{\mathrm{ASD}}) \to g^+g^+g^+$

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Reason 1: inspiration from $\mathrm{Tr}(F^2_{\mathrm{ASD}}) \to g^+ g^+ g^+$

$\mathrm{Tr}(F^3_{\mathrm{ASD}})$ appears as a subleading term in m^2_{top} in SM effective lagrangian

-- Neill / Dawson, Lewis, Zeng --

$$\mathcal{L}_{\mathrm{eff}} = H\mathrm{Tr}(F^2) + \frac{1}{m^2_{\mathrm{top}}} \sum_{i=1}^{4} c_i O_i + \mathcal{O}\left(\frac{1}{m^4_{\mathrm{top}}}\right)$$

Dimension 7 operators

$$\supset H\mathrm{Tr}(F^3) = H\mathrm{Tr}(F^3_{\mathrm{SD}}) + H\mathrm{Tr}(F^3_{\mathrm{ASD}})$$

Q: Does the observation for $\mathrm{Tr}(F^2)$ carries over to $\mathrm{Tr}(F^3)$?

Reason 2: SUSY as an organisational principle

Since this operator involves only field strengths,
can be considered in SYM with any $\mathcal{N}$

Q: Can we find universal structures which are invariant among different form factors and theories?

Supersymmetric decomposition:

Ex: 1 loop, external gluons:

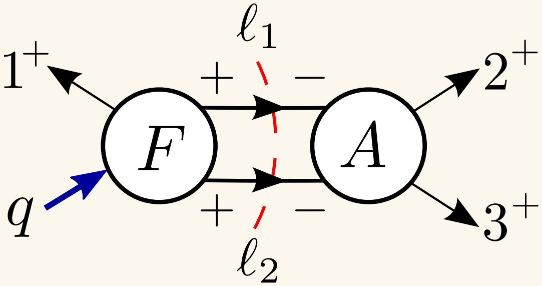
-- Bern, Dixon, Dunbar, Kosower --

Non cut-constructible

$$A_n^{\mathrm{QCD}} = A_n^{\mathcal{N}=4} - 4A_n^{\mathcal{N}=1,\mathrm{chiral}} + A_n^{\phi}$$

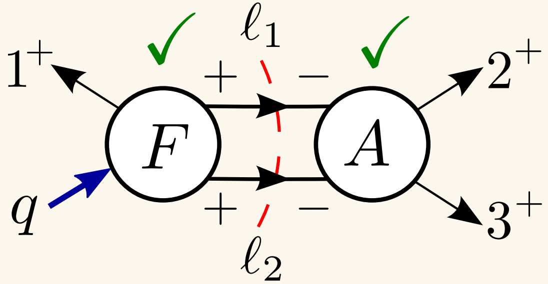
Trade more complicated calculations for
supersymmetric building blocks

Method: Construct integrand using generalised unitarity - 1 loop



Method: Construct integrand using generalised unitarity - 1 loop

Theory independent



Method: Construct integrand using generalised unitarity - 1 loop

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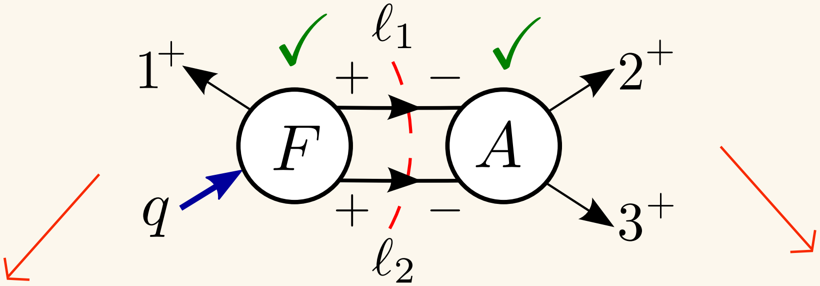
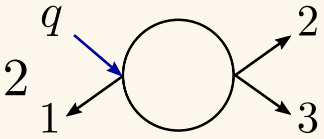


