The space of EFT and CFT: behind the walls of cyclic polytopes

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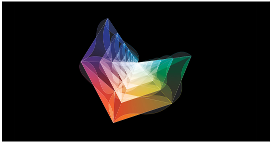
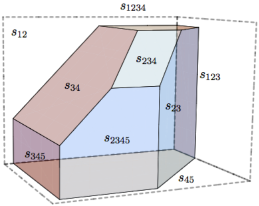
Positive geometry $\rightarrow$ emergent locality and unitarity see Tomasz, Nima and Song's talk





Essentially one is asking:

What is the question for which these amplitudes are the answer to?

This is not a first, in many case positivity IS unitarity

- Positivity in the OPE:

$$\langle\phi(1)\phi(2)\phi(3)\phi(4)\rangle = \sum_i \mathfrak{p}_i K_{\Delta_i,\ell_i}(z,\bar{z}), \quad \mathfrak{p}_i > 0$$

- Optical theorem:

$$Dis[M_4(s,0)] = E_{cm}^2\sigma > 0$$

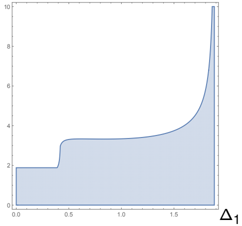
This is not a first. For a long time positivity IS unitarity

- Positivity in the OPE:

$$\langle\phi(1)\phi(2)\phi(3)\phi(4)\rangle = \sum_i \mathfrak{p}_i K_{\Delta_i,\ell_i}(z,\bar{z}), \quad \mathfrak{p}_i > 0$$

  via crossing

Δ_2 :



- Optical theorem:

$$Dis[M_4(s,0)] = E_{cm}^2\sigma > 0$$

  via the eyes of higher-dimension operators $a(\partial\phi)^4$

We should expect more: these are special functions, constrained by factorization and symmetries

- : CFTs:

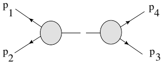
$$\langle\phi(1)\phi(2)\phi(3)\phi(4)\rangle = \sum_i \mathfrak{p}_i g_{\Delta_i,\ell_i}(z,\bar{z}), \quad \mathfrak{p}_i > 0$$

  Symmetries constrain

$$(z^2(1-z)\partial_z^2 - z^2\partial_z) g_{\Delta,\ell} = \Delta(\Delta-1) g_{\Delta,\ell}$$

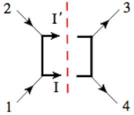
- QFTs:

$$Dis[M_4(s,t)] = \sum_i \mathfrak{p}_i G^{\alpha}_{\ell_i}(\cos\theta)$$



$$= p_{12}^{\mu_1}\cdots p_{12}^{\mu_\ell} P_{\mu_1\cdots\mu_\ell\nu_1\cdots\nu_\ell} p_{34}^{\mu_1}\cdots p_{34}^{\mu_\ell} = G^{\alpha}_{\ell}\left(1+\frac{2t}{m^2}\right)$$

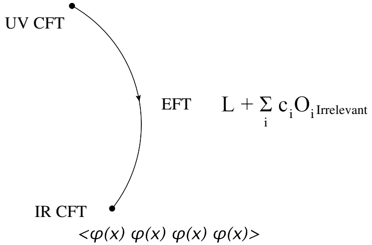
Loops:



$$= \sum_\ell |f_\ell|^2 G_\ell^{\frac{1}{2}}(\cos\theta),$$

We now ask:

What is the question for which the space of consistent QFT/CFT are the answer to?

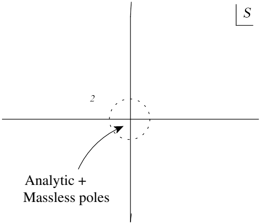


What is the geometric property from which unitarity, locality and symmetries emerge as a union.

The space of QFT from M_4

At low energies, we only have photons and gravitons. Consider general QFT whose UV completion is weakly coupled (in M_{pl}),

(*surpressed*)

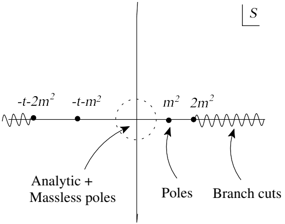


$$M^{IR}(s,t) = \{poles\} + \sum_{k,i} g_{k-i,i} s^{k-i} t^i$$

Different QFTs (standard model) leads to different $\{g_{i,j}\}$

Why might the space be non-trivial?

