Amplituhedron meets Jeffrey-Kirwan Residue

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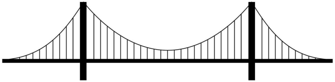
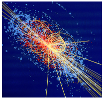
with Livia Ferro and Matteo Parisi – arXiv:1805.01301

Scattering amplitudes in Quantum Field Theory

$$\int \mathcal{D}\phi \, e^{iS[\phi]}$$





$\longrightarrow$ modern on-shell methods: generalized unitarity, recursion relations, ...

Geometrization of scattering amplitudes:

[Hodges]

"Amplitude = Volume of a geometric space"

Main example: $\mathcal{N}=4$ super Yang-Mills (SYM)

- $\longrightarrow$ supersymmetric cousin of QCD
- $\longrightarrow$ numerous dualities and correspondences
- $\longrightarrow$ simplicity: "Hydrogen atom of 21st century"

Geometrization of amplitudes in $\mathcal{N}=4$ SYM: **Amplituhedron** $\mathcal{A}_{n,k}^{(m)}$

[Arkani-Hamed, Trnka]

Introduction

The **Jeffrey-Kirwan residue** is a powerful concept in

→ *Mathematics:* localization of non-abelian group actions in equivariant cohomology [Jeffrey, Kirwan]

→ *Physics:* (supersymmetric) localization in gauge theories [Witten]

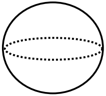
Supersymmetric localization:

→ Gauge theories with fermionic symmetries: $Q^2 = 0$

→ Supersymmetric path integrals can be evaluated explicitly

Example:

→ $\mathcal{N} = (2,2)$ SUSY gauge theory in two dimensions with the gauge group: $U(p)$

partition function on  $\xrightarrow{\text{Coulomb branch localization}}$ $\langle 1 \rangle = \int_{\gamma} \frac{d^p \sigma}{W_{X_1}(\sigma) \cdot \ldots \cdot W_{X_s}(\sigma)}$

The contour γ given by the **Jeffrey-Kirwan residue** — fixed by **gauge charges** of the fields X_i.

Introduction

More generally: Jeffrey-Kirwan residue is an **operation on rational forms** [Brion, Vergne]

For tree-level Amplituhedron $\mathcal{A}_{n,k}^{(m)}$: define a volume function $\Omega_{n,k}^{(m)}$. It can be obtained as

$$\Omega_{n,k}^{(m)} = \int_{\gamma} \omega_{n,k}^{(m)}$$

Our work: [Ferro, TL, Parisi]

Apply the Jeffrey-Kirwan residue to find the Amplituhedron volume function

We showed that:

- JK-residue gives the correct contour for cyclic polytopes, $k = 1$:

$$\Omega_{n,1}^{(m)} = \text{JKRes}\, \omega_{n,1}^{(m)} \qquad \text{for any } n \text{ and any } m$$

- JK-residue gives the correct contour for conjugates to cyclic polytopes, $k = n - m - 1$:

$$\Omega_{n,n-m-1}^{(m)} = \text{JKRes}\, \omega_{n,n-m-1}^{(m)} \qquad \text{for any } n \text{ and even } m$$

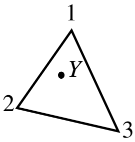
Positivity plays a crucial role in getting the correct result!

Amplituhedron $\mathcal{A}_{n,k}^{(m)}$: the space

$\mathcal{A}_{3,1}^{(2)}$: Triangle

$$Y^A = \sum_{i=1}^{3} c_i \, Z_i^A$$

$A = 1, 2, 3$



$c_i > 0$

$k = 1$

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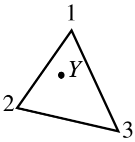
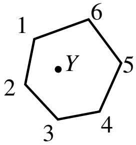