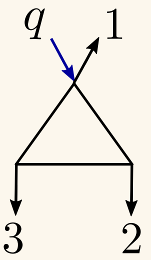
$$F_{\mathcal{O}}(1^+,\ell_1^+,\ell_2^+;q) = -[1\ell_1][\ell_1\ell_2][\ell_2 1]$$

$$A(2^+,3^+,\ell_2^-,\ell_1^-) = \frac{\langle \ell_2\ell_1\rangle^3}{\langle 23\rangle\langle 3\ell_2\rangle\langle \ell_1 1\rangle}$$

$$F_{\mathcal{O}}^{(1)}(1^+,2^+,3^+;q) = i\,F_{\mathcal{O}}^{(0)}\Bigg($$



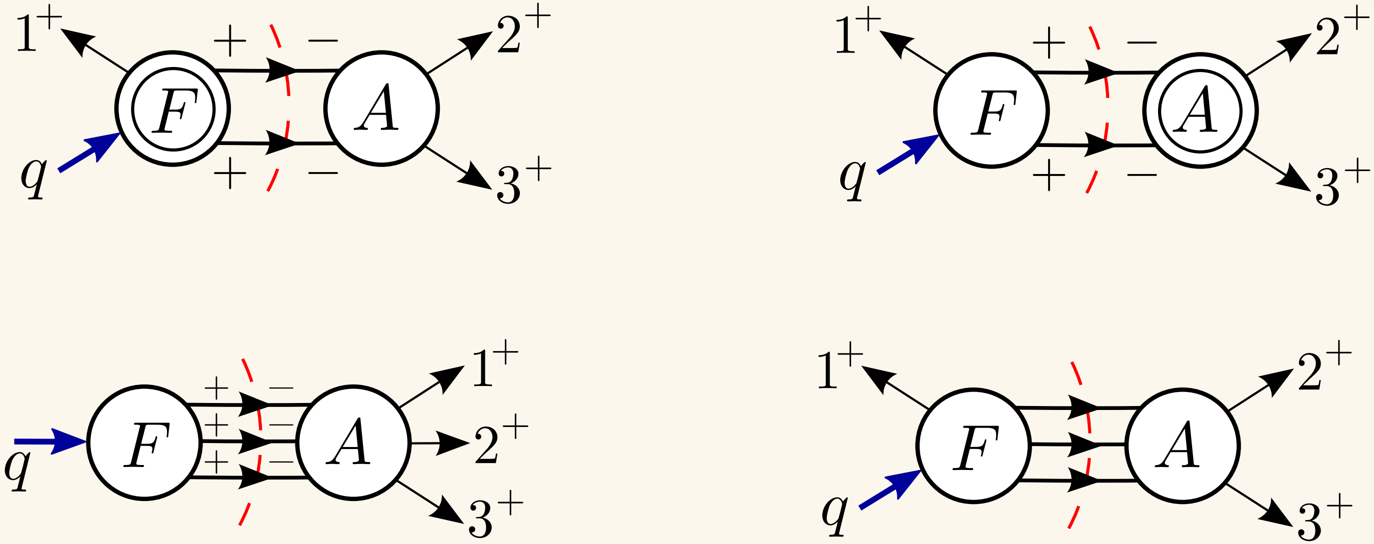
$$+\,s_{23}$$



$$\Bigg)$$

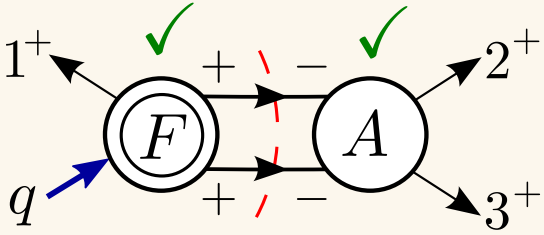
$$+\,\text{cyclic}(1,2,3)$$

Method: Construct integrand using generalised unitarity - 2 loops

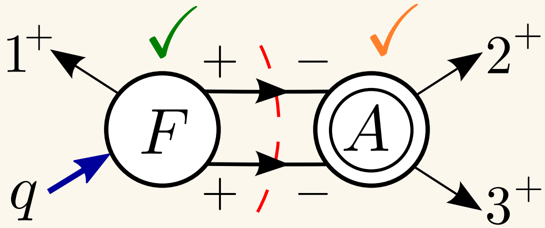


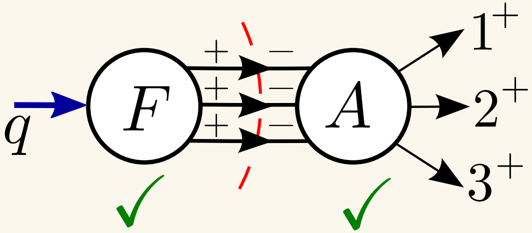
Method: Construct integrand using generalised unitarity - 2 loops

