Positive Geometries, Canonical Forms and Scattering Amplitudes

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Motivations

Search for "holographic" S-matrix theory: fascinating geometric structures underlying scattering amplitudes, in some auxiliary space

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What questions to ask, directly in the "kinematic space", to generate local, unitary dynamics? Avatar of these geometries? [Nima's talk]

Amplitudes as Forms

Scattering amps as differential forms on kinematic space $\rightarrow$ a new picture for amplituhedron [Arkani-Hamed, Thomas, Trnka] (see [Thomasz' talk])

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$\eta_i \to dZ_i \implies \Omega_n^{(4k)}$ for N^{k}MHV tree; similarly $\Omega^{(2n-4)}(\lambda, \tilde{\lambda})$.

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This talk: identical structure for wide variety of theories in any dim:

- Bi-adjoint ϕ^3 from kinematic and worldsheet associahedra
- YM/NLSM: "geometrizing" color & its duality to kinematics
- Real and complex integrals, double-copy & projectivity

Kinematic Associahedron

$\mathcal{A}_n := \Delta_n \ \cap \ H_n$. Δ_n: all planar variables $s_{i,i+1,\cdots,j} \geq 0$ (top-dim)
H_n: $-s_{i\,j} = c_{i,j}$ (positive constants) for all non-adjacent $1 \leq i, j < n$

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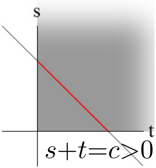

$$e.g.\ \mathcal{A}_4 = \{s > 0, t > 0\} \cap \{-u = \text{const} > 0\}$$

$$\mathcal{A}_5 = \{s_{12}, \cdots, s_{51} > 0\} \cap \{-s_{13}, -s_{14}, -s_{24} = \text{const} > 0\}$$

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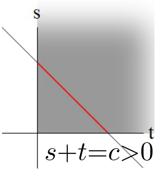

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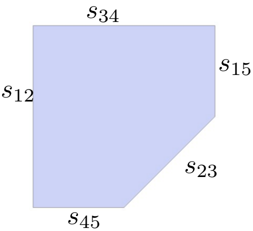
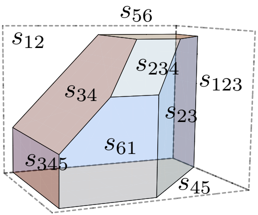


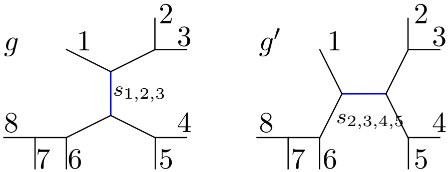


Planar Scattering Forms

The planar scattering form is a sum of $d\log$'s for planar cubic graphs:

$$\Omega_n^{(n-3)} := \sum_{\text{planar } g} \text{sgn}(g) \bigwedge^{n-3} d\log s_{i,i+1,\cdots,j}\,, \qquad e.g.\ \Omega_4 = \frac{ds}{s} - \frac{dt}{t}$$

with $\text{sgn}(g) = -\text{sgn}(g')$ for any g, g' related by a *mutation*



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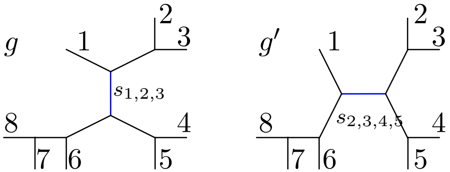

Sign-flip rule fixed by projectivity: invariant under *local* $\text{GL}(1)$ transf. $s_{i,\cdots,j} \to \Lambda(s) s_{i,\cdots,j}$ (only depends on ratios of variables). Remarkable property not true for each (or any subsets of) Feynman diagrams!