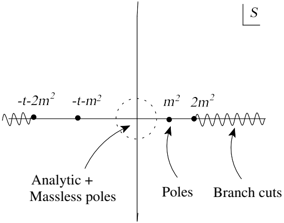
The space of QFT from M_4



$$M^{IR}(s,t) = \{poles\} + \sum_{k,i} g_{k-i,i} s^{k-i} t^i$$

The space of QFT from M_4

Why is the space non-trivial (set $D = 4$ $G_\ell^\alpha \to P_\ell$)?



$$M(s,t) = -\sum_a \mathfrak{p}_a P_{\ell_a}\left(1+\frac{2t}{m_a^2}\right)\left(\frac{1}{s-m_a^2}+\frac{1}{u-m_a^2}\right)$$

$$= \sum_{k,q}\sum_a \mathfrak{p}_a \frac{1}{m_a^{2k+2}} v_{k,\ell_a}^q s^{k-q} t^q$$

so we have

$$\sum_{k,q} g_{k-q,q} s^{k-q} t^q = \sum_{k,q}\left(\sum_a \mathfrak{p}_a \frac{1}{m_a^{2k+2}} u_{k,\ell_a}^q\right) s^{k-q} t^q$$

Why is the space non-trivial?

$$\sum_{k,q} g_{k-q,q} s^{k-q} t^q = \sum_{k,q} \left(\sum_a \mathfrak{p}_a \frac{1}{m_a^{2k+2}} u^q_{k,\ell_a} \right) s^{k-q} t^q$$

Organizing the higher dimension operators as

$$\begin{array}{cccccc}
 & m^0 & \frac{1}{m^2} & \frac{1}{m^4} & \frac{1}{m^6} & \cdots \\
t^0 & g_{0,0} & g_{1,0} & g_{2,0} & g_{3,0} & \cdots \\
t^1 & & g_{0,1} & g_{1,1} & g_{2,1} & \cdots \\
t^2 & & & g_{0,2} & g_{1,2} & \cdots \\
t^3 & & & & g_{0,3} & \cdots
\end{array}$$

The space of QFT from M_4

Why is the space non-trivial?

$$\sum_{k,q} g_{k-q,q} s^{k-q} t^q = \sum_{k,q} \left(\sum_a \mathfrak{p}_a \frac{1}{m_a^{2k+2}} u^q_{k,\ell_a} \right) s^{k-q} t^q$$

Organizing the higher dimension operators as

	m^0	$\frac{1}{m^2}$	$\frac{1}{m^4}$	$\frac{1}{m^6}$	$\cdots$
t^0	$g_{0,0}$	$g_{1,0}$	$g_{2,0}$	$g_{3,0}$	$\cdots$
t^1		$g_{0,1}$	$g_{1,1}$	$g_{2,1}$	$\cdots$
t^2			$g_{0,2}$	$g_{1,2}$	$\cdots$
t^3				$g_{0,3}$	$\cdots$

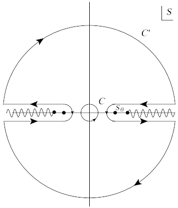
Take $k = 2$ (dimension 8 operators)

$$\vec{g}_2 = \begin{pmatrix} g_{2,0} \\ g_{1,1} \\ g_{0,2} \end{pmatrix} \in \sum_a \mathfrak{p}'_\ell \vec{u}_{2,\ell} \quad \mathfrak{p}'_\ell > 0$$

The coefficients must live in the convex hull of the vectors $\vec{u}_{2,\ell}$, i.e. the inside of a polytope.

The space of QFT from M_4

Flat out unitarity tells us Adams, Arkani-Hamed, Dubovsky, Nicolis and Rattazzi,



	m^0	$\frac{1}{m^2}$	$\frac{1}{m^4}$	$\frac{1}{m^6}$	$\cdots$
t^0	$g_{0,0}$	$g_{1,0}$	$g_{2,0}$	$g_{3,0}$	$\cdots$
t^1		$g_{0,1}$	$g_{1,1}$	$g_{2,1}$	$\cdots$
t^2			$g_{0,2}$	$g_{1,2}$	$\cdots$
t^3				$g_{0,3}$	$\cdots$

$$\{g_{0,0}, g_{2,0}, g_{4,0}, \cdots\} > 0$$

Now we know that

$$\vec{g}_k \in \sum_a \mathfrak{p}'_\ell \vec{u}_{k,\ell} \quad \mathfrak{p}'_\ell > 0$$

and the above is simply due to

$$P_\ell(1) > 0 \to u^0_{k,\ell} > 0$$

"the tip of an iceberg"

The space of CFT from $\langle\phi(1)\phi(2)\phi(3)\phi(4)\rangle$

Consider the a 1D four-point function:

$$\langle\phi(1)\phi(2)\phi(3)\phi(4)\rangle \equiv F(z)$$

$$\mathbf{F}(z) = \sum_{\Delta} \mathfrak{p}_{\Delta} C_{\Delta}(z), \quad C_{\Delta}(z) = z^{\Delta}\,{}_2F_1(\Delta, \Delta, 2\Delta, z)$$