Theory independent

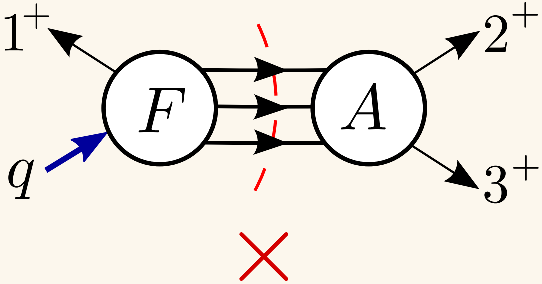


Theory independent Theory dependent, but known





Theory independent



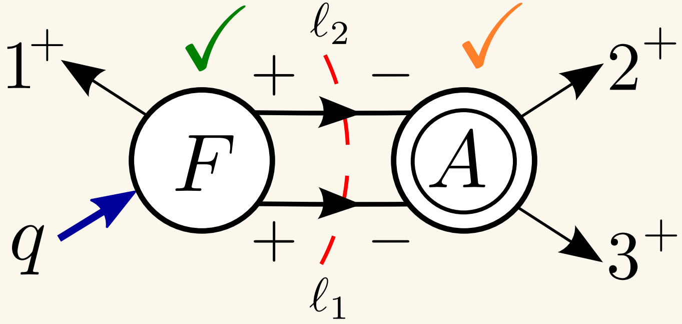
New information

Transcendentality and $\mathcal{N}$

Contribution to maximally trascendental part from theory dependent cuts

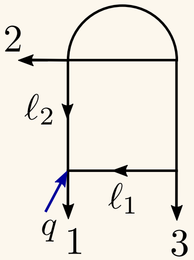
-- Bern, Dixon, Dunbar, Kosower --

Theory independent $(\mathcal{N}=4) - \beta_0 \times (\mathcal{N}=1 \text{ chiral})$



$\propto$ 1 loop bubble

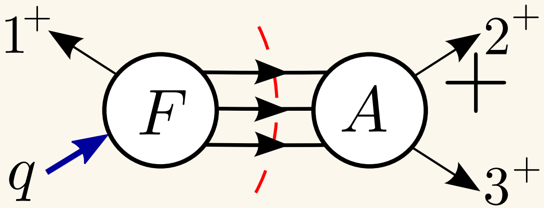
$$\longrightarrow \quad \beta_0 \frac{\mathrm{Tr}_+(1\ell_2\ell_1 132)}{s_{12}s_{13}} \times$$



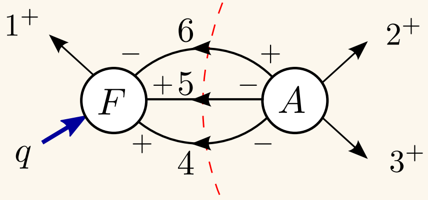
$\longrightarrow$ transcendentality < 4

Transcendentality and $\mathcal{N}$

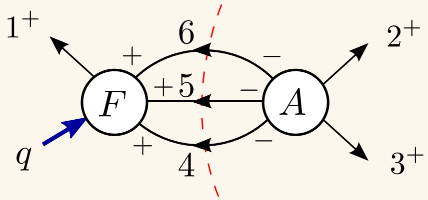
Contribution to maximally trascendental part from theory dependent cuts