Canonical Form of $\mathcal{A}_n$

Unique form of positive geometry= "volume" of the dual: $\Omega(A)$ only has $d\log$ singularities o boundaries ∂A with $\mathrm{Res} = \Omega(\partial A)$

For simple polytopes: $\sum_v \pm \wedge\, d\log F$ for faces $F = 0$ adjacent to v

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$$e.g. \qquad \Omega(\mathcal{A}_4) = \Omega_4^{(1)}|_{H_4} = \left(\frac{ds}{s} - \frac{dt}{t}\right)|_{-u=c>0} = \left(\frac{1}{s} + \frac{1}{t}\right) ds$$

$$\Omega(\mathcal{A}_5) = \Omega_5^{(2)}|_{H_5} = \left(\frac{1}{s_{12}s_{34}} + \cdots + \frac{1}{s_{51}s_{23}}\right) ds_{12} \wedge ds_{34}$$

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Similarly for $m(\alpha|\beta)$: "volume" of degenerate $\mathcal{A}_n$ (faces at infinity). Also given by intersection of $\mathcal{A}_n^*$'s in dual kinematic space [Frost]

Triangulations & ϕ^3 Amps

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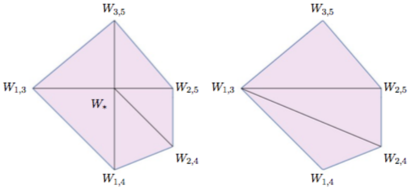

$$\Omega(\mathcal{A}_5) = d^2\mathbf{s}\,\left(\frac{1}{s_{12}s_{34}} + \cdots + \frac{1}{s_{51}s_{23}}\right)$$
$$= d^2\mathbf{s}\left(\frac{s_{12}+s_{51}}{s_{12}s_{34}s_{51}} + \frac{s_{12}+s_{51}}{s_{12}s_{51}s_{23}} + \frac{s_{12}-s_{45}+s_{23}}{s_{12}s_{23}s_{45}}\right)$$
$$= \text{sum of 3 triangles of } \mathcal{A}_5 \text{ itself}$$

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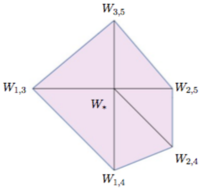

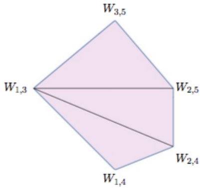

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Similar to "local" or "BCFW" triangulations of the amplituhedron: manifest new symmetries of ϕ^3 obscured by Feynman diagrams!

Worldsheet Associahedron & Scattering Eqs

A well-known associahedron: minimal blow-up of the open-string worldsheet $\mathcal{M}^+_{0,n} := \{\sigma_1 < \sigma_2 < \cdots < \sigma_n\}/\mathrm{SL}(2,\mathbb{R})$ [Deligne, Mumford]

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The *canonical form* of $\overline{\mathcal{M}}^+_{0,n}$ is "Parke-Taylor" form (see also [Mizera])

$$\omega_n^{\text{WS}} := \frac{1}{\text{vol }[\text{SL}(2)]} \prod_{a=1}^{n} \frac{d\sigma_a}{\sigma_a - \sigma_{a+1}} := \text{PT}(1,2,\cdots,n)\, d\mu_n$$

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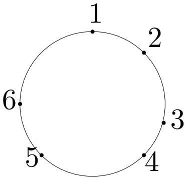
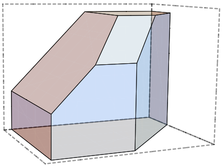


scattering equations
$\longrightarrow$
as a map from $\overline{\mathcal{M}}^{+}_{0,n}$ to $\mathcal{A}_n$



Pushforward from the wolrdsheet

On H_n, scattering eqs provide a diffeomorphism from $\overline{\mathcal{M}}^+_{0,n}$ to $\mathcal{A}_n$:

$$\sum_{b\neq a}\frac{s_{ab}}{\sigma_a-\sigma_b}=0 \implies s_{a,a+1}=\sigma_{a,a+1}\sum_{1<i+1\leq a\leq j<n}\frac{c_{i,j}}{\sigma_{i,j}} \qquad (\sigma_n\to\infty)$$

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Diff $A \to B \implies$ pushforward $\Omega(A) \to \Omega(B)$ [Arkani-Hamed, Bai, Lam]

$$y = f(x) \qquad \implies \qquad \Omega(B)_y = \sum_{x=f^{-1}(y)} \Omega(A)_x$$

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Canonical form of $\mathcal{A}_n$, $\Omega_n^{\phi^3}$ is the pushforward of $\omega_n^{\rm WS}$ by summing over $(n-3)!$ sol. of scattering eqs (geometric origin of CHY)

$$\sum_{\text{sol.}} d\mu_n \, \text{PT}(\alpha)|_{H(\alpha)} = m(\alpha|\alpha) \, d^{n-3}\mathbf{s}$$