Expand the four-point function, around $z = \frac{1}{2}$

$$\mathbf{F}\left(\frac{1}{2} + y\right) = \sum_{q=0}^{\infty} f_q y^q$$

We consider the space $\{f_q\}$

Crossing symmetry

$$z^{-2\Delta_\phi} F(z) = (1-z)^{-2\Delta_\phi} F(1-z) \to F(z) = \left(\frac{z}{1-z}\right)^{2\Delta_\phi} F(1-z)$$

implies the four-point function lies in a subplane X

The space of CFT from $\langle\phi(1)\phi(2)\phi(3)\phi(4)\rangle$

The 1-D blocks also yield an infinite set of vectors

$$C_\Delta\left(\frac{1}{2}+y\right)=\sum_{q=0}^{\infty}c_{\Delta,q}y^q$$

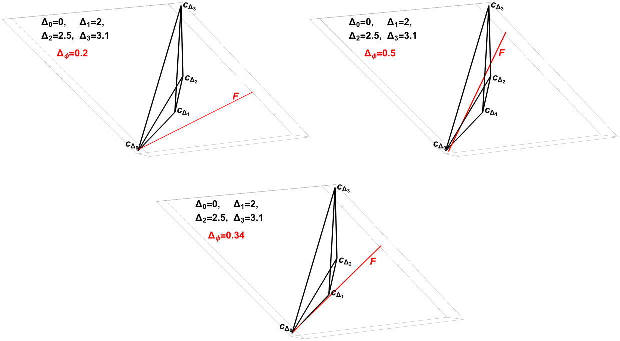
Unitarity then requires that

$$\mathbf{F}=\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{L-1}\end{pmatrix}\in\sum_\Delta \mathfrak{p}_\Delta\begin{pmatrix} c_{\Delta,0} \\ c_{\Delta,1} \\ \vdots \\ c_{\Delta,L-1}\end{pmatrix}\quad \mathfrak{p}_\Delta>0$$

The space of CFT from $\langle\phi(1)\phi(2)\phi(3)\phi(4)\rangle$

$$\mathbf{F} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{L-1} \end{pmatrix} \in \sum_{\Delta} \mathfrak{p}_\Delta \begin{pmatrix} c_{\Delta,0} \\ c_{\Delta,1} \\ \vdots \\ c_{\Delta,L-1} \end{pmatrix} \qquad \mathfrak{p}_\Delta > 0$$

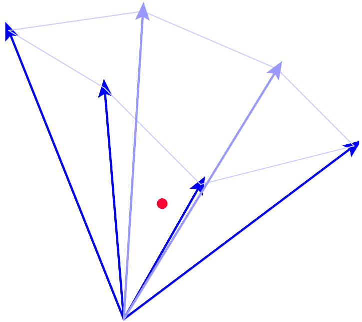
For a given CFT spectrum have the polytope $P(\Delta_i) = \sum_i \mathfrak{p}_{\Delta_i} \vec{c}_{\Delta_i}$ and a crossing plane $X(\Delta_\phi)$, and they must intersect. For example:



If $\vec{u}_{k,\ell}$ for EFT and $\vec{c}_{\Delta_i}$ for CFT are just random vectors, our geometric problem becomes hopeless rapidly:

Let's say given n vectors $\vec{u}$, to compute the region of the polytope we need to

- Determine which one of these $\vec{u}$s are vertices
- Amongst the vertices, determine all the set that constitute boundary facets



The complexity is $\sim n^{d/2}$

But $\vec{u}_{k,\ell}$ and $\vec{c}_{\Delta_i}$ are not random vectors!

EFT

The $\vec{u}_{k,\ell}$, arrises from Taylor expand

$$M(s,t) = \sum_a \mathfrak{p}_a P_{\ell_a}\left(1+\frac{2t}{m_a^2}\right)\left(\frac{1}{s-m_a^2}+\frac{1}{u-m_a^2}\right)$$

Define

$$P_\ell(1+x) = \sum_q v_{\ell,q} x^q$$

The vector $\vec{v}_\ell = (v_{\ell,0},\ v_{\ell,1},\ v_{\ell,2},\cdots)$ take the form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 6 & 10 & 15 & 21 & 28 \\ 0 & 0 & \frac{3}{2} & \frac{15}{2} & \frac{45}{2} & \frac{105}{2} & 105 & 189 \\ 0 & 0 & 0 & \frac{5}{2} & \frac{35}{2} & 70 & 210 & 525 \\ 0 & 0 & 0 & 0 & \frac{35}{8} & \frac{315}{8} & \frac{1575}{8} & \frac{5775}{8} \\ 0 & 0 & 0 & 0 & 0 & \frac{63}{8} & \frac{693}{8} & \frac{2079}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{231}{16} & \frac{3003}{16} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{429}{16} \end{pmatrix}.$$

Note that all v is positive! But there is more,

$$det[\vec{v}_{\ell_1}\,\vec{v}_{\ell_2}\cdots] > 0, \quad \text{for} \ell_1 > \ell_2 > \cdots$$

All ordered minors are positive!

EFT

$$det[\vec{v}_{\ell_1}\vec{v}_{\ell_2}\cdots] > 0, \quad \mathrm{for} \ell_1 > \ell_2 > \cdots$$

All ordered minors are positive!