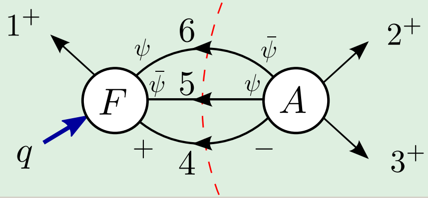
× New information



$+$ 2 configurations

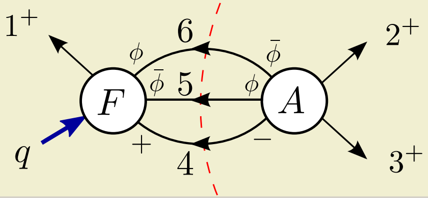


# of Weyl fermions ← × c_F



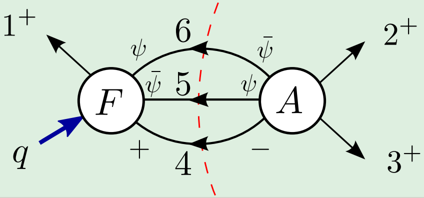
$+$ 3 configurations

of (real) scalars ← × c_S



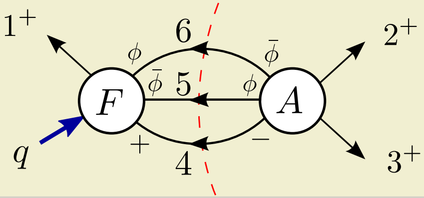
$+$ 3 configurations

$\times\ c_F$



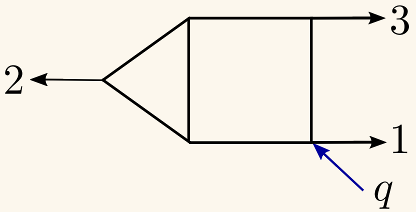
$+$ 3 configurations

$\times\ c_S$



$+$ 3 configurations

$\mathcal{N}$	c_F	c_S
4	4	6
2	2	2
1	1	0
0	0	0



$\times \mathrm{Num}(c_F, c_S)$

fermions

scalars

→ transcendentality < 4

$\xcancel{\mathcal{N} =} 4$ (no triangle)

Finite remainder function

Running of the coupling - beta function

$$a^U S_\epsilon = \left(\frac{\mu}{\mu_0}\right)^{2\epsilon} a(\mu) \left[1 - a(\mu)\frac{\beta_0}{\epsilon} + a^2(\mu)\left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon}\right)\right] + \cdots$$

N (adj of $SU(N)$)

$$\beta_0 = \frac{11}{3} - \frac{1}{6}\sum_i \frac{C_i}{N} - \frac{2}{3}\sum_j \frac{\tilde{C}_j}{N} \, ,$$

-- Gross, Wilczek --

Finite remainder function

Running of the coupling - beta function

UV divergences - operator renormalisation

$$\lambda^U = \lambda(\mu)\left[1 - a(\mu)\frac{\gamma_0}{\epsilon} + \frac{a^2(\mu)}{2}\left(\frac{\rho_0^2}{\epsilon^2} - \frac{\rho_1}{\epsilon}\right)\right] + \cdots$$

$$F_{\mathcal{O}}^R = \lambda(\mu)\left[(F_{\mathcal{O}}^R)^{(0)} + a(\mu)(F_{\mathcal{O}}^R)^{(1)} + a^2(\mu)(F_{\mathcal{O}}^R)^{(2)}\right] + \cdots$$

$$= \lambda^U\left[(F_{\mathcal{O}}^U)^{(0)} + a^U(F_{\mathcal{O}}^U)^{(1)} + (a^U)^2(F_{\mathcal{O}}^U)^{(2)}\right] + \cdots$$

Finite remainder function

Running of the coupling - beta function

UV divergences - operator renormalisation

IR subtraction - Catani remainder (guise of Bern, Dixon, Kosower)

Renormalised form factors

$$\mathcal{R}^{(2)}(\epsilon) \; = \; (\mathcal{F}^R_{\mathcal{O}})^{(2)}(\epsilon) - \frac{1}{2}\left[(\mathcal{F}^R_{\mathcal{O}})^{(1)}(\epsilon)\right]^2 + \frac{\beta_0}{\epsilon}(\mathcal{F}^R_{\mathcal{O}})^{(1)}(\epsilon)$$

$$-e^{-\gamma_E \epsilon}\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}(\mathcal{F}^R_{\mathcal{O}})^{(1)}(2\epsilon)\left(\frac{\beta_0}{\epsilon} + K\right) + \frac{n\, e^{\gamma_E \epsilon}}{4\epsilon\,\Gamma(1-\epsilon)} H^{(2)}$$

$$K_{\text{SYM}} = 2\left[(4-\mathcal{N}) - \zeta_2\right]$$

$$H^{(2)}_{\text{SYM}} = 2\zeta_3 + \frac{(4-\mathcal{N})}{2}\,\zeta_2$$

Findings I: $\mathcal{N}=4$

Transcendentality 4,3,2,1,0

Maximal piece identical to BPS operator $R^{(2)}_{\mathrm{Tr}(\phi^3)}(1^\phi,2^\phi,3^\phi)$