Projective Scattering Forms

General scattering forms: sum over all cubic graphs with numerators

$$\Omega[N] = \sum_g N(g) \bigwedge_{I=1}^{n-3} d\log s_I\,, \qquad e.g.\ N_s\, d\log s + N_t\, d\log t + N_u\, d\log u$$

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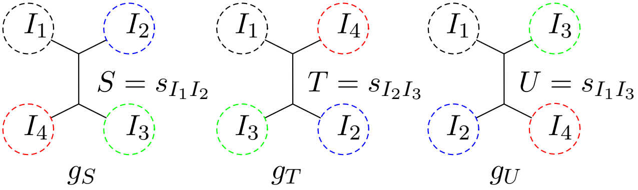
$\Longrightarrow$ N's (numerators) satisfy Jacobi identities [Bern, Carrasco, Johansson]

$$N(g_S) + N(g_T) + N(g_U) = 0\,, \quad e.g.\ N_s + N_t + N_u = 0$$



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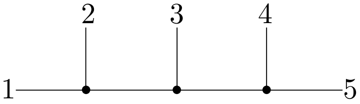

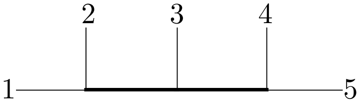

$$f^{a_1 a_2 b} f^{b a_3 c} f^{c a_4 a_5} \quad \leftrightarrow \quad \mathrm{d}s_{12} \wedge \mathrm{d}s_{45}$$

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Duality between *color factors* and *differential forms on* $\mathcal{K}_n$ for cubic graphs: $C(g)$ and $W(g)$ satisfy the same algebra.

$$\textbf{Claim}: \; W(g) := \pm \bigwedge_{I=1}^{n-3} ds_I \quad \Longrightarrow \quad W(g_S) + W(g_T) + W(g_U) = 0$$

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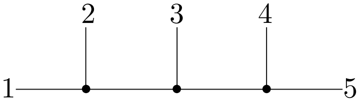

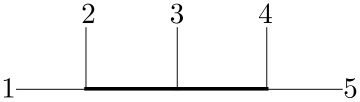

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Scattering forms are color-dressed amps without color factors.

For U(N), partial amps are pullbacks to subspaces (same as Ω^{ϕ^3}).

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Key: forms are projective $\implies$ unique $\Omega^{\rm YM}$ and $\Omega^{\rm NLSM}$!

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Alternatively they are pushforward of rigid worldsheet objects

$$\Omega_n^{\rm YM} = \sum_{\rm sol.} d\mu_n {\rm Pf}' \Psi_n \,, \quad \Omega_n^{\rm NLSM} = \sum_{\rm sol.} d\mu_n \det{}' A_n$$

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The regulators ensure no log divergences, and the leading term in ϵ comes from vertices, *e.g.* for simple polytopes (d facets for each):

$$\lim_{\epsilon \to 0} \mathcal{I}_P(\{X\}) = \sum_{\text{vertex } i} \prod_a^d \frac{1}{X_a} \,, \quad e.g. \; \lim_{\epsilon \to 0} \mathcal{I}_{[0,1]} = \frac{1}{A} + \frac{1}{B}$$

$$\lim_{\epsilon \to 0} \epsilon^2 \int \frac{dx dy}{x(y-x)(1-y)} x^{\epsilon A} (y-x)^{\epsilon B} (1-y)^{\epsilon C} = \frac{1}{AB} + \frac{1}{BC} + \frac{1}{CA} \,.$$

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Natural to integrate Ω_P in P provided regulators for all facets W_a's

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The regulators ensure no log divergences, and the leading term in ϵ comes from vertices, *e.g.* for simple polytopes (d facets for each):

$$\lim_{\epsilon \to 0} \mathcal{I}_P(\{X\}) = \sum_{\text{vertex } i} \prod_{a}^{d} \frac{1}{X_a} \,, \quad e.g.\ \lim_{\epsilon \to 0} \mathcal{I}_{[0,1]} = \frac{1}{A} + \frac{1}{B}$$

$$\lim_{\epsilon \to 0} \epsilon^2 \int \frac{dx dy}{x(y-x)(1-y)} x^{\epsilon A} (y-x)^{\epsilon B} (1-y)^{\epsilon C} = \frac{1}{AB} + \frac{1}{BC} + \frac{1}{CA} \,.$$

Looks like *canonical function* of some polytope in X space.