Tells us that the convex hull of $\{\vec{v}_\ell\}$ is a cyclic polytope

- All $\vec{v}_\ell$ are vertices
- The co-dimension 1 boundaries are known. For $\vec{v}_\ell = (v_{\ell,0}, \cdots, \ v_{\ell,q})$

$$q \in even \quad (i, i{+}1), \quad (i, i{+}1, j, j{+}1), \quad (i, i{+}1, \cdots, j, j{+}1)$$

$$q \in odd \quad (1, i, i{+}1), \quad (1, i, i{+}1 \cdots j, j{+}1), \quad (i, i{+}1, n), \quad (i, i{+}1 \cdots j, j{+}1, n)$$

EFT

But $\vec{v}_\ell$ is not $\vec{u}_{k,\ell}$,

$$M(s,t) = \sum_a \mathfrak{p}_a P_{\ell_a}\left(1+\frac{2t}{m_a^2}\right)\left(\frac{1}{s-m_a^2}+\frac{1}{u-m_a^2}\right)$$

$\vec{u}_{k,\ell}$ recieves are contributions from propagators, message from locality

EFThedron

Let's consider s-channel pole only (large-N YM),

$$
\begin{aligned}
M(s,t) &= -\sum_a \mathfrak{p}_a \frac{P_{\ell_a}\left(1+\frac{2t}{m_a^2}\right)}{s-m_a^2} \\
&= \sum_a \mathfrak{p}_a \frac{1}{m_a^2}\left(1+\frac{s}{m_a^2}+\left(\frac{s}{m_a^2}\right)^2+\cdots\right)_{\text{locality}} \left(v_{\ell_a,0}+v_{\ell_a,1}t+v_{\ell_a,2}t^2\cdots\right)_{\text{unitarity}}
\end{aligned}
$$

We find that locality and unitarity leads to two separate positive geometry!

	m^0	$\frac{1}{m^2}$	$\frac{1}{m^4}$	$\frac{1}{m^6}$	$\cdots$
t^0	$g_{0,0}$	$g_{1,0}$	$g_{2,0}$	$g_{3,0}$	$\cdots$
t^1		$g_{0,1}$	$g_{1,1}$	$g_{2,1}$	$\cdots$
t^2			$g_{0,2}$	$g_{1,2}$	$\cdots$
t^3				$g_{0,3}$	$\cdots$

$$
\vec{g}_2 = \begin{pmatrix} g_{2,0} \\ g_{1,1} \\ g_{0,2} \end{pmatrix} \in \sum_a \mathfrak{p}'_\ell \vec{v}_\ell \quad \mathfrak{p}'_\ell > 0
$$

EFThedron

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\begin{aligned}
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&= \sum_a \mathfrak{p}_a \frac{1}{m_a^2} \underbrace{\left(1+\frac{s}{m_a^2}+\left(\frac{s}{m_a^2}\right)^2+\cdots\right)}_{locality} \underbrace{\left(v_{\ell_a,0}+v_{\ell_a,1}t+v_{\ell_a,2}t^2\cdots\right)}_{unitarity}
\end{aligned}
$$

We find that locality and unitarity leads to two separate positive geometry!

$$
\begin{array}{cccccc}
 & m^0 & \frac{1}{m^2} & \frac{1}{m^4} & \frac{1}{m^6} & \cdots \\
t^0 & g_{0,0} & g_{1,0} & g_{2,0} & g_{3,0} & \cdots \\
t^1 & & g_{0,1} & g_{1,1} & g_{2,1} & \cdots \\
t^2 & & & g_{0,2} & g_{1,2} & \cdots \\
t^3 & & & & g_{0,3} & \cdots
\end{array}
$$

$$
\vec{g}_2 = \begin{pmatrix} g_{2,0} \\ g_{1,1} \\ g_{0,2} \end{pmatrix} \in \sum_a \mathfrak{p}'_\ell \vec{v}_\ell \quad \mathfrak{p}'_\ell > 0
$$

EFThedron

Let's consider s-channel pole only (large-N YM),

$$M(s,t) = -\sum_a \mathfrak{p}_a \frac{P_{\ell_a}\left(1+\frac{2t}{m_a^2}\right)}{s-m_a^2}$$

$$= \sum_a \mathfrak{p}_a \frac{1}{m_a^2}\left(1+\frac{s}{m_a^2}+\left(\frac{s}{m_a^2}\right)^2+\cdots\right)_{locality} \left(v_{\ell_a,0}+v_{\ell_a,1}t+v_{\ell_a,2}t^2\cdots\right)_{unitarity}$$

We find that locality and unitarity leads to two separate positive geometry!

	m^0	$\frac{1}{m^2}$	$\frac{1}{m^4}$	$\frac{1}{m^6}$	$\cdots$
t^0	$g_{0,0}$	$g_{1,0}$	$g_{2,0}$	$g_{3,0}$	$\cdots$
t^1		$g_{0,1}$	$g_{1,1}$	$g_{2,1}$	$\cdots$
t^2			$g_{0,2}$	$g_{1,2}$	$\cdots$
t^3				$g_{0,3}$	$\cdots$

$$\vec{g}_2 = \begin{pmatrix} g_{2,0} \\ g_{1,1} \\ g_{0,2} \end{pmatrix} \rightarrow Det[\vec{g}_2, \vec{v}_\ell, \vec{v}_{\ell+1}] > 0$$

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Let's consider s-channel pole only (large-N YM),

$$M(s,t) = -\sum_a \mathfrak{p}_a \frac{P_{\ell_a}\left(1+\frac{2t}{m_a^2}\right)}{s-m_a^2}$$

$$= \sum_a \mathfrak{p}_a \frac{1}{m_a^2}\left(1+\frac{s}{m_a^2}+\left(\frac{s}{m_a^2}\right)^2+\cdots\right)_{locality}\left(v_{\ell_a,0}+v_{\ell_a,1}t+v_{\ell_a,2}t^2\cdots\right)_{unitarity}$$

