integrands and amplitudes for two-loop five-point scattering in QCD

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in collaboration with Christian Brønnum-Hansen,
Bayu Hartanto and Tiziano Peraro

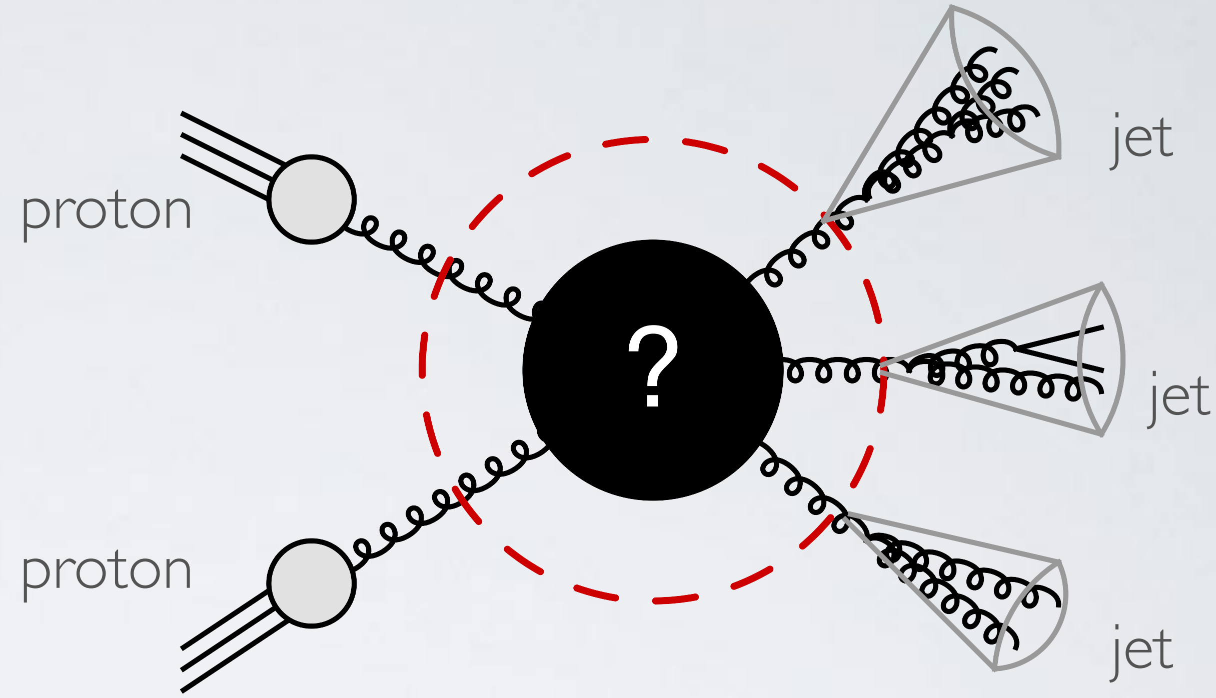
Amplitudes
SLAC, 19th June 2018

new precision frontier: 2 → 3 scattering

(2 → 3)/(2 → 2) ratio quantities become accessible

systematic errors cancel

high precision observables

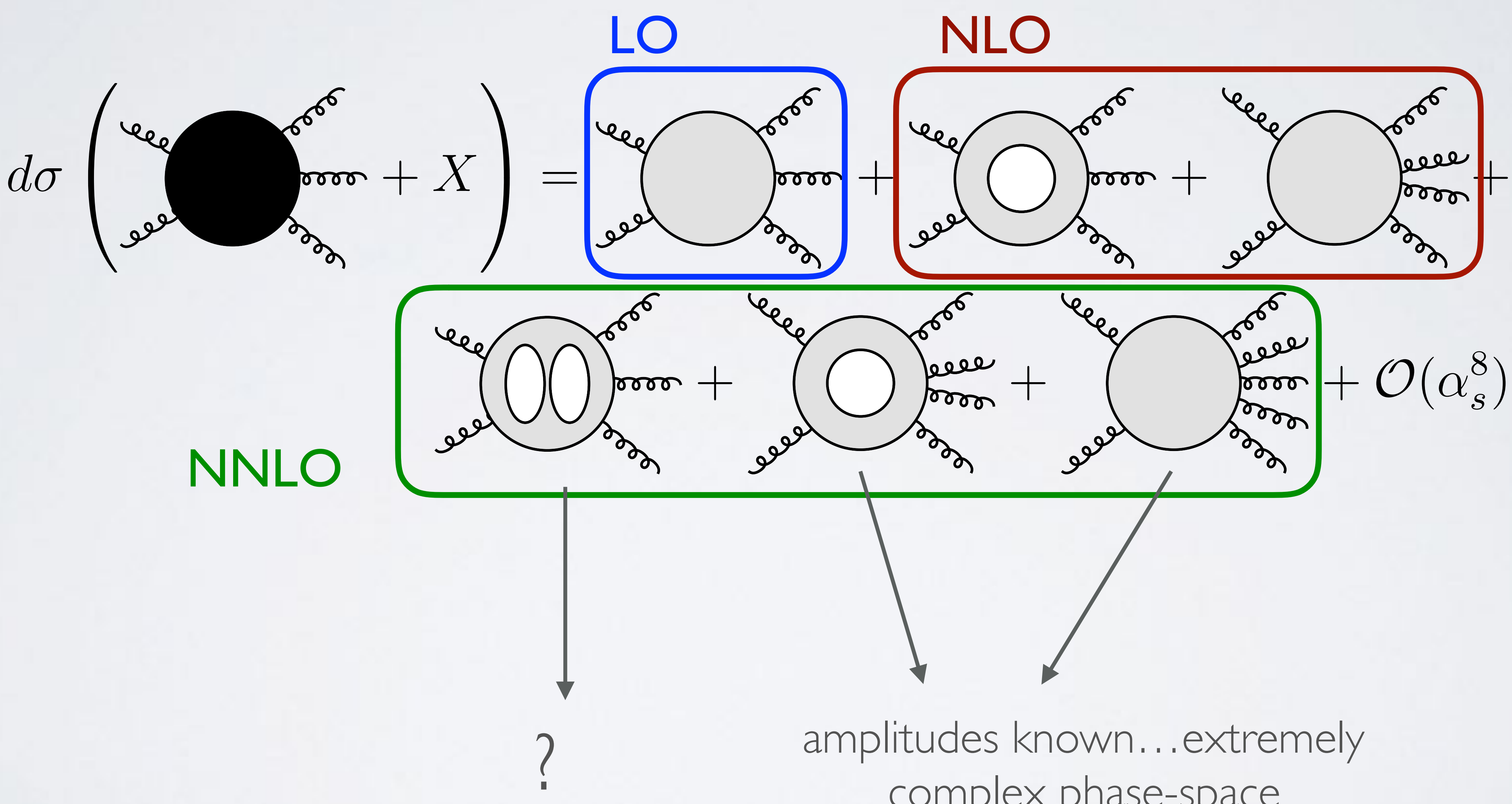


process	precision observables
$pp \to 3j$	jet multiplicity ratios, α_s at high energies, 3-jet mass
$pp \to \gamma\gamma + j$	background to Higgs p_T, signal/background interference effects
$pp \to H + 2j$	Higgs p_T, Higgs coupling through vector boson fusion (VBF)
$pp \to V + 2j$	Vector boson p_T, W^+/W^- ratios and multiplicity scaling