more points $\longrightarrow$



$\mathcal{A}_{n,1}^{(2)}$: Polygon

$$Y^A = \sum_{i=1}^{n} c_i Z_i^A$$

$A = 1, 2, 3$

$c_i > 0$

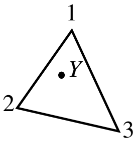
$k = 1$

Amplituhedron $\mathcal{A}^{(m)}_{n,k}$: the space

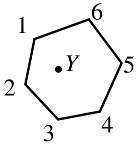
$\mathcal{A}^{(2)}_{3,1}$: Triangle

$$Y^A = \sum_{i=1}^{3} c_i \, Z_i^A$$

$$A = 1, 2, 3$$



more points →



$\mathcal{A}^{(2)}_{n,1}$: Polygon

$$Y^A = \sum_{i=1}^{n} c_i \, Z_i^A$$

$$A = 1, 2, 3$$

$c_i > 0$

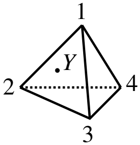
↓ higher dimension

$k = 1$

$\mathcal{A}^{(m)}_{m+1,1}$: Simplex

$$Y^A = \sum_{i=1}^{m+1} c_i \, Z_i^A$$

$$A = 1, \ldots, m+1$$



Amplituhedron $\mathcal{A}_{n,k}^{(m)}$: the space

$\mathcal{A}_{3,1}^{(2)}$: Triangle

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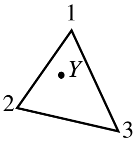
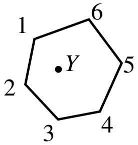


more points →



$\mathcal{A}_{n,1}^{(2)}$: Polygon

$$Y^A = \sum_{i=1}^{n} c_i \, Z_i^A$$

$A = 1, 2, 3$

higher dimension ↓

$c_i > 0$

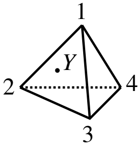
$k = 1$

higher dimension ↓

$\mathcal{A}_{m+1,1}^{(m)}$: Simplex

$$Y^A = \sum_{i=1}^{m+1} c_i \, Z_i^A$$

$A = 1, \ldots, m+1$



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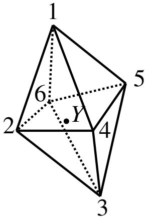

$\mathcal{A}_{n,1}^{(m)}$: Cyclic polytope

$$Y^A = \sum_{i=1}^{n} c_i \, Z_i^A$$

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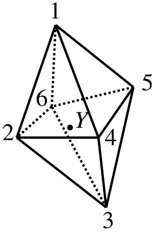
Amplituhedron $\mathcal{A}_{n,k}^{(m)}$: the space



$\mathcal{A}_{n,1}^{(m)}$: Cyclic polytope

$$Y^A = \sum_{i=1}^{n} c_i Z_i^A$$

$$A = 1, \ldots, m+1$$

higher helicities $\longrightarrow$

[Illustration by Andy Gilmore]

$\mathcal{A}_{n,k}^{(m)}$: Amplituhedron

$$Y_\alpha^A = \sum_{i=1}^{n} c_{\alpha i} Z_i^A$$

$$A = 1, \ldots, m+k$$

$$\alpha = 1, \ldots, k$$

The Amplituhedron

$$\boxed{Y = c \cdot Z}$$

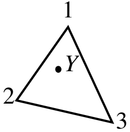
$$\begin{cases} Z \in M_+(k+m, n) & \text{fixed positive external data} \\ c \in G_+(k,n) & \text{vary over all positive matrices} \\ Y \in G(k, k+m) & \end{cases}$$

Amplituhedron: the volume form

For each amplituhedron $\mathcal{A}_{n,k}^{(m)}$ one defines a volume form $\widetilde{\Omega}_{n,k}^{(m)} = \prod_\alpha \langle Y_1 \ldots Y_k\, d^m Y_\alpha \rangle \Omega_{n,k}^{(m)}$

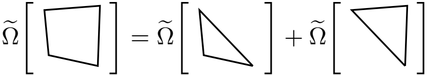
$\widetilde{\Omega}_{n,k}^{(m)}$ has **logarithmic singularities** at **all boundaries** of $\mathcal{A}_{n,k}^{(m)}$

→ **Example:**



$$\rightarrow \boxed{Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3} \rightarrow \widetilde{\Omega}_{3,1}^{(2)} = \frac{dc_2}{c_2}\frac{dc_3}{c_3} = \frac{\langle Y\, dY\, dY \rangle \langle 123 \rangle^2}{\langle Y12 \rangle \langle Y23 \rangle \langle Y31 \rangle}$$

→ Compatible with triangulations



Why do we care?