We find that locality and unitarity leads to two separate notion of locality!

	m^0	$\frac{1}{m^2}$	$\frac{1}{m^4}$	$\frac{1}{m^6}$	$\cdots$
t^0	$g_{0,0}$	$g_{1,0}$	$g_{2,0}$	$g_{3,0}$	$\cdots$
t^1		$g_{0,1}$	$g_{1,1}$	$g_{2,1}$	$\cdots$
t^2			$g_{0,2}$	$g_{1,2}$	$\cdots$
t^3				$g_{0,3}$	$\cdots$

$$\begin{pmatrix} g_{0,1} \\ g_{1,1} \\ g_{2,1} \end{pmatrix} \in \sum_a \mathfrak{p}'_a \begin{pmatrix} \frac{1}{m_a^2} \\ \frac{1}{m_a^4} \\ \frac{1}{m_a^6} \end{pmatrix} \qquad \mathfrak{p}'_a > 0$$

The vector is in the convex hull of points on the half-moment curve!

$$(t, t^2, t^3 \cdots, t^a), \quad t \in R^+$$

EFThedron

$$(t, t^2, t^3 \cdots, t^a), \quad t \in R^+$$

Organizing the couplings for fixed t power into the Hankel matrix ($g'_k \equiv g_{k,i}$)

$$K(g') = \begin{pmatrix} 1 & g'_1 & \cdots & g'_{p-1} \\ g'_1 & g'_2 & \cdots & g'_p \\ \vdots & \vdots & \vdots & \vdots \\ g'_{p-1} & g'_p & \cdots & g'_{2p-2} \end{pmatrix},$$

The constraint is the statement that

$$i \in even: \quad Det \begin{pmatrix} 1 & g'_1 & \cdots & g'_{\frac{i}{2}} \\ g'_1 & g'_2 & \cdots & g'_{\frac{i}{2}+1} \\ \vdots & \vdots & \vdots & \vdots \\ g'_{\frac{i}{2}} & g'_{\frac{i}{2}+1} & \cdots & g'_i \end{pmatrix} \geq 0, \qquad i \in odd: \quad Det \begin{pmatrix} g'_1 & g'_2 & \cdots & g'_{\frac{i+1}{2}} \\ g'_2 & g'_3 & \cdots & g'_{\frac{i+3}{2}} \\ \vdots & \vdots & \vdots & \vdots \\ g'_{\frac{i+1}{2}} & g'_{\frac{i+3}{2}} & \cdots & g'_i \end{pmatrix} \geq 0$$

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Indeed consider

$$\frac{\Gamma[-s]\Gamma[-t]}{\Gamma[1-s-t]}|_{s\to 0} = \cdots + \frac{\pi^2}{6} + \zeta_3 s + \frac{\pi^4}{90}s^2 + \cdots$$

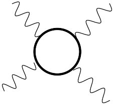
$$Det\begin{pmatrix} \frac{\pi^2}{6} & \zeta_3 \\ \zeta_3 & \frac{\pi^4}{90} \end{pmatrix} = 0.33541 > 0$$

$$\frac{\Gamma[-s]\Gamma[-t]}{\Gamma[1-s-t]}\left(1 - \frac{tu}{1+s}\right)|_{s\to 0} = \cdots + \frac{\pi^2}{6} + (1+\zeta_3)s + (\frac{\pi^4}{90} - 1)s^2 + \cdots$$

$$Det\begin{pmatrix} \frac{\pi^2}{6} & 1+\zeta_3 \\ 1+\zeta_3 & \frac{\pi^4}{90} - 1 \end{pmatrix} = -4.71364 < 0$$

EFThedron

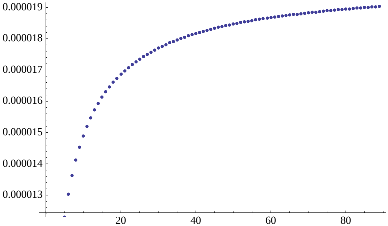
Consider the EFT of a scalar coupled to gravitons Congkao Wen, Wei-Ming Chen, Y-t



$$\Big|_{t=0} = \frac{\langle 23\rangle^4[14]^4}{M_{pl}^4}\left(\frac{1}{50400} + \frac{1}{17297280}\frac{s}{m^2} + \cdots\right)$$

$$= \frac{\langle 23\rangle^4[14]^4}{M_{pl}^4}\left(\sum_j 3\sqrt{\pi}4^{-2j-3}\frac{\Gamma[2j-1]}{(4+j)\Gamma[2j+\frac{7}{2}]}s^{j-1}\right)$$

