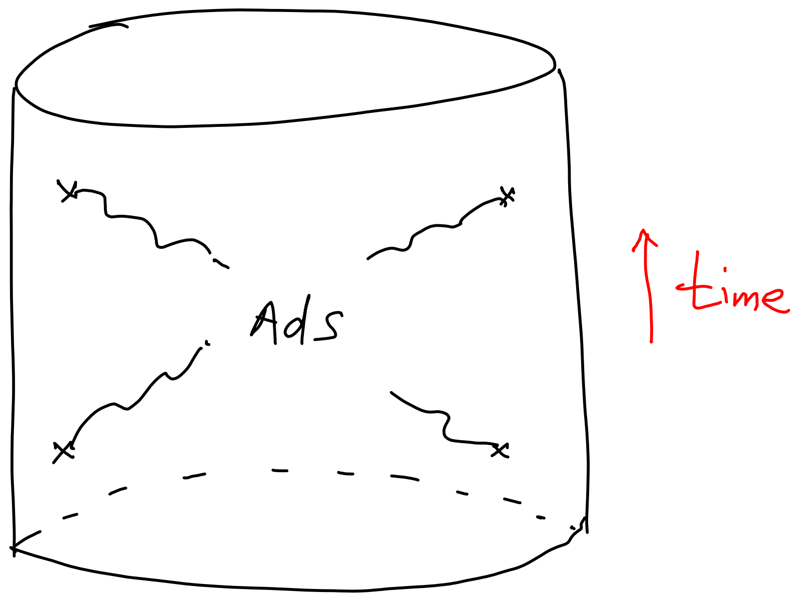
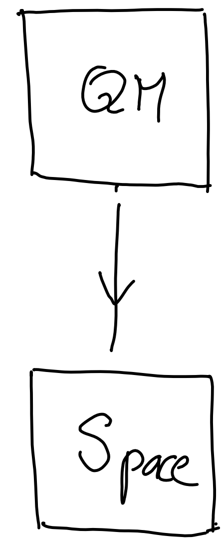
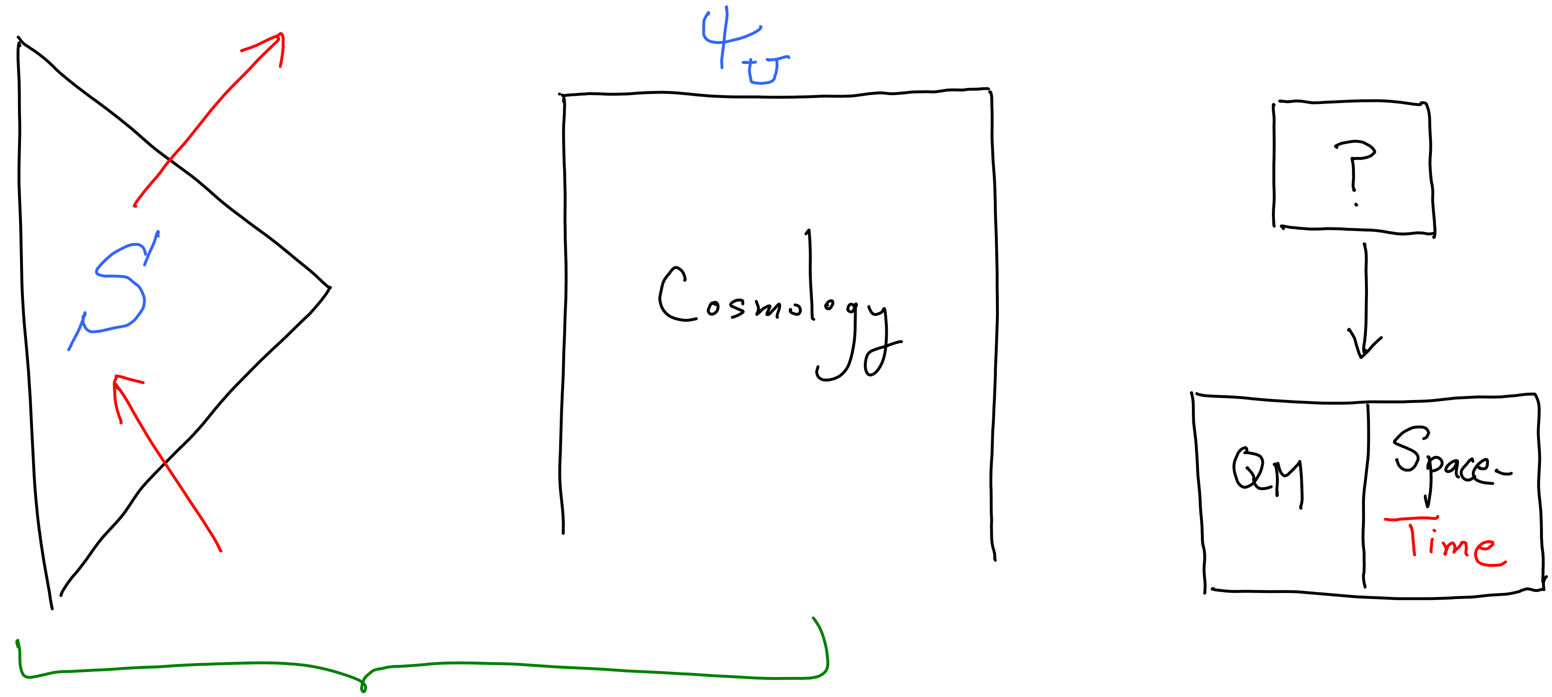
Scattering Amplitudes From

Combinatorial Geometry at Infinity


Ads
time


QM
Space



Unlike ∂AdS, no obvious notion of time + locality

The Canvas

* Physical momenta — Kin. Space ~ ∂ Minkowski
* "Twistor" variables — (Amps)
* "Celestial Sphere"

⋮

* Spatial Future — (ψ_{Univ})

WHAT IDEAS BREATHE PHYSICS-LIFE INTO THIS SPACE?

The Canvas

* Physical momenta
* "Twistor" variables
* "Celestial Sphere"

⋮

* Spatial Future

Kin. Space $\sim \partial$ Minkowski

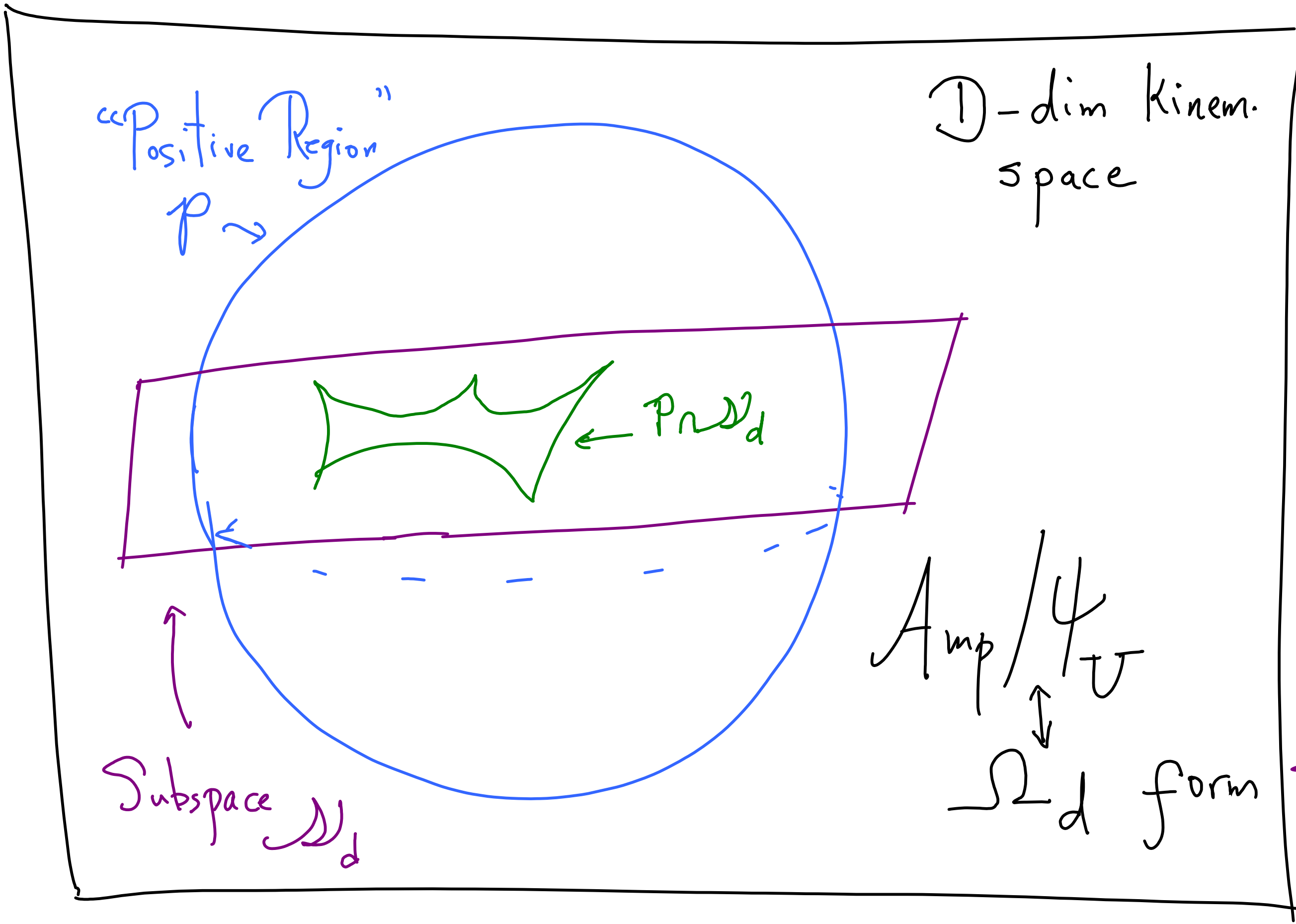
(Amps)