Brandhuber, Travaglini, Penante, Wen

$$\mathcal{R}^{(2)}_{\mathcal{O};4} = -\frac{3}{2}\,\mathrm{Li}_4(u) + \frac{3}{4}\,\mathrm{Li}_4\left(-\frac{uv}{w}\right) - \frac{3}{2}\log(w)\,\mathrm{Li}_3\left(-\frac{u}{v}\right)$$

$$+\frac{1}{16}\log^2(u)\log^2(v) + \frac{\log^2(u)}{32}\left[\log^2(u) - 4\log(v)\log(w)\right]$$

$$+\frac{\zeta_2}{8}\log(u)\left[5\log(u) - 2\log(v)\right] + \frac{\zeta_3}{2}\log(u) + \frac{7}{16}\zeta_4 + \mathrm{perms}\,(u,v,w)$$

$$u=\frac{s_{12}}{q^2} \qquad v=\frac{s_{23}}{q^2} \qquad w=\frac{s_{13}}{q^2} \qquad u+v+w=1$$

Findings I: $\mathcal{N} = 4$

Sub-maximal transcendentality - pure and non-pure terms

Degree 3: $\frac{u}{v} \times \mathrm{Li}_3 \ldots$, pure_3

Degree 2: $\frac{u^2}{v^2} \times \mathrm{Li}_2 \ldots$, pure_2

Degree 1: $\frac{u^2}{vw} \times \log \ldots$, $\frac{u}{v} \times \log \ldots$, pure_1

Degree 0: $\frac{1}{uvw} \times \ldots$, pure_0

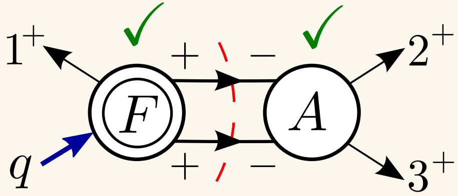
→ mixing with $\mathrm{Tr}(DFDF)$

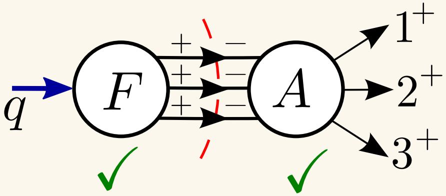
Terms with rational prefactors combine in such a way that in soft and collinear limits only logarithmic divergences survive

$$F^{(0)} \sim [12][23][31] \to 0 \qquad \mathcal{R}^{(2)} \to 0$$

Findings II: $\mathcal{N} < 4$

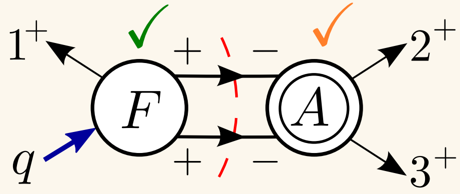
Theory independent

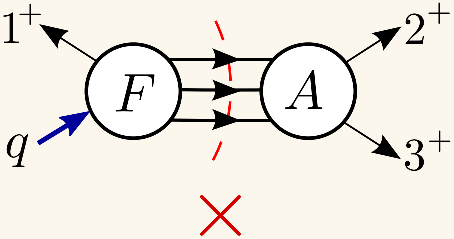




Theory independent

Theory independent Theory dependent, but known





New information

Findings II: $\mathcal{N} < 4$

difference from $\mathcal{N} = 4$ result pure and quite simple

$\mathcal{R}^{\mathcal{N}=2} - \mathcal{R}^{\mathcal{N}=4}$	$\mathcal{R}^{\mathcal{N}=1} - \mathcal{R}^{\mathcal{N}=4}$
$-\frac{5}{2}\zeta_2 \log(uvw) - \frac{11}{2}\zeta_3$	$-\frac{15}{4}\zeta_2 \log(uvw) - \frac{33}{4}\zeta_3$
$18\zeta_2$	$\frac{243}{8}\zeta_2$
$3 \log(uvw)$	$\frac{9}{2} \log(uvw)$
$-\frac{45}{4}$	$-\frac{135}{8}$

All terms multiplied by rational kinematic prefactors are identical

Some observations I: Hidden maximal transcendetality

Conjecture by Loebbert , Sieg, Wilhelm, Yang

The full form factor for any given operator in $\mathcal{N}=4$ SYM theory has hidden uniform transcendentality

Form factors in SL(2) sector: $\mathcal{O}_{\vec{n}} = \mathrm{Tr}\left[\prod_{j=1}^{L} \frac{(D_+)^{n_j}}{n_j!} X\right]$