Integral vs. Pushforward

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Conjecture: Leading term of the integral $\mathcal{I}_P$ equals canonical function of Q, $F_Q(X)$, which is given by pushforward of $\Omega(P)$:

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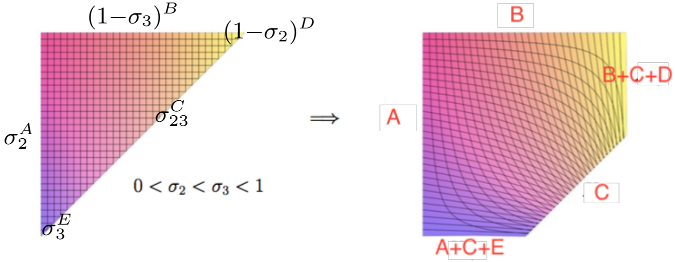
We believe this to be the general mechanism behind CHY: field-theory limit of string integral = pushforward via scattering eqs

Same for integrals with extra walls passing through vertices of P: Q can be thought of as "blowup" of P at those boundaries, *e.g.*

$$\lim_{\epsilon\to 0}\int \frac{d\sigma_2 d\sigma_3}{\sigma_2\sigma_{23}(1-\sigma_3)}\text{“KN”} = \frac{1}{AB}+\left(\frac{1}{A}+\frac{1}{C}\right)\frac{1}{(A+C+E)}+\left(\frac{1}{B}+\frac{1}{C}\right)\frac{1}{(B+C+D)}$$



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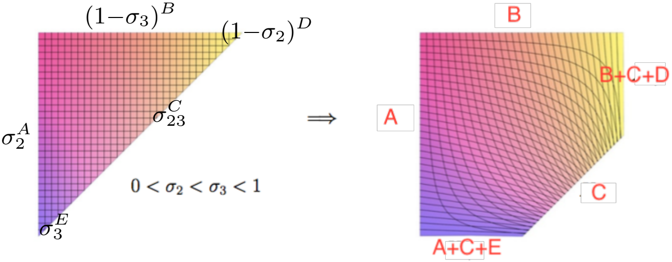

The leading term, and remarkably the pushforward, only depends on the combinatorics, not any details of the extra walls.

Complex Integrals vs. CHY formula

Another natural integral: $|\Omega_P(Y)|^2$ on $\mathbb{C}^d$. Same limit as the real one:

$$\mathcal{I}_P^{\mathbb{C}} := (\frac{\epsilon}{2\pi i})^d \int_{\mathbb{C}^d} |\Omega_P(Y)|^2 \prod_{\text{facets}} |Y \cdot W_a|^{\epsilon X_a}, \quad \lim_{\epsilon\to 0} \mathcal{I}_P^{\mathbb{C}} = \lim_{\epsilon \to 0} \mathcal{I}_P \,,$$

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Key: the limit again comes from every vertex, which gives the same residue as the real case: $\lim_{\epsilon\to 0} \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{dz d\bar{z}}{|z|^2} |z|^{\epsilon A} = \lim_{\epsilon \to 0} \int \frac{r dr}{r^2} r^{\epsilon A} = \frac{1}{A}$.

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The same holds for complex integrals with extra walls, and the limit equals to "CHY formula" (self-intersection number [Mizera])

$$\lim_{\epsilon \to 0} \mathcal{I}_P^{\mathbb{C}} = \frac{1}{(2\pi i)^d} \oint_{|E^I| = \varepsilon} \frac{\Omega_P(Y)\, \hat{\Omega}_P(Y)}{\prod_{I=1}^d E^I} \,, \quad E^I := \sum_a \frac{X_a W_a^I}{Y \cdot W_a} \,.$$

Integrals of General Forms

These integrals extract rational functions in X space from forms, and we can apply them to general (non-$d\log$) forms like $\Omega_n^{\mathrm{YM/NLSM}}$.

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Consider pullback of $\Omega^{\rm YM}(x) = \sum_g N_g^{\rm YM}(\epsilon, k) \wedge_{i,j} d\log x_{ij}$ to H_n: the regulated integral in $\mathcal{A}_n$ (all $x_{ij} > 0$) gives the partial amplitude!