(Ψ_{Univ})

EMERGING PICTURE:

Combinatorics $\longleftrightarrow$ "Positive Geometries" $\longleftrightarrow$ Canonical Forms

General Picture



Ω_d fixed thusly:

$\mathcal{S}_d$ intersects $\mathcal{P}$ in a POSITIVE GEOMETRY

Ω_d Pulls Back to CANONICAL FORM

Amplituhedron: $dZ_a^I \longleftrightarrow \eta_a^I$; $d\lambda_a^\alpha, d\tilde{\lambda}_a^{\dot\alpha} \longleftrightarrow \eta_a^\alpha, \tilde{\eta}_a^{\dot\alpha}$

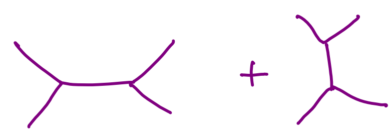
Geometrizes Helicity

Associahedron: $ds \wedge ds \ldots \longleftrightarrow f\,f$

Geometrizes Color

Amplituhedron: $[12345] = \dfrac{\left(\langle 1234\rangle \eta_5 + \text{cyclic}\right)^4}{\langle 1234\rangle \cdots \langle 5123\rangle}$

$$\Longrightarrow \frac{\left(\langle 1234\rangle dZ_5 + \text{cyclic}\right)^4}{\langle 1234\rangle \cdots \langle 5123\rangle} = d\log \frac{\langle 2345\rangle}{\langle 1234\rangle} \cdots d\log \frac{\langle 5123\rangle}{\langle 1234\rangle}.$$

Associahedron:  $\quad \dfrac{1}{s} + \dfrac{1}{t} \Rightarrow d\log\left(s/t\right)$

Remarkable Projectivity: $(s,t) \to \Lambda(s,t)\,(s,t)$

Color — Kinematics

Planar $\mathcal{N}=4$ SYM Amps

ϕ^3 theory, Pions, Gluons...

4_σ [φ^N theories]

Amplituhedra

(Generalized) Associahedra

Cosmological Polytopes

Effective Field Theory

Conformal Field Theory

"EFThedron"
Hidden Positive Geometry
of Causality + Unitarity

"CFThedra"
Positive Geometry of Conformal Bootstrap.

Outline

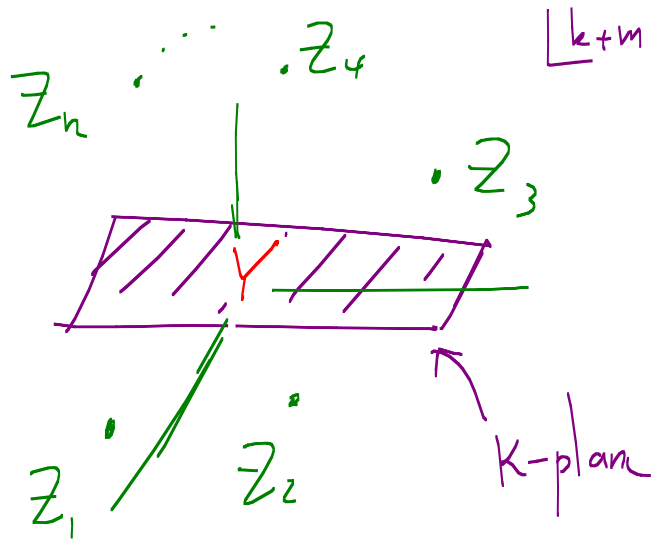
① Theory + Practice of Amplituhedron in Kinematic (Mom-Twistor) space + natural speculation for Amplitudehedron

② Amplitudes from "UV regulated" integration of canonical forms + the Kinematic space of the World-sheet; Associahedra avatar of the "UV completion".
+ string amps from

③ $\Psi_U \longrightarrow$ Scattering Amplitude : emergence of Lorentz-Invariance + Unitarity from "Amplitude-facet" of Cosmological Polytopes

The Kinematic-space Amplituhedron

Are You In or Out?



When is Y in the amplituhedron?

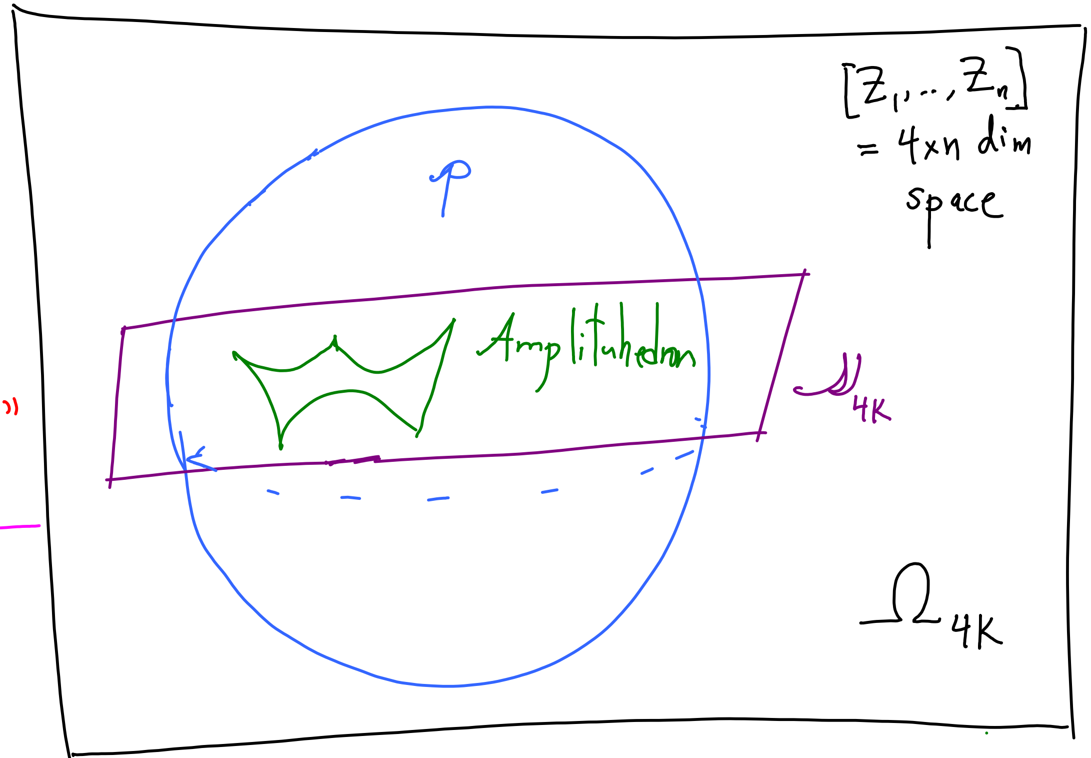
Project through $Y \to m$ dim. picture. What does this picture look like?