Assign transc k to numerical/kinematic prefactors:

$$S_{n_j}^{(k)} \qquad \frac{1}{(m_j-n_j)^k} \qquad \frac{1}{(m_j-n_j)^k}\frac{u_i^k}{v_i^k}$$

Then, complete remainder has uniform hidden transc 4 at 2 loops

Some observations I: Hidden maximal transcendetality

Likewise, take the remainder of $\mathrm{Tr}(F^3_{\mathrm{ASD}}) \to g^+ g^+ g^+$

$$\sum_{j=1}^{m} \frac{1}{j} \frac{(1-u_i)^j}{v_i^j} \xrightarrow[m \to \infty]{} -\log\left(1 - \frac{1-u_i}{v_i}\right)$$

Degree 3: $\frac{u}{v} \times \mathrm{Li}_3 \ldots \, , \, \mathrm{pure}_3$

$$\frac{u}{v} \to \frac{(1-v-w)}{v}$$

$$\to \xcancel{\mathrm{pure}_3}$$

(well, almost)

Consider operators with $(D_+)^{n_j}$ to test the conjecture.

Some observations II: $\mathcal{N} = 0$

Jin and Yang: Higgs amplitudes to 3 gluons in the EFT including dim 7 operators

$$\mathcal{L}_{\text{eff}} = H\,\text{Tr}(F^2) + \frac{1}{m_{\text{top}}^2}\sum_{i=1}^{4} c_i O_i + \mathcal{O}\left(\frac{1}{m_{\text{top}}^4}\right)$$

Dimension 7 operators

Max transc piece is identical to $\mathcal{N} = 4$

All non-pure terms in the remainder identical to $\mathcal{N} = 4$

$$\frac{u}{v}, \frac{u^2}{v^2}$$

$\mathcal{N} = 4$ knows more than just max transc piece!

... to conclude

By considering form factors of $\mathrm{Tr}(F^3_{\mathrm{ASD}})$ in SYM theories with various $\mathcal{N}$, we've discovered that $\mathcal{N}=4$ contains most of the result, not only the maximally transcendental part (but why?)

Found one more instance where maximal transcendentality principle apply (but what is the meaning of it?)

... a side comment on super form factors

Konishi super multiplet & super cuts

... to do

Consider family of form factors with magnons to test hidden trancendentality

Non-minimal form factors

Thank you!

$$\begin{aligned}\mathcal{R}^{(2)}_{\mathcal{O};3}\Big|_{\text{pure}} = {} & \text{Li}_3(u)+\text{Li}_3(1-u)-\frac{1}{4}\log^2(u)\log\left(\frac{vw}{(1-u)^2}\right)\\ & +\frac{1}{3}\log(u)\log(v)\log(w)+\zeta_2\log(u)-\frac{5}{3}\zeta_3+\text{perms}\,(u,v,w)\,.\end{aligned}$$

$$\mathcal{R}^{(2)}_{\mathcal{O};2}\Big|_{\text{pure}} = -\text{Li}_2(1-u)-\log^2(u)+\frac{1}{2}\log(u)\log(v)-\frac{13}{2}\zeta_2 + \text{perms}\,(u,v,w)\,,$$

$$\begin{aligned}\mathcal{R}^{(2)}_{\mathcal{O};3}\Big|_{u/w} = {} & \Big[-\text{Li}_3\left(-\frac{u}{w}\right)+\log(u)\text{Li}_2\left(\frac{v}{1-u}\right)-\frac{1}{2}\log(1-u)\log(u)\log\left(\frac{w^2}{1-u}\right)\\ & +\frac{1}{2}\text{Li}_3\left(-\frac{uv}{w}\right)+\frac{1}{2}\log(u)\log(v)\log(w)+\frac{1}{12}\log^3(w)+(u\leftrightarrow v)\Big]+\text{Li}_3(1-v)-\text{Li}_3(u)\\ & +\frac{1}{2}\log^2(v)\log\left(\frac{1-v}{u}\right)-\zeta_2\log\left(\frac{uv}{w}\right)\,.\end{aligned}$$

$$\mathcal{R}^{(2)}_{\mathcal{O};2}\Big|_{u^2/w^2} = \text{Li}_2(1-u)+\text{Li}_2(1-v)+\log(u)\log(v)-\zeta_2\,.$$

$$\mathcal{R}^{(2)}_{\mathcal{O};1} = \left(-4+\frac{v}{w}+\frac{u^2}{2vw}+2\log(uvw)\right)$$