Tree amplitudes in $\mathcal{N} = 4$ SYM can be extracted from $\widetilde{\Omega}_{n,k}^{(4)}$

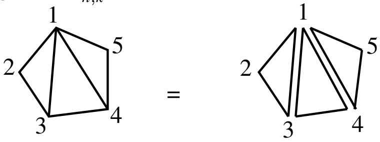
- (4) ← *space-time dimension*
- k ← *helicity sector*
- n ← *number of particles*

Amplituhedron: the volume form

How to find the volume function?

→ *Geometrically:* Triangulate $\mathcal{A}_{n,k}^{(m)}$ and sum over the known volumes of each triangles

e.g.:



$$\Omega_{5,1}^{(2)} = [123] + [134] + [145]$$

→ *Analytically:* Evaluate the contour integral

$$\Omega_{n,k}^{(m)} = \int_{\gamma} \frac{d^{k\cdot n}c}{(12\ldots k)\ldots(n1\ldots k-1)} \prod_{\alpha,A} \delta(Y_\alpha^A - \sum_i c_{\alpha i} Z_i^A) = \int_{\gamma} \omega_{n,k}^{(m)}$$

e.g.: $\gamma = \{c_4 = c_5 = 0\} \cup \{c_2 = c_5 = 0\} \cup \{c_2 = c_3 = 0\}$

different contours ↔ different triangulations

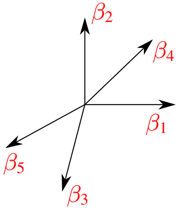
Finding γ is a highly non-trivial task!

Jeffrey-Kirwan residue: definition

The Jeffrey-Kirwan residue is an operation on rational differential forms

$$\omega = \frac{dx_1 \wedge \ldots \wedge dx_r}{\beta_1(x) \ldots \beta_n(x)}, \qquad \beta_i(x) = \beta_i \cdot x + \alpha_i$$

Example



Ingredients

- $B = \{\beta_i\}$ is the set of r-dimensional vectors: **charges**
- A **cone** is the positive span of r **charges**
- Fixed reference vector $\eta \in \mathbb{R}^r$

Jeffrey-Kirwan residue

$$\text{JKRes}^{B,\eta}\,\omega = \sum_{cone \ni \eta} \text{Res}_{cone}\,\omega$$

→ $\text{Res}_{cone}\,\omega$ is the multivariate residue, up to a sign

→ the signs are determined by the orientation of cones

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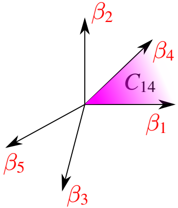

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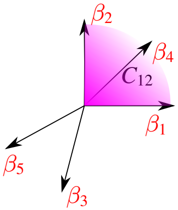

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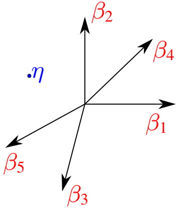

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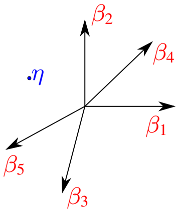

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Jeffrey-Kirwan residue: properties

$$\boxed{\mathrm{JKRes}^{B,\eta}\,\omega = \sum_{cone\ni\eta} \mathrm{Res}_{cone}\,\omega}$$

Example

$$\omega = \frac{dx_1 \wedge dx_2}{\beta_1(x)\ldots\beta_5(x)}$$

The JK-residue provides a particular contour on which we integrate ω

1. Any η

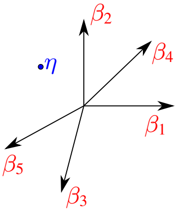
$$\mathrm{JKRes}^{\mathrm{B},\eta}\,\omega = \mathrm{Res}_{C_{25}}\,\omega + \mathrm{Res}_{C_{45}}\,\omega + \mathrm{Res}_{C_{23}}\,\omega$$

2. For η' close to η

$$\underline{\mathrm{JKRes}^{\mathrm{B},\eta'}\,\omega} = \mathrm{Res}_{C_{25}}\,\omega + \mathrm{Res}_{C_{45}}\,\omega + \mathrm{Res}_{C_{23}}\,\omega = \underline{\mathrm{JKRes}^{\mathrm{B},\eta}\,\omega}$$