Let's suppose we don't know the constant piece. The positivity of the Hankel matrix yields $\mathcal{O}(s^0) \geq 0.0000190301$



while $\frac{1}{50400} = 0.0000198413$

EFThedron

We see that the constraint from unitarity, locality and Lorentz invariance forces the EFT to live in a union of two positivie geometries

$$M(s,t) = -\sum_a \mathfrak{p}_a \frac{P_{\ell_a}\left(1+\frac{2t}{m_a^2}\right)}{s-m_a^2}$$

$$= \sum_a \mathfrak{p}_a \frac{1}{m_a^2}\left(1+\frac{s}{m_a^2}+\left(\frac{s}{m_a^2}\right)^2+\cdots\right)_{locality}\left(v_{\ell_a,0}+v_{\ell_a,1}t+v_{\ell_a,2}t^2\cdots\right)_{unitarity}$$

$$(Conv[moment\ curve])_{locality}(Conv[cyclic\ polytope])_{unitarity}$$

The EFTHedron

There is a much cleaner way to state the positive geometry. Lets start with

$$g_{k,q} = \sum_a \mathsf{p}_a \left[\frac{1}{m_a^{2(k+1)}} 2^q v_{\ell_a,q} \right] \equiv \sum_\ell C_{k,\ell} G_{\ell,q},$$

where $C_{k,\ell}$ is given as $\sum_{\{a:\ell_a=\ell\}} \frac{p_a}{m_a^{2(k+1)}}$. In other words it is an $k \times \infty$ matrix, whose column vectors are points on a degree k moment curve.

EFThedron $g_{k,q} \in C_{k,\ell} G_{\ell,q}$ $\leftrightarrow$ *Amplituhedron* $Y^I_\alpha = C_{+,\alpha,i} Z^I_i$

This space can be conveniently defined through its boundaries (or walls). For fixed k, consider an infinite set of walls $\mathcal{W}^q_I$:

$$\mathcal{W}_I = \{0, 0, 1, \cdots, 0\}$$

$$k = 2: \quad \mathcal{W}_I = \langle *ii{+}1 \rangle, \quad k = 3: \quad \mathcal{W}_I = \langle *1, i, i{+}1 \rangle, \ \langle *i, i{+}1, n \rangle$$

They satisfy

$$\sum_{q=0}^{k} \mathcal{W}^q_I G_{\ell,q} \geq 0.$$

We then find that the space is simply

$$A_{k,I} \equiv \sum_q g_{k,q} \mathcal{W}_I^q = \sum_\ell C_{k,\ell} \left(\sum_q G_{\ell,q} \mathcal{W}_I^q \right)$$

is a point inside the convex hull of half moment curves

$$A_{0,I} \geq 0, \quad A_{1,I} \geq 0, \quad det \begin{pmatrix} A_{0,I} & A_{1,I} \\ A_{1,I} & A_{2,I} \end{pmatrix} \geq 0, \quad det \begin{pmatrix} A_{1,I} & A_{2,I} \\ A_{2,I} & A_{3,I} \end{pmatrix} \geq 0$$

$$det \begin{pmatrix} A_{0,I} & A_{1,I} & A_{2,I} \\ A_{1,I} & A_{2,I} & A_{3,I} \\ A_{2,I} & A_{3,I} & A_{4,I} \end{pmatrix} \geq 0, \quad det \begin{pmatrix} A_{1,I} & A_{2,I} & A_{3,I} \\ A_{2,I} & A_{3,I} & A_{4,I} \\ A_{3,I} & A_{4,I} & A_{5,I} \end{pmatrix} \geq 0, \cdots e.t.c.$$

The EFTHedron

Thus the space of allowed EFT, is given by the EFThedron

$$M[K[\vec{A}_I]] \geq 0$$

For $\mathcal{W}_I = \{0, 0, 1, \cdots, 0\}$, the constraint says

$$g_{k,q} = \sum_a x_{k,q,a} \frac{1}{m_a^{2(k+1)}}, \quad x_{k,q,a} > 0$$

For $k = 2: \quad \mathcal{W}_I = \langle * i i{+}1 \rangle, \quad k = 3: \quad \mathcal{W}_I = \langle *1, i, i{+}1 \rangle, \ \langle *i, i{+}1, n \rangle$ the positivity of individual *A*

$$g_{k,q} = \sum_a x_{k,a} \frac{v_{\ell_a,q}}{m_a^{2(k+1)}}, \quad x_{k,a} > 0.$$

while for the whole Hankel matrix

$$g_{k,q} = \sum_a \mathsf{p}_a \frac{v_{\ell_a,q}}{m_a^{2(k+1)}}, \quad \mathsf{p}_a > 0.$$

The EFTHedron in the real world

Including the *u*-channel contribution:

$$M(s,t) = -\sum_a \mathfrak{p}_a P_{\ell_a}\left(1+\frac{2t}{m_a^2}\right)\left(\frac{1}{s-m_a^2}+\frac{1}{u-m_a^2}\right)$$

$$\to M(z,t) = \sum_a \mathfrak{p}_a P_{\ell_a}\left(1+\frac{2t}{m_a^2}\right)\left(\frac{1}{-\frac{t}{2}-z-m_a^2}+\frac{1}{-\frac{t}{2}+z-m_a^2}\right)$$

Upon Taylor expansion we have

$$\sum_{k-q\in even,\,q}\sum_a p_a\left[\frac{1}{m_a^{2(k+1)}}u_{\ell_a,k,q}\right]z^{k-q}t^q$$

where the new vectors are given by

$$u_{\ell,k,q} = \sum_{a+b=q}(-)^a\frac{(k-q+1)_a}{a!}2^{b-a}v_{\ell,b}$$

The EFTHedron in the real world

$$u_{\ell,k,q} = \sum_{a+b=q} (-)^a \frac{(k-q+1)_a}{a!} 2^{b-a} v_{\ell,b}$$

The new vectors are a k-dependent projection of $\vec{v}_\ell$ to half-dimension subspace, one might expect all structures are lost.