{Physics $m=4$: This is precisely conf. of momentum twistors}

$\mathcal{P}$: Config. of $\{Z_1, \ldots, Z_n\}$ has fixed "binary code"

$\Rightarrow$ physical poles > 0

$+$ maximal "winding #"



In full:

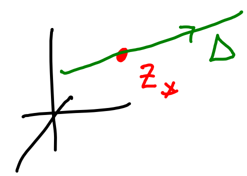
$\langle i\, i{+}1\, j\, j{+}1 \rangle > 0$

$\{\langle 1234 \rangle, \ldots, \langle 123n \rangle\}$ has k sgn flips

$\langle AB_\alpha\, i\, i{+}1 \rangle > 0$, $\langle AB_\alpha AB_\beta \rangle > 0$

$\{\langle AB_\alpha 12 \rangle, \ldots, \langle AB_\alpha 1n \rangle\}$ has $k+2$ sgn flips

$\mathcal{S}_{4K}$: Affine subspace



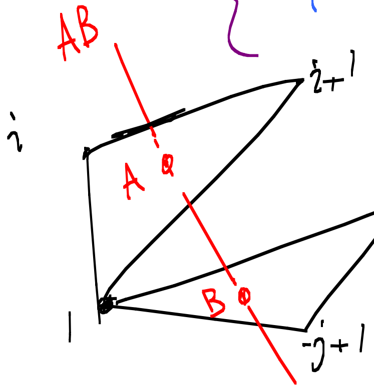
$$Z^I_a = Z^I_{*a} + y^I_\alpha \Delta^\alpha_a$$

$$\begin{pmatrix} Z_* \\ \hline \Delta \end{pmatrix} \cap G_+(4+K, n)$$

Example: 1-loop MHV

$*\ \{\langle AB|2\rangle, \dots, \langle AB|n\rangle\}$ has $k+2=2$ sgn flips

$$\begin{array}{cccccccccccc} & & & i & i+1 & & & j & j+1 & & & \\ \{+ & + & \cdots & + & - & \cdots & - & + & \cdots & + \} \end{array}$$



$$A = Z_1 + \alpha_i Z_i + \alpha_{i+1} Z_{i+1} \qquad \alpha_{i,i+1,j,j+1} > 0$$
$$B = -Z_1 + \alpha_j Z_j + \alpha_{j+1} Z_{j+1}$$

$$\Rightarrow \Omega = \sum_{i,j} d\log\alpha_i \, d\log\alpha_{i+1} \, d\log\alpha_j \, d\log\alpha_{j+1}$$

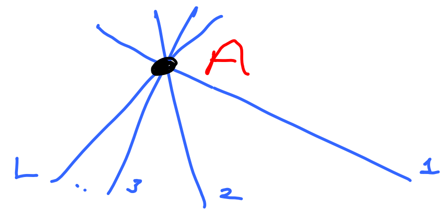
$$= \sum_{i,j} d\log\frac{\langle AB\,1i\rangle}{\langle AB\,i\,i+1\rangle} \, d\log\frac{\langle AB\,1\,i+1\rangle}{\langle AB\,i\,i+1\rangle} \, d\log\frac{\langle AB\,1j\rangle}{\langle AB\,j\,j+1\rangle} \, d\log\frac{\langle AB\,1\,j+1\rangle}{\langle AB\,j\,j+1\rangle}$$

Infinite Class of Cuts

w/ Langer
Trnka
Yelleshpur-Srikant

* "Maximal Intersection"

  $\langle AB_\alpha \, AB_\beta \rangle \longrightarrow 0$



4L full form
$\downarrow$
$(2L+3)$ on cut

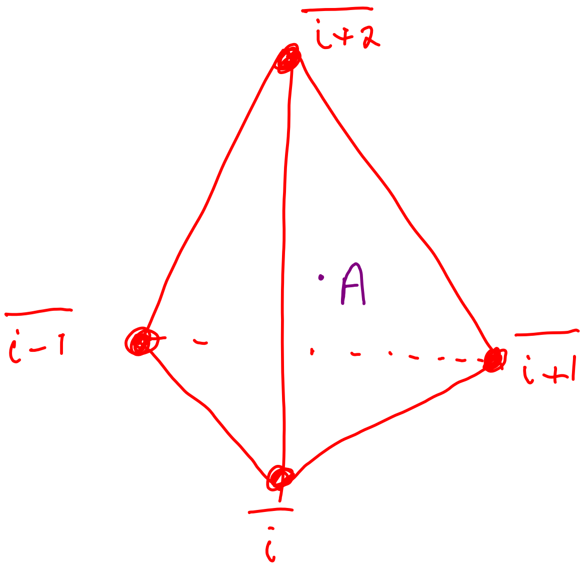
* Deep in "interior" – most complicated local integrands contribute.

* Momentum Twistor Amplituhedron: $\langle (AB)_\alpha \, i j \rangle > 0$!

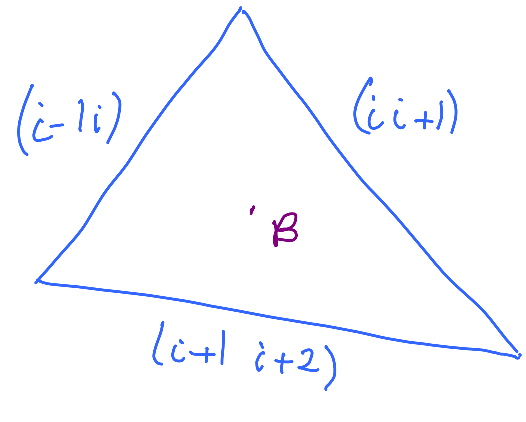
Remarkable prediction: solve "1-loop" (AB) geometry

$\Longrightarrow$ cut to ∞ loop order!

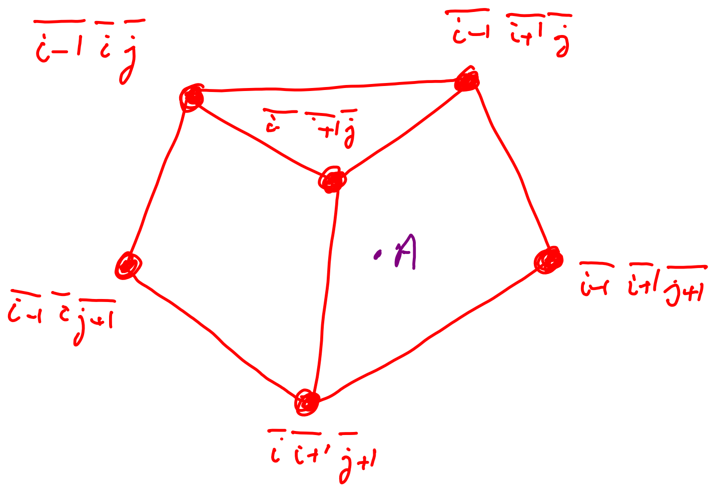
A - Geometry



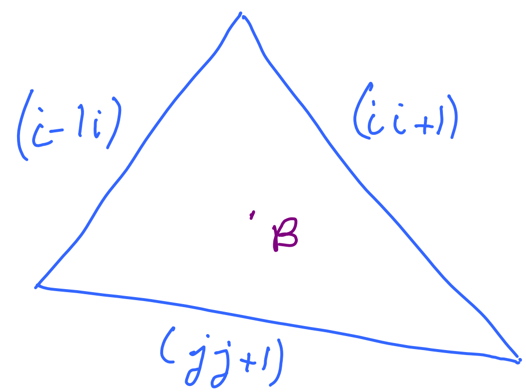
AB - Geometry



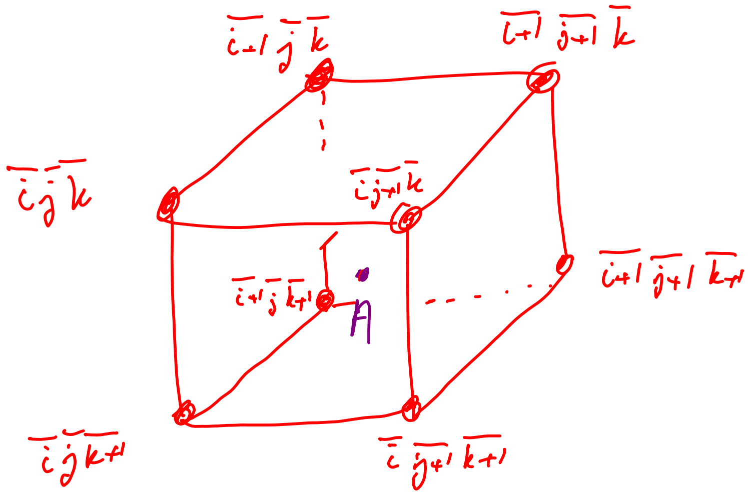
A - Geometry



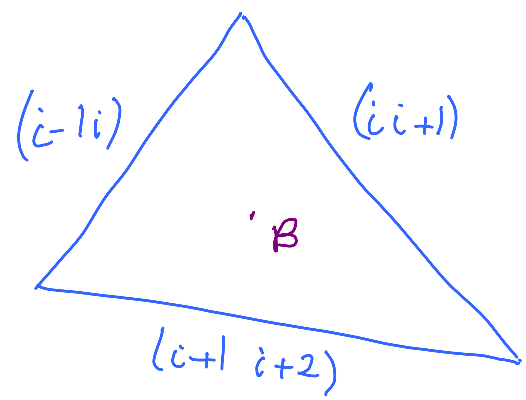
AB - Geometry



A - Geometry



AB - Geometry



Final Result

$$\underbrace{\frac{1}{2}\sum_{i<j<k}\frac{\langle A(ii{+}1)\cap(Ajj{+}1)kk{+}1\rangle\langle i{-}1ii{+}1i{+}2\rangle\langle j{-}1jj{+}1j{+}2\rangle\langle k{-}1kk{+}1k{+}2\rangle}{\langle Ai{-}1ii{+}1\rangle\langle Aii{+}1i{+}2\rangle\langle Aj{-}1jj{+}1\rangle\langle Ajj{+}1j{+}2\rangle\langle Ak{-}1kk{+}1\rangle\langle Akk{+}1k{+}2\rangle}}_{A \text{ geometry}}\ \underbrace{\prod_{\alpha}\frac{\langle A(ii{+}1)\cap(Ajj{+}1)kk{+}1\rangle}{\langle AB_\alpha ii{+}1\rangle\langle AB_\alpha jj{+}1\rangle\langle AB_\alpha kk{+}1\rangle}}_{\text{factorized } (AB)_\alpha \text{ geom.}}$$

All these poles spurious!