$pp \to VV + j$	backgrounds to p_T spectra for new physics decaying via vector boson

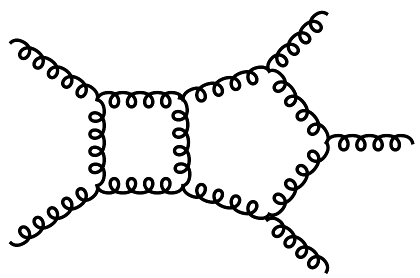
key example: 3j/2j ratio at the LHC can probe of the running of α_s in a new energy regime

e.g. CMS @ 7 TeV $\alpha_s(m_Z^2) = 0.1148 \pm 0.0014(\text{exp.}) \pm 0.0018(\text{PDF}) \pm 0.0050(\text{theory})$

new precision frontier: 2 → 3 scattering



complexity for 2⟶3 processes



planar gluon scattering

		$2 \to 2$		$2 \to 3$	
		$\mathcal{N}=4$	QCD	$\mathcal{N}=4$	QCD
one loop	integrand basis	1	65	5	175
	master integrals	1	2	1	2
two loops	integrand basis	2	15360	15	55580
	master integrals	1	7	3	61

these types of amplitudes might not be so impressive these days in SUSY theories…

very important in the development of on-shell methods such as unitarity, leading singularities, etc.