The geometry is richer

	m^0	$\frac{1}{m^2}$	$\frac{1}{m^4}$	$\frac{1}{m^6}$	$\cdots$
t^0	$g_{0,0}$	$g_{1,0}$	$g_{2,0}$	$g_{3,0}$	$\cdots$
t^1		$g_{0,1}$	$g_{1,1}$	$g_{2,1}$	$\cdots$
t^2			$g_{0,2}$	$g_{1,2}$	$\cdots$
t^3				$g_{0,3}$	$\cdots$

- For fixed mass-dimension, there is a critical spin above which it becomes cylic (all ordered minors are positive)
- The boundaries are determined from the cylicity

$$\langle X, i, i+1\rangle > 0 \text{ for,} \quad i \geq 5, \langle X,4,3\rangle > 0, \quad \langle X,3,5\rangle > 0$$

The EFTHedron in the real world

$$u_{\ell,k,q} = \sum_{a+b=q} (-)^a \frac{(k-q+1)_a}{a!} 2^{b-a} v_{\ell,b}$$

The new vectors are a k-dependent projection of $\vec{v}_\ell$ to half-dimension subspace, one might expect all structures are lost.

The geometry is richer

$$\begin{array}{cccccc} & m^0 & \frac{1}{m^2} & \frac{1}{m^4} & \frac{1}{m^6} & \cdots \\ t^0 & g_{0,0} & \color{red}{g_{1,0}} & \color{red}{g_{2,0}} & \color{red}{g_{3,0}} & \cdots \\ t^1 & & \color{red}{g_{0,1}} & \color{red}{g_{1,1}} & \color{red}{g_{2,1}} & \cdots \\ t^2 & & & g_{0,2} & g_{1,2} & \cdots \\ t^3 & & & & g_{0,3} & \cdots \end{array}$$

- The boundaries of the Minkowski sum is always given by that of the highest k

$$\partial\left[\begin{pmatrix} g_{1,0} \\ g_{0,1} \end{pmatrix} \oplus \begin{pmatrix} g_{2,0} \\ g_{1,1} \end{pmatrix} \oplus \begin{pmatrix} g_{3,0} \\ g_{2,1} \end{pmatrix}\right] = \partial\begin{pmatrix} g_{3,0} \\ g_{2,1} \end{pmatrix}$$

- The moment curve constraint is generalized to rescaled moment curve

The EFTHedron in the real world

Spinning polytopes:

The same structure is found for when the external states are massless with spins: photons, gauge bosons, and gravitons.

- Lorentz-symmetry: In the form of fixing the residue basis to be Wigner $d^{\ell}_{h_1-h_2,\,h_3-h_4}(\theta) = \langle \ell, h_1 - h_2 | e^{-i\theta \mathcal{J}_y} | \ell, h_3 - h_4 \rangle$. For $(-h, h, h, -h)$ we simply have
$$d^{\ell}_{-2h,2h}(\theta) = \mathcal{J}(\ell + 4h, 0, -4h, \cos\theta)$$
- Unitarity: In the form of residue having positive coefficients
- Locality: In the form of
$$\frac{1}{s - m_a}, \quad \text{or} \int ds' \frac{1}{s - s'}$$

The EFTHedron in the real world

Consider the configuration (-2, +2, +2, -2) where we have

$$\langle 14\rangle^4[23]^4\left(\sum_{i,j} g_{i,j}z^it^j\right) \tag{8}$$

The exchanged spin begins with spin-4

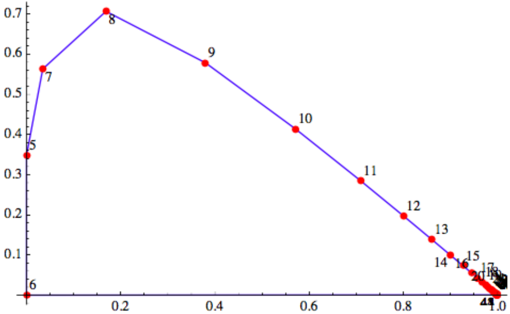
- (z^2, t^2): The space is one-dimensional, and the bound is simply

$$-\frac{11}{36} < \frac{g_{2,0}}{g_{0,2}}$$

- (z^4, z^2t^2, t^4): The critical spin is $s_c = 6$, spin-4 is inside the hull, i.e. not a vertex. The boundaries are:

$$\langle X, i, i+1\rangle > 0 \text{ for}, \quad i \geq 7, \langle X, 6, 5\rangle > 0, \quad \langle X, 5, 7\rangle > 0 \tag{9}$$

Out[175]=



(10)

The CFTHedron

Is there a similar structure?