First non-trivial $\mathcal{N}=4$ computation for $n \to \infty$, $L \to \infty$

New "Kinematic Space" defn of Amplituhedron critical for this

Natural Conjecture for Bruhat Location for all k, n:

$$m = 4 \left[Z_1, \dots, Z_n \right]$$

$$\mathrm{Conf}_{\mathrm{flip}\ k}\left[m = 4, n\right]$$

$\langle i\, i{+}1\ j\, j{+}1 \rangle > 0$

k sgn flips

Integrand Form dlog sing $\longleftrightarrow$ Boundaries of $C_K[n]$

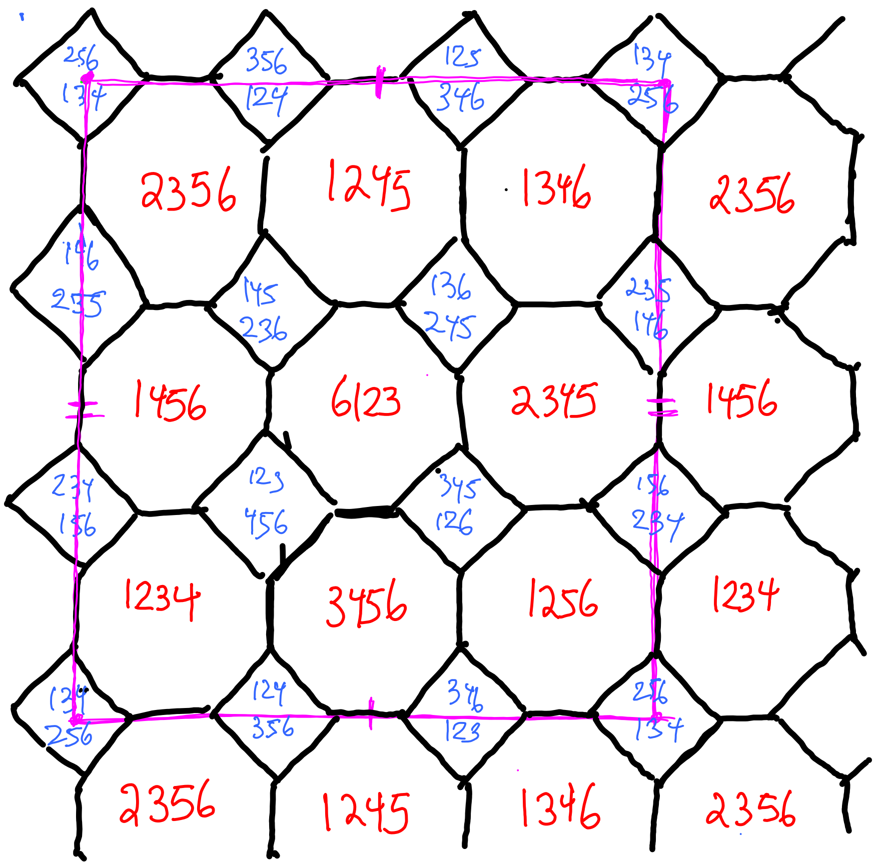
Sing of Amp Itself Branch Points $\longleftrightarrow$ Boundaries of $C_K[n] / \text{Little Grp}$

$\{K = 0$ "Cluster co-ordinates in symbol"$\}$

Boundary Structure for 6pt NMHV

w/ He
Spradlin

A
Toms!



$G(4,6) = G(2,6)$

Flip $= 1 \Rightarrow$

(odd even) > 0

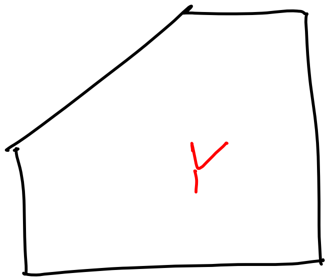
$1234 = $ 1234 56

$\begin{matrix}123\\456\end{matrix} = $ 123 456

Associahedra + Kinematic-space

Avatar of the Worldsheet

Integrating Canonical Forms $\to$ Canonical Functions

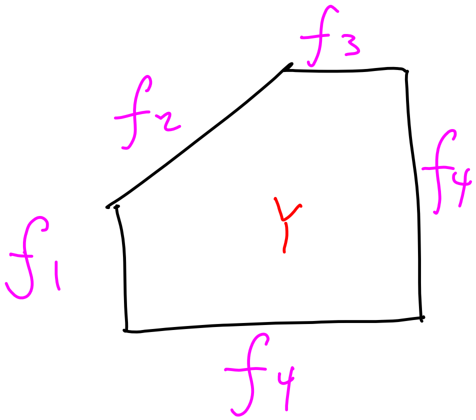


$$\alpha^{\mathcal{P}} \int_{\mathcal{P}} \Omega_{\mathcal{P}}$$

DIVERGES

Integrating Canonical Forms → Canonical Functions

w/ Bai
He
Thomas



$$\alpha'^{D}\int_{P} \Omega_P \prod_a \left(\frac{W_a\cdot Y}{W_\infty\cdot Y}\right)^{\alpha' f_a}$$

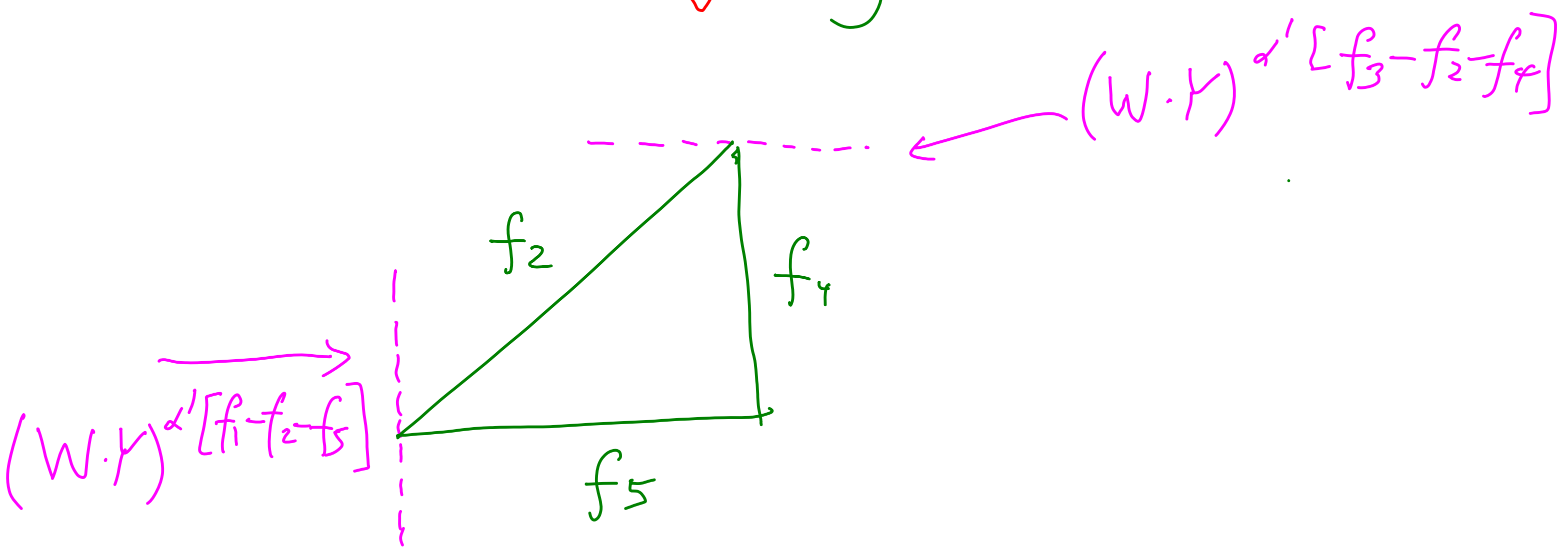
Canonical Regulator

$$\alpha' \to 0 : \quad \sum_i \frac{1}{f_i f_{i+1}}$$

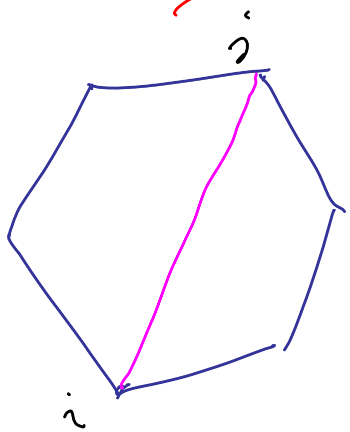
"Canonical Function"