Bern, Rozowsky, Yan, Czakon, Dixon, Kosower, Cachazo, Spradlin, Volovich…

$$(\text{amplitude}) = \sum_c (\text{colour})_c (\text{ordered amplitude})_c$$

strip colour factors

$$(\text{ordered amplitude}) = \sum_i (\text{kinematic})_i (\text{integral})_i$$

rational function of kinematics

special basis of functions

summary of state-of-the-art

first results for planar 2→3 parton scattering amplitudes

2→3 master integrals

[Papadopoulos, Tommasini, Wever arXiv:1511.09404]

[Gehrmann, Henn, Lo Presti arXiv:1511.05409]

[Chicherin, Henn, Mitev arXiv:1712.09610]

a first look at two-loop five-gluon amplitudes in QCD

[SB, Brønnum-Hansen, Hartanto, Peraro arXiv:1712.02229]

Efficient integrand reduction for particles with spin

[Boels, Jin, Luo arXiv:1802.06761]

Planar two-loop five-gluon amplitudes from numerical unitarity

[Abreu, Febres-Cordero, Ita, Page, Zeng arXiv:1712.03946]

Two-loop five-point massless QCD amplitudes within the IBP approach

[Chawdhry, Lim, Mitov arXiv:1805.09182]

an on-shell toolbox for multi-loop integrands

momentum twistors

[Hodges (2009)]

six-dimensional spinor-helicity

[Cheung, O'Connell (2009)]

finite field reconstruction

[von Manteuffel, Schabinger (2014)]

[Peraro (2016)]

$$A = \int_k \sum_i \frac{\Delta_i(k,p)}{(\text{propagators})_i}$$

(generalised) unitarity cuts

[Bern, Rozowsky, Yan, Dixon, Kosower, de Freitas, Wong…(1997-)]

integrand reduction

[Ossola, Papadopoulos, Pittau, Mastrolia, SB, Frellesvig, Zhang, Peraro, Mirabella, …(2005-)]

$$\mathcal{A} = \sum_i S_i \frac{C(\Delta_i)\Delta_i}{\prod D_\alpha}$$

colour/kinematics relations

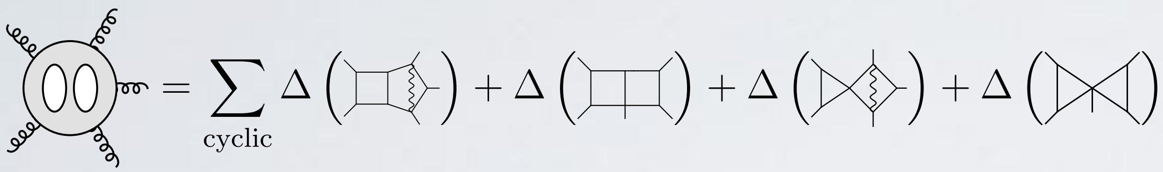
[Bern, Carrasco, Johansson (2008)]

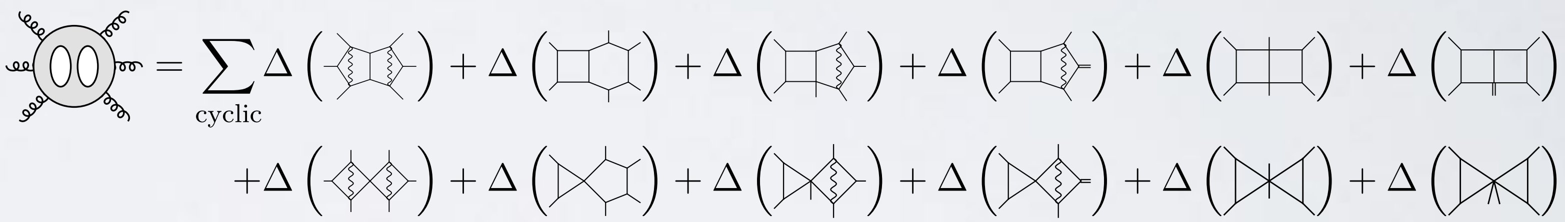
previously…all-plus test cases