$\rightarrow$ A **chamber** Δ is a maximal non-empty intersection of cones

$\rightarrow$ JK-residue gives the **same form** of the answer in each chamber



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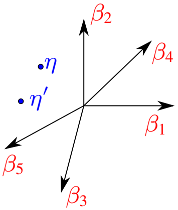

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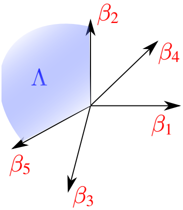
Example

$$\omega = \frac{dx_1 \wedge dx_2}{\beta_1(x)\ldots\beta_5(x)}$$



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$$\text{JKRes}^{B,\eta}\,\omega = \sum_{cone \ni \eta} \text{Res}_{cone}\,\omega$$

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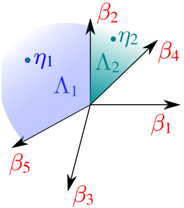


3. JKRes gives **different form** of the answer in various chambers

$$\text{JKRes}^{B,\eta_1}\,\omega = \text{Res}_{C_{25}}\,\omega + \text{Res}_{C_{45}}\,\omega + \text{Res}_{C_{23}}\,\omega$$

$$\parallel$$

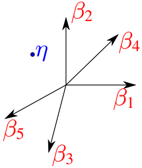
$$\text{JKRes}^{B,\eta_2}\,\omega = \text{Res}_{C_{45}}\,\omega + \text{Res}_{C_{12}}\,\omega + \text{Res}_{C_{42}}\,\omega$$

However, the function is **independent** of the chamber

$\rightarrow$ This equality follows from the **global residue theorem**

$$\int_\Gamma \frac{dx_1 \wedge dx_2}{\beta_1(x)\ldots\beta_5(x)} = 0\,, \qquad \text{with } \Gamma : \begin{cases} |\beta_2(x)| = \epsilon \\ |\beta_1(x)\beta_3(x)\beta_4(x)\beta_5(x)| = \epsilon \end{cases}$$

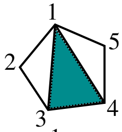
Amplituhedron meets Jeffrey-Kirwan residue



- Apply Jeffrey-Kirwan residue to: $\Omega^{(m)}_{n,1} = \int_{\gamma} \omega^{(m)}_{n,1}$
- E.g. $n = 5$: **positivity** of Z implies that vectors

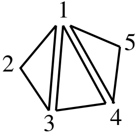
  $$\beta_1, -\beta_2, \beta_3, -\beta_4, \beta_5, -\beta_1, \beta_2, -\beta_3, \beta_4, -\beta_5, \beta_1$$

  are ordered clockwise



- Each $\text{Res}_{cone}\, \omega^{(2)}_{5,1}$ computes the volume of a triangle

  $$\text{Res}_{C_{25}}\, \omega^{(2)}_{5,1} = [134]$$

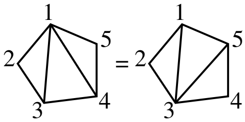


- Each chamber provides a **triangulation**

  $$\text{JKRes}^{B,\eta}\, \omega^{(2)}_{5,1} = [123] + [134] + [145]$$

- Adjacent chambers: related by **bistellar flip**
  - → *Geometrically:* 

  - → *Analytically:* Global residue theorem



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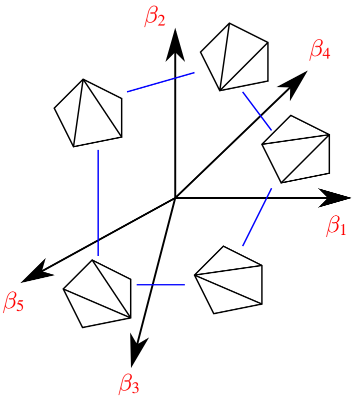
  $$\text{Res}_{C_{25}}\, \omega^{(2)}_{5,1} = [134]$$