Canonical Function only depends on shape:

$\Downarrow$ Degenerate



Kinematical Associahedron



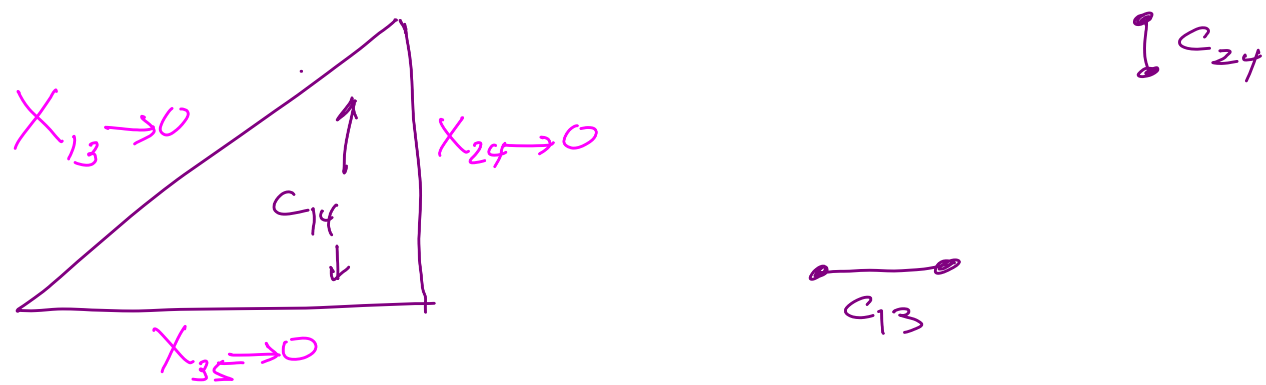
$$X_{ij} = (p_i + \cdots + p_{j-1})^2$$

$$X_{ij} \geq 0$$

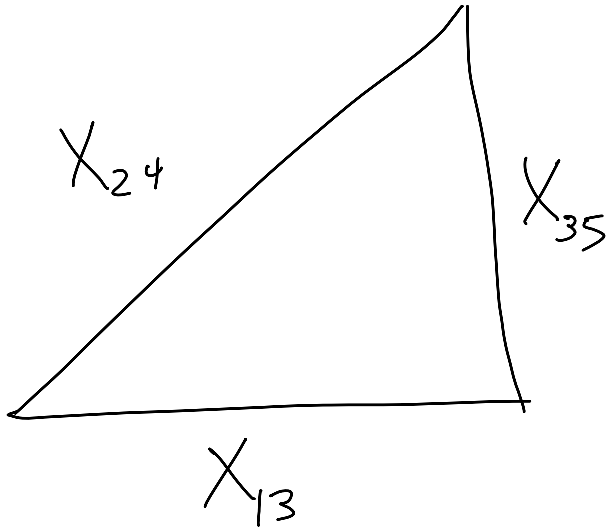
$$X_{ij} + X_{i+1\,j+1} - X_{i\,j+1} - X_{i+1\,j} \equiv c_{ij} \geq 0$$

$$i < j \leq n-1 \quad [\text{leave out } n]$$

... turn off all but one c_{ij}'s, space drops dimension, EXCEPT for C_{1n-1} : SIMPLEX



Full Associahedron is Minkowski Sum of all these!



$$X_{13} + X_{24} + X_{34} = c_{14}$$

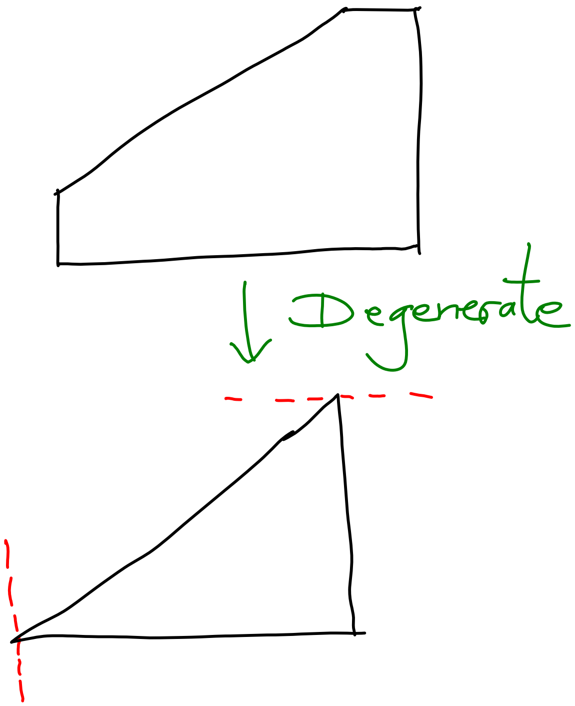
$$\Downarrow$$

$$X_{i\,i+2} = c_{1n-1}\left(\sigma_{i+1} - \sigma_i\right)$$

$$0 = \sigma_1 < \sigma_2 \cdots < \sigma_{n-1} < 1$$

Kin Space Avatar of the Gauge-Fixed WS!

Canonical Function = Amplitude



$$\alpha'^2 \int \Omega \, \left| W_{a\cdot Y} \right|^{\alpha' x_{ij}} = Amp$$

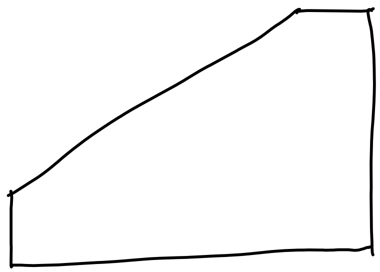
$$\downarrow$$

$$\alpha'^2 \int PT \, (\sigma_i - \sigma_j)^{\alpha' s_{ij}} \, !$$

WS, PT + KN, from Kin Space

[$Z[\alpha|\alpha]$ theory Amps @ finite α']

Q: What's wrong with



$\alpha'^2 \int \Omega \, |W_{a\text{-}Y}|^{\alpha' x_{ij}}$ @ finite α'?

A: "Canonical Function" doesn't map to itself on poles as $x_{ij} \to 0$! Physics: failure to factorize for finite α'

Q: Which polytopes P have canonical functions that can be consistently "α'" deformed: "UV completed"?

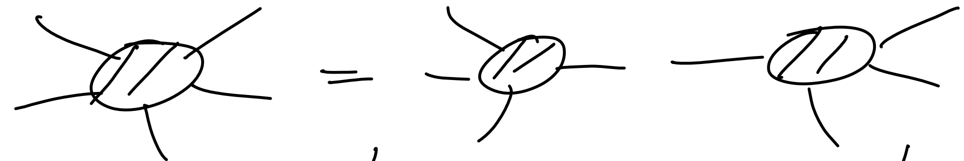
Q: Which Polytopes can be given a "curvy, binary" description of the form

$$F_a = 1 - \prod_{\substack{\alpha \text{ not} \\ \text{touching } a}} F_\alpha$$

... it appears the answer is

PRODUCTS of ASSOCIAHEDRA!

Remarkable: ∂



emerges from all possible polytopes from consistency of α' deformation!

[Natural conjecture: allow general powers $\longleftrightarrow$ discover polytopes of cluster algebras...]

$\Psi_{\mathcal{U}}$, Amplitudes

and

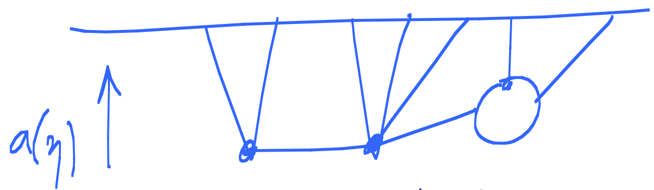
"Amplitude Facet" of Cosmological Polytopes

What are Rules for Ψ_{Universe}?

(I) How is consistent "unitary time evolution physics" encoded in late-time correlators?

(II) Is there an autonomous object that satisfies these rules without bulk time evolution?