Indeed there is!

$$Det \begin{pmatrix} C_{\Delta_1}(z_1) & C_{\Delta_2}(z_1) & \cdots & C_{\Delta_n}(z_1) \\ C_{\Delta_1}(z_2) & C_{\Delta_2}(z_2) & \cdots & C_{\Delta_n}(z_2) \\ \vdots & \vdots & \vdots & \vdots \\ C_{\Delta_1}(z_n) & C_{\Delta_2}(z_n) & \cdots & C_{\Delta_n}(z_n) \end{pmatrix} >$$

for $z_1 < z_2 < \cdots < z_n$ and $\Delta_1 < \Delta_2 < \cdots < \Delta_n$

The convex hull of the block vectors is again a cyclic polytope!

The CFTHedron

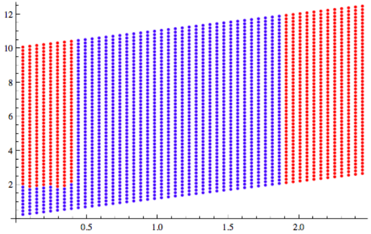
This gives us the control over the relevant boundaries

$$\mathbf{F} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{L-1} \end{pmatrix} \in \sum_{\Delta} \mathfrak{p}_{\Delta} \begin{pmatrix} c_{\Delta,0} \\ c_{\Delta,1} \\ \vdots \\ c_{\Delta,L-1} \end{pmatrix} \quad \mathfrak{p}_{\Delta} > 0$$

For example with (f_0, f_2) the relevant boundaries are

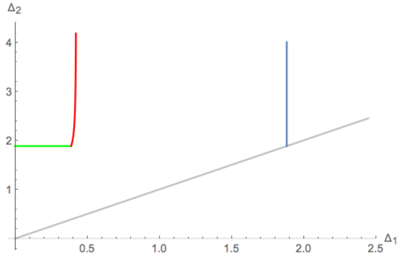
$$\begin{aligned} W_1 &= (1\Delta_1\Delta_2)\,, \\ W_2 &= (\infty 1\Delta_1)\,, \\ W_3 &= (\infty\Delta_1\Delta_2)\,, \\ W_4 &= (1\Delta_2\dot{\Delta}_2)\,, \\ W_5 &= (\infty\Delta_2\dot{\Delta}_2)\,. \end{aligned}$$

The resulting carved out space is



The CFTHedron

The boundaries of this plot is understandable in terms of walls



$$
\begin{aligned}
C_1 &: \; \vec{W}_1 = -\vec{W}_4 \,, \\
C_2 &: \; \vec{W}_1 = -\vec{W}_3 \,, \\
C_3 &: \; \vec{W}_1 = -\vec{W}_2 \,.
\end{aligned}
$$

$$
\begin{aligned}
&\vec{W} \equiv (a, b) \,, \\
&\text{where } \; \langle W \vec{F} \rangle = aF_0 + bF_2 = \vec{W} \cdot (F_0, F_2) \,.
\end{aligned}
$$

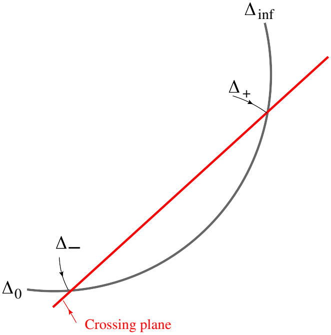
Each boundary correspond to a set of wall pointing in opposite directions

The CFTHedron

This can be simply understood as

$$\langle 1 F \Delta_i \Delta_{i+1} \rangle > 0, \quad \langle F 1 \Delta_i \infty \rangle > 0$$

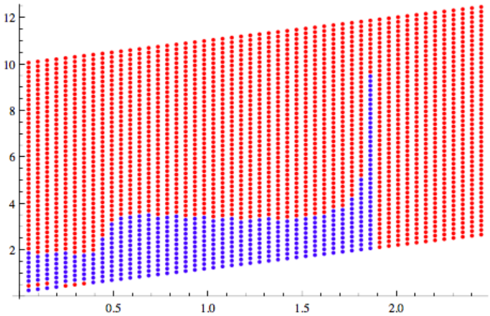
Consider the result from projecting through 1,



where $\Delta_\pm \to \langle F, 1, \Delta \rangle = 0$

The CFTHedron

Going to higher dimensions gives further constraint! (f_0, f_2, f_4) Exp, given $\Delta_\phi = 0.3$, in the space of possible lowest first two operators (Δ_1, Δ_2) are given by:



The allows us to “carve” out the space of consistent CFTs geometrically

Conclusions

The constraint of unitarity, locality and symmetries manifest itself as positive geometry on the space of consistent QFTs. The “external” data ARE positive.

This is just preliminary!

- We need to understand the generalized moment curve for $s-u$ EFThedron.
- Explore the space for mixed graviton photon scattering
- Proof of various conjectures (Weak gravity) for the land scape.
- Solving the 1D CFT geometry at higher dimensions (in external data)
- Extensions to CFT with $D > 1$