- Each chamber provides a **triangulation**

  $$\text{JKRes}^{B,\eta}\, \omega^{(2)}_{5,1} = [123] + [134] + [145]$$

- Adjacent chambers: related by **bistellar flip**



→ *Geometrically:*

→ *Analytically:* Global residue theorem

General Statement: Cyclic Polytopes

For cyclic polytopes ($k = 1$, any m, any n):

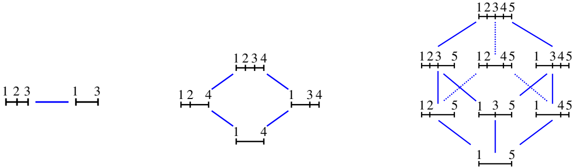
$$\Omega_{n,1}^{(m)} = \text{JKRes}^{B,\eta}\omega_{n,1}^{(m)}$$

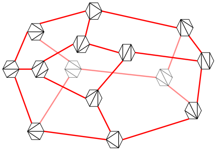
- For each chamber
  - → *Geometrically:* different triangulations of $\mathcal{A}_{n,1}^{(m)}$
  - → *Analytically:* different representation of $\Omega_{n,1}^{(m)}$
- For adjacent chamber
  - → *Geometrically:* triangulations related by bistellar flip
  - → *Analytically:* representations related by global residue theorem

Vertices of the **secondary polytope** $\Sigma(\mathcal{P})$ are triangulations of a given polytope $\mathcal{P}$

Examples of secondary polytopes

- For $m = 1$: combinatorially equivalent to the **hypercube** in $n - 2$ dimensions





- For an n-gon, $m = 2$: **Associahedron**
- In general very complicated
  - → e.g. physical case, eight particles: 3-dimensional with 40 vertices
  - → e.g. physical case, nine particles: 4-dimensional with 357 vertices

It is fully captured by the Jeffrey-Kirwan residue applied to $\omega_{n,1}^{(m)}$!

General statement: Conjugates to cyclic polytopes

Parity conjugation

→ **Amplitudes:** conjugate the helicities of the particles

→ **Amplitudhedron:** $k \leftrightarrow n-m-k$

Conjugates to cyclic polytopes are not polytopes!

Jeffrey-Kirwan residue prescription for conjugates to cyclic polytopes:

$$\Omega^{(m)}_{n,n-m-1} = \text{JKRes}^{B,\eta}\,\omega^{(m)}_{n,n-m-1}\,, \qquad \text{for even } m$$

Cyclic polytopes	and	their conjugates	
$Y \in G(1, m+1)$		$Y \in G(n-m-1, n-1)$	
Triangles in $\mathcal{A}^{(m)}_{n,1}$	$\leftrightarrow$	Triangles in $\mathcal{A}^{(m)}_{n,n-m-1}$	all m
Triangulations of $\mathcal{A}^{(m)}_{n,1}$	$\leftrightarrow$	Triangulations of $\mathcal{A}^{(m)}_{n,n-m-1}$	even m
$\Sigma(\mathcal{A}^{(m)}_{n,1})$	$\leftrightarrow$	$\Sigma(\mathcal{A}^{(m)}_{n,n-m-1})$	even m

Conclusions

The Jeffrey-Kirwan residue

- provides a **triangulation-independent** formula for the volume forms $\widetilde{\Omega}^{(m)}_{n,1}$ and $\widetilde{\Omega}^{(m)}_{n,n-m-1}$
- strengthen the connection between
  - $\rightarrow$ the complicated **analytic/algebraic** nature of scattering amplitudes
  - $\rightarrow$ the rich **geometric/combinatorial** structure of the amplituhedron
- naturally leads to the study of the **secondary amplituhedron**
  - $\rightarrow$ encodes **all triangulations** of the amplituhedron
  - $\rightarrow$ generalises the notion of secondary polytope

Outlook

Generalisations:

- What is the **generalisation** of the Jeffrey-Kirwan residue for all other amplituhedra?
- Can we find the **secondary amplituhedron** in all these cases?

  → many **new representations** of scattering amplitudes

  → new and rich structure of **all triangulations**

The origin:

- Why Jeffrey-Kirwan residue provides the correct answer?
- Can one understand the amplituhedron from some **underlying localization perspective**?
- Can we learn some new lessons for **supersymmetric localization**?

Thank you for your attention!