Toy Model for Cosm. WF of U

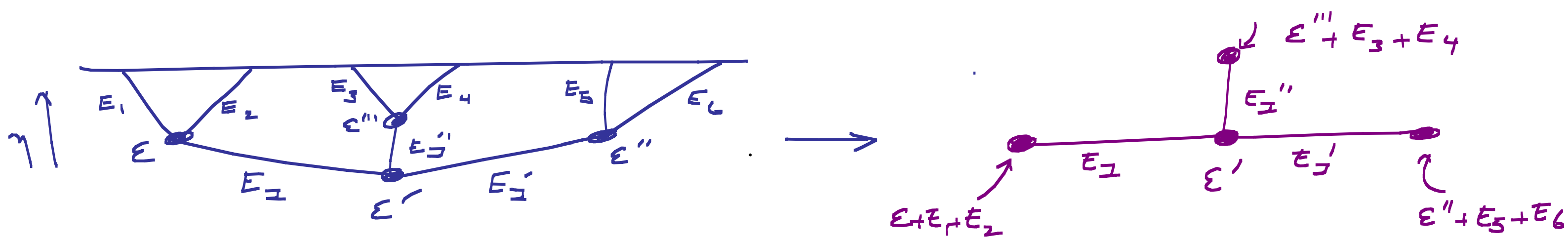


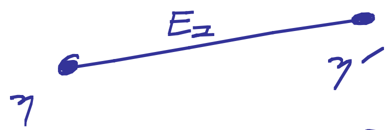
conf. coupled scalar, non-conf. polynomial interactions

$\Psi[\phi]$

$$S = \int d\eta \, (\partial\phi)^2 + g_3(\eta)\,\phi^3 + g_4(\eta)\,\phi^4 + \ldots$$

$$g_i(\eta) = \int d\varepsilon \, e^{i\varepsilon\eta} \, \tilde{g}(\varepsilon)$$





$$G_B[\eta,\eta'] = \frac{1}{E_I}\Big[\underbrace{e^{iE_I(\eta'-\eta)}\,\theta(\eta-\eta') + (\eta\leftrightarrow\eta')}_{\text{Feynman Prop}}\ \underbrace{-\,e^{iE_I(\eta+\eta')}}_{\text{BC @ }\eta=0\text{ to compute }\psi}\Big]$$

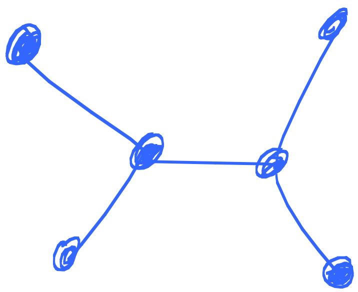
3 terms

$$F_G[x_i; y_I] = \int \prod_v d\eta_v\, e^{i\eta_v x_v} \prod_{\text{edges}} G_B[\eta_i, \eta_j, y_{ij}] ,$$

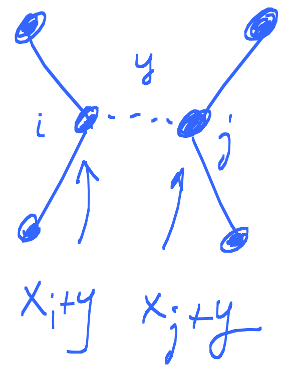
Time Evol

$O(3^E)$ terms

* When couplings are η-independent, there is a non-η dependent way of computing Ψ: just solve the Sch. Eqn! Generalizes to "Old-fashioned pert theory":



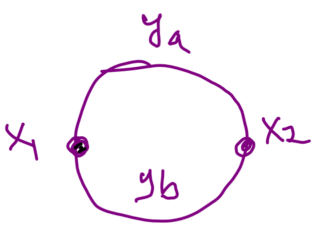
$$= \frac{1}{\left(\sum X_i\right)} \sum_{\text{cuts}}$$



$$\frac{1}{x},$$

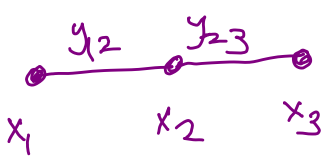
$$\frac{1}{(x_1+x_2)(x_1+y)(y+x_2)},$$

[3 terms]



$$\frac{2(x_1+x_2+y_a+y_b)}{(x_1+x_2)(x_1+y_a+y_b)(x_2+y_a+y_b)(x_1+x_2+2y_a)(x_1+x_2+2y_b)},$$

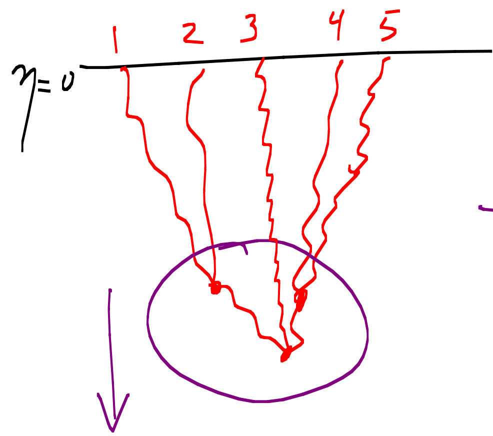
[7 terms]



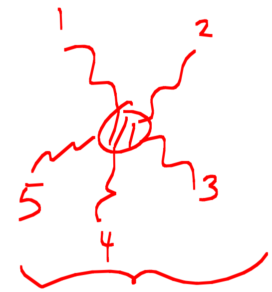
$$\frac{x_1+2x_2+x_3+y_{12}+y_{23}}{(x_1+x_2+x_3)(x_1+y_{12})(y_{12}+x_2+y_{23})(y_{23}+x_3)(x_1+x_2+y_{23})(y_{12}+x_2+x_3)}$$

[9 terms]

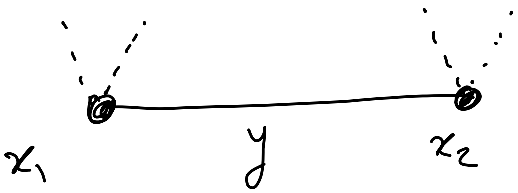
Ψ_U Contains Scattering Amplitudes!



$$\left(\frac{1}{\underbrace{E_1+\cdots+E_5}_{\to\, 0}}\right)\times$$

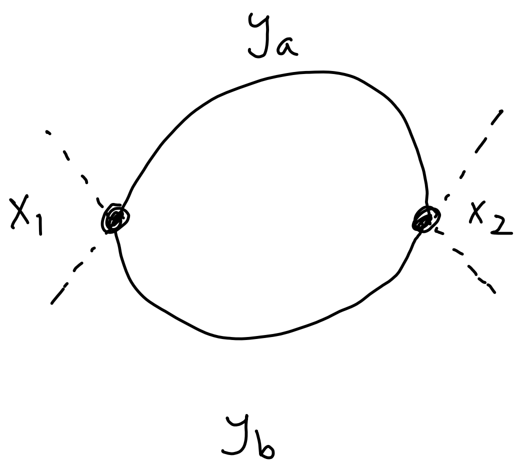


Scatt. Ampl.



$$\frac{1}{x_1 + x_2} \to 0$$

$$\underbrace{\frac{1}{x_1+y}\,\frac{1}{x_2+y}}_{\Downarrow} \qquad \frac{1}{x^2-y^2} = \frac{1}{s} \; !$$



$$\xrightarrow{x_1+x_2\to 0} \frac{1}{(y_a+y_b)^2 - x_1^2}\left(\frac{1}{2y_a} + \frac{1}{2y_b}\right)$$

= Feynman Tree Thm Loop Integrand!

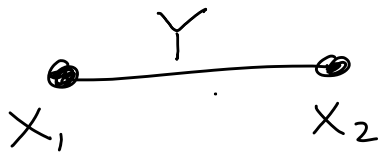
(l_0 integral done w/ $i\epsilon$)

* In Flat Space: the rational functions give $\Psi_{vac}[\varphi]$

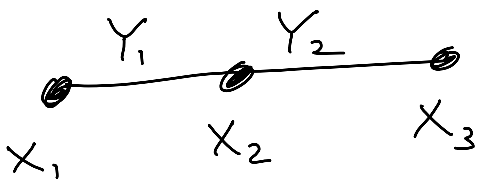
* In dS, further int. gives $\Psi_{dS}[\varphi]$.

(In AdS, ∂ correlators)

* We get "Cosmological POLYLOGS" even @ tree level!



$$S: \quad \frac{X_1+Y}{X_1-Y} \otimes \frac{X_2+Y}{X_2+X_1} \quad + \quad 1 \longleftrightarrow 2$$



$S:$

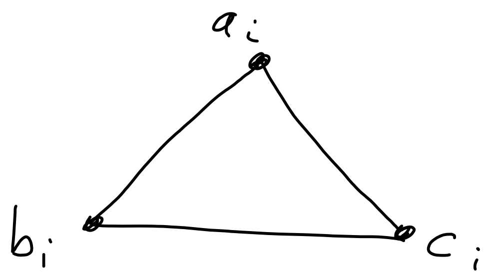
{{-1, X1 - Y1, X2 + Y1 + Y2, X2 + X3 + Y1}, {1, X1 - Y1, X1 + X2 + Y2,
X1 + X2 + X3}, {1, X1 - Y1, X2 + Y1 - Y2, X2 + X3 + Y1}, {-1,
X1 - Y1, X1 + X2 - Y2, X1 + X2 + X3}, {1, X1 - Y1, X1 + X2 - Y2,
X3 + Y2}, {-1, X1 - Y1, X2 + Y1 - Y2, X3 + Y2}, {1, X1 - Y1,
X2 + Y1 + Y2, X3 + Y2}, {-1, X1 - Y1, X1 + X2 + Y2, X3 + Y2}, {1,
X1 + Y1, X1 + X2 - Y2, X1 + X2 + X3}, {-1, X1 + Y1, X1 + X2 + Y2,
X1 + X2 + X3}, {-1, X1 + Y1, X1 + X2 - Y2, X3 + Y2}, {1, X1 + Y1,
X1 + X2 + Y2, X3 + Y2}, {-1, X1 + Y1, X2 + Y1 - Y2,
X2 + X3 + Y1}, {1, X1 + Y1, X2 + Y1 + Y2, X2 + X3 + Y1}, {1,
X1 + Y1, X2 + Y1 - Y2, X3 + Y2}, {-1, X1 + Y1, X2 + Y1 + Y2,
X3 + Y2}, {1, X1 - Y1, X3 - Y2, X2 + X3 + Y1}, {-1, X1 - Y1,
X3 - Y2, X1 + X2 + X3}, {1, X1 - Y1, X3 - Y2, X1 + X2 + Y2}, {-1,
X1 - Y1, X3 - Y2, X2 + Y1 + Y2}, {-1, X1 - Y1, X3 + Y2,

+ . . .