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$$\lim_{\epsilon\to 0} \epsilon^{n-3} \int_{x_{ij}>0} \Omega^{\mathrm{YM}}(x)|_H \prod_{i,j} x_{ij}^{\epsilon X_{ij}} = \sum_g \frac{\mathrm{Res}_g \Omega_n^{\mathrm{YM}}}{\prod X_{ij}^{(g)}} = M_n^{\mathrm{YM}}\,,$$

$$e.g. \qquad \lim_{\epsilon\to 0} \int_{0<x<c} (N_s d\log x + N_t d\log(c-x)) x^{\epsilon s} (c-x)^{\epsilon t} = \frac{N_s}{s} - \frac{N_t}{t}\,.$$

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Each vertex gives $n-3$ planar poles $X_{i\,j}$, times the residue. Here X's are given by the kinematic data defining residues, $N_g^{\text{YM}}(\epsilon, p)$.

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If we take $\Omega_n^{\rm YM}$ in BCJ form, it gives exactly the gravity amplitude!

$$\lim_{\epsilon\to 0}\int_{\mathbb{C}^d} |\Omega_n^{\rm YM}(x)|^2 \prod_I^{2^{n-2}-1} |x_I|^{\epsilon s_I} = \sum_g \frac{|\text{Res}_g \Omega_n^{\rm YM}|^2}{\prod s_I^{(g)}} = M_n^{\rm GR}\,, e.g.$$

$$\lim_{\epsilon\to 0}\int_{\mathbb{C}} |\frac{N_s dx}{x} + \frac{N_t dy}{y} + \frac{N_u dz}{z}|^2 |x|^{\epsilon s}|y|^{\epsilon t}|z|^{\epsilon u} = \frac{|N_s|^2}{s} + \frac{|N_t|^2}{t} + \frac{|N_u|^2}{u}\,.$$

Projectivity as the Key for Double Copy

It is crucial to start with a *projective form*, otherwise the integral also has non-vanishing residue at infinity $\sim N_s + N_t + N_u$. In general, projectivity ensures the absence of pole at infinity along any direction!

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Any projective form admits $\Omega = \sum_\alpha N_\alpha \Omega^{\phi^3}(\alpha)$, and integral gives

$$\lim_{\epsilon\to 0} \int_{\mathbb{C}^{n-3}} \Omega_L \, \Omega_R^* \prod_I |x_I|^{\epsilon s_I} = \sum_{\alpha,\beta} N_L(\alpha) N_R(\beta) \, m(\alpha|\beta) \,.$$

which in particular implies KLT if expanded in a $(n-3)!$ basis.

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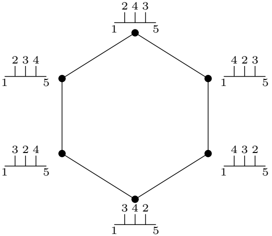
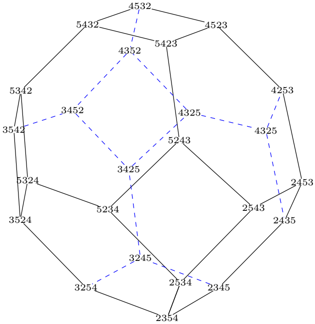
$$\mathcal{P}_n = \Delta_n \cap H_n\,, \quad \Delta_n = \{s_{1,\pi(2),\cdots,\pi(a)} > 0\}$$
$$H_n = \{s_{ij} \text{ negative const for } 2 \leq i < j \leq n-1\}$$





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Each has a subspace from natural rewriting of scattering eqs and the form gives a nice sum of cubic trees; further generalization of WS forms to non-planar leading singularities of $\mathrm{G}(2,n)$ [w. Yan, Zhang, Zhang]

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$\rightarrow$ graph associahedra [Devadoss], generalized permutohedra [Postnikov], "polytopes" with combinatoric factorization [Cachazo][Early] *etc.*

$\rightarrow$ cluster polytopes, with generalized "scattering-eq maps" [Thomas]

Outlook

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- Loops: halohedra *etc.* at one loop [Salvatori] ; picture similar to amplituhedron? relations with ambitwistor strings? [Yvonne's talk]
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Thank you for your attention!