Already Pretty Complicated....

Cosmological Polytopes

w/ Benincasa
Postnikar

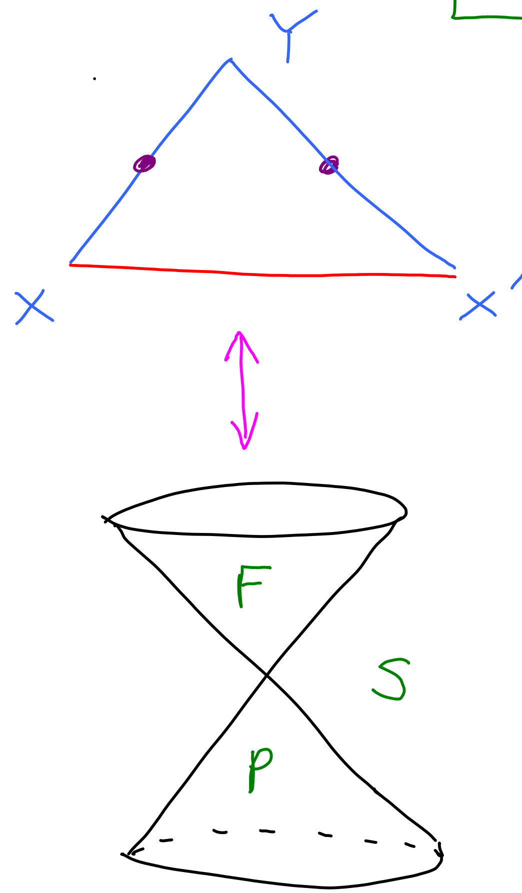


Collection of Triangles
Allowed to intersect
on 2 of 3 sides

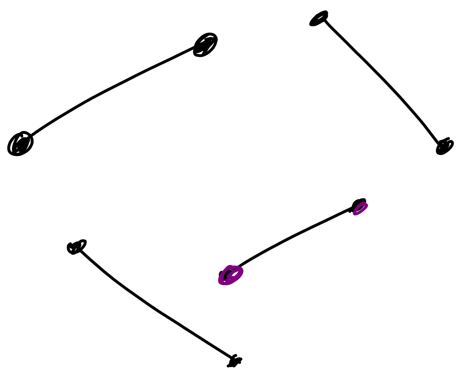
$$a_i + b_i = a_j + b_j$$

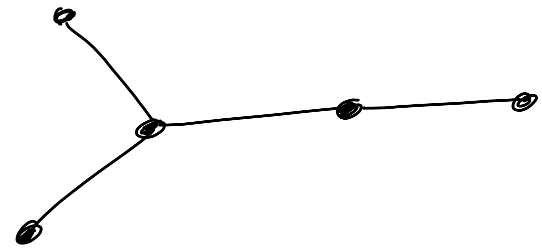
and/or $a_i + c_i = a_j + c_j$

NOT $b_i + c_i = b_j + c_j$

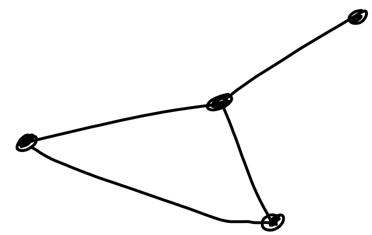


Polytopes Associated w/ Graphs

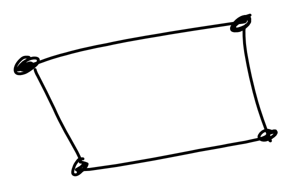


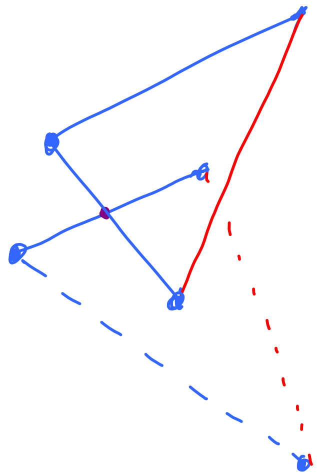


or

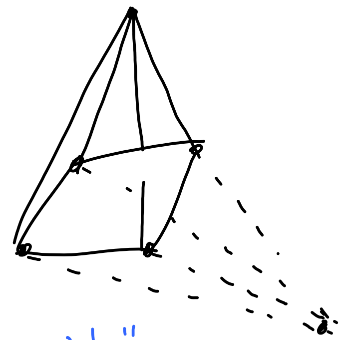


or

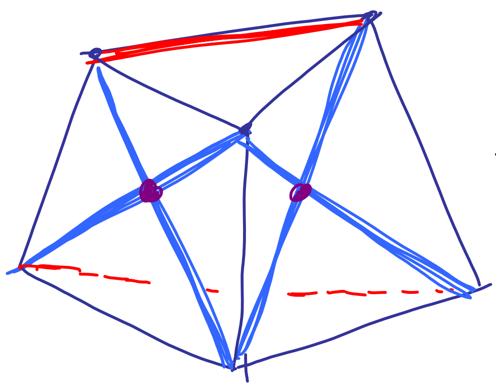




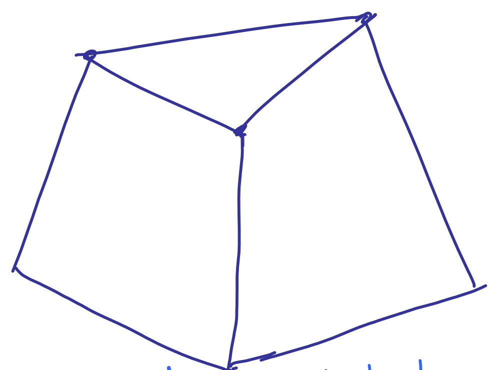
→



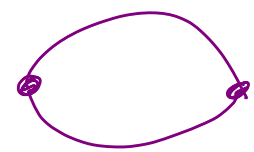
"Double" square pyramid

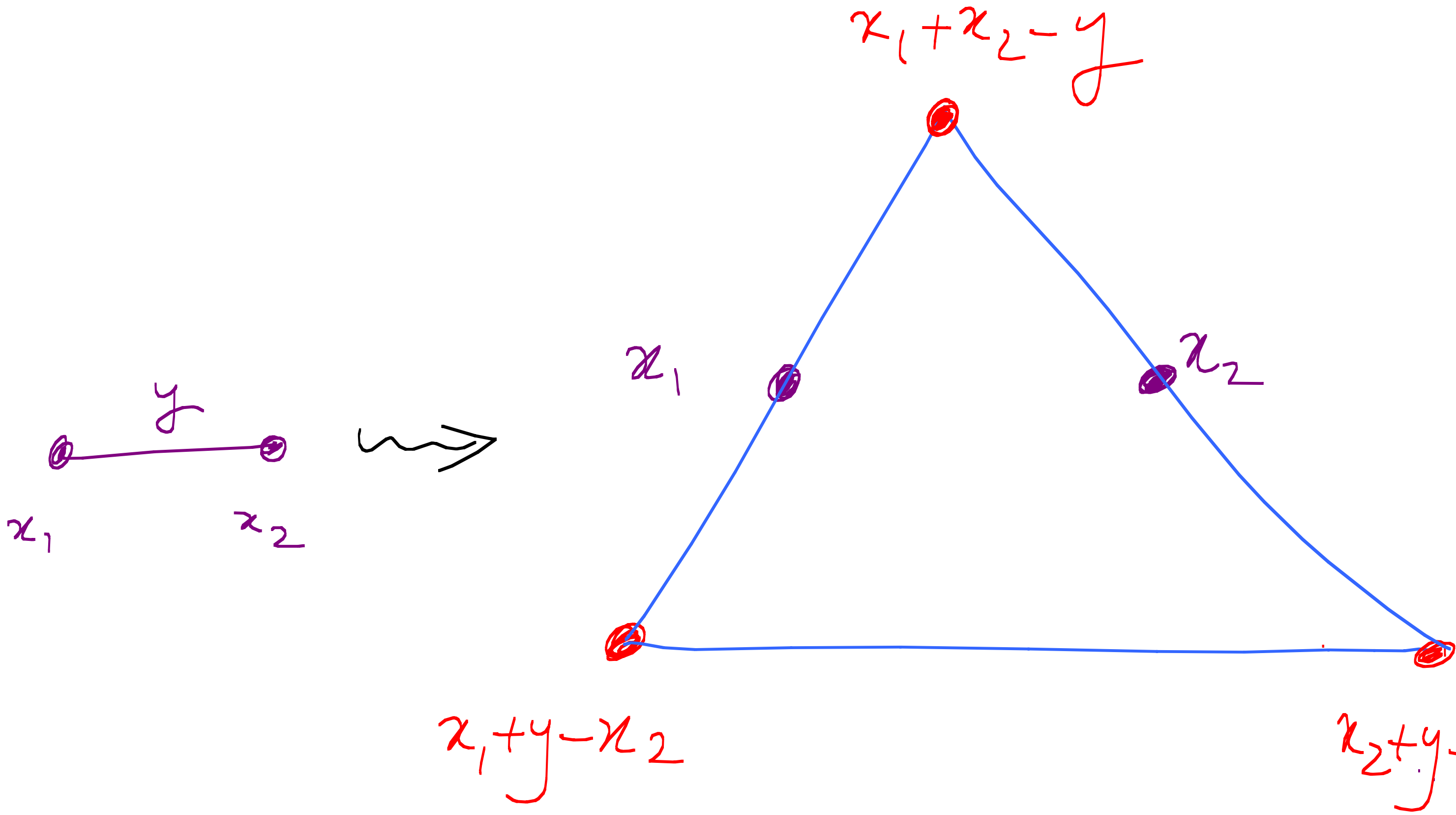


→



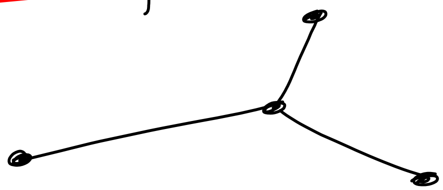
Truncated Tetrahedron





y
x_1
x_2
x_1+x_2-y
x_1
x_2
x_1+y-x_2
x_2+y-x_1

Wavefunction = Canonical Form!



$$\mathcal{X} = \sum x\, X + y\, Y$$

$$\Omega\left[\mathcal{X},\ \text{Cosm. Polytope}[G]\right] = \left(\prod dx\, dy\right) \times \psi_G[x, y]$$

* Natural triangulations give → Time Int. Rep
→ Old-fash. pert. th.

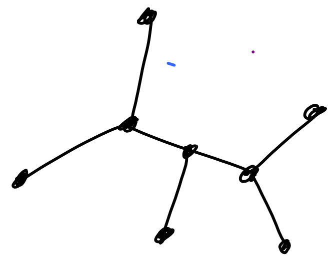
* That ψ_G satisfies Sch. Eqn → Emergent from Polytope!

* [New triangulations → more efficient expressions for ψ_G]

Scattering Amplitude Facet

w/ Benincasa

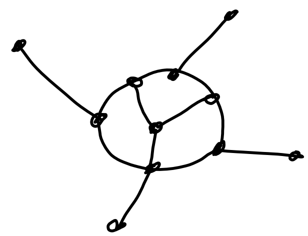
* Tree level :



Face where $\sum_a x_a \to 0$ is a SIMPLEX

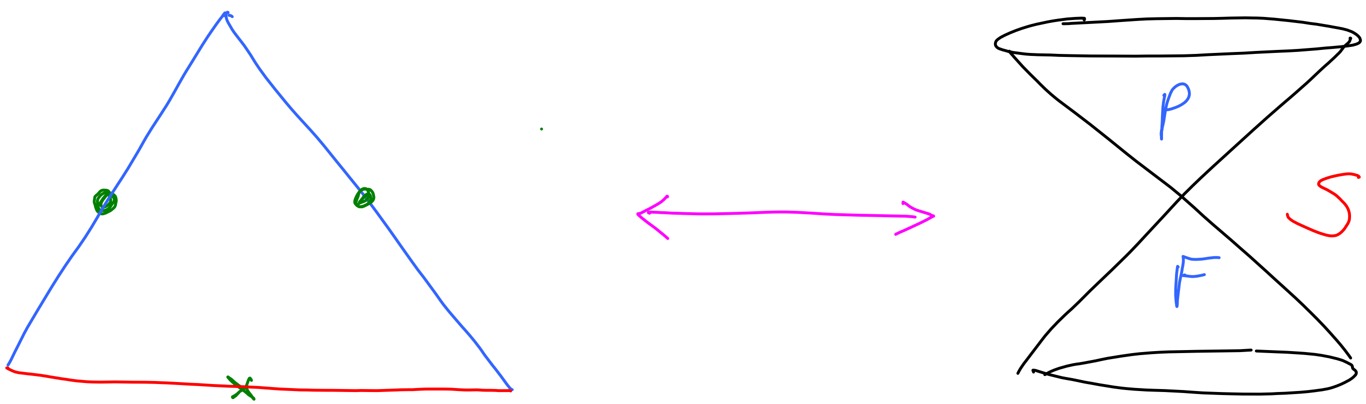
$$\left(\prod_{\partial E} \frac{1}{\text{Energies}}\right) \to \prod_E \frac{1}{S_E}$$

* Loop level :



"Feynman Tree Thm Polytope"

Not a simplex... but,
All triangulations w/ polytope vertices = All ways of doing lo integrals!



Lorentz-Invariance
Unitarity

} On "Amplitude Face", emerge from this combinatorial geometry

Symbol of Cosmological Polylog $=$ Record of Projective Paths in Cosm. Polytope!

$\{V_1, V_2, \ldots, V_{V+E}\}$ s.t.

$$V_1 \longrightarrow V_2 \xrightarrow[\pi_{V_1}]{} V_3 \xrightarrow[\pi_{V_2}]{}$$

All connected by edges

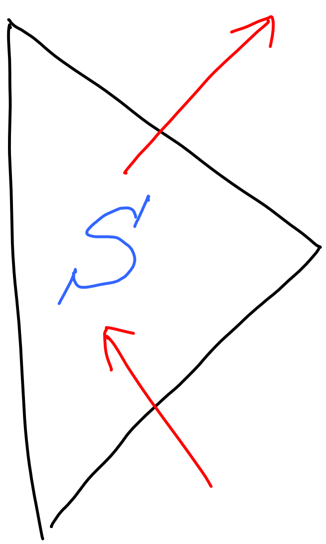
$$\{\tilde{W}_1, \ldots, \tilde{W}_{V+E} ; V_1, \ldots, V_{V+E}\}$$

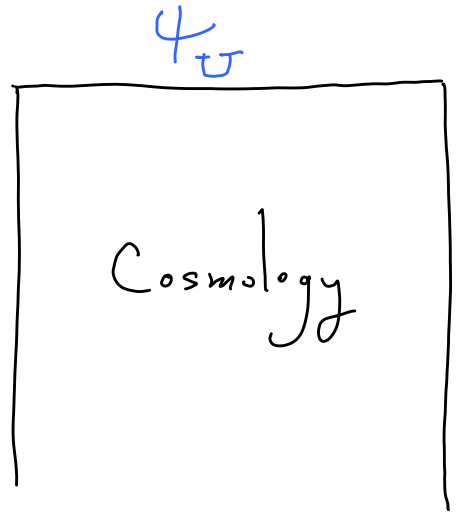
$$= \text{Matrix}[\tilde{W}_I \cdot V_J]$$

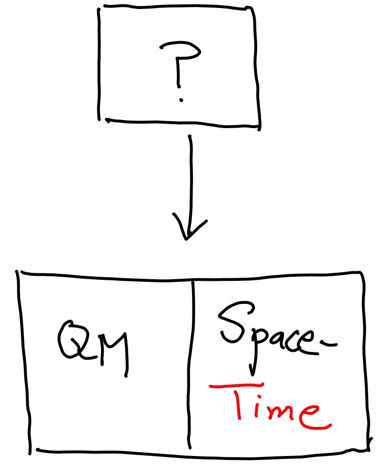
$$S = \sum_{\substack{\{\tilde{X}_{i_1}, \ldots, \tilde{X}_{i_V}\} \\ \{V_1, \ldots, V_{V+E}\}}} (\pm) \prod \begin{array}{c} \tilde{y}_1 \cdots \ \tilde{y}_E \ \tilde{x}_{i_1} \cdots \ \tilde{x}_{i_V} \\ \begin{pmatrix} & & & \\ & & \cdots & \\ & & & \\ & & & \end{pmatrix} \end{array} \begin{array}{c} V_1 \\ \vdots \\ V_{V+E} \end{array}$$

symbol $=$ cons. nested minors

w/ Yuan







Much more progress in past 10 years than I imagined was possible... can't wait for next 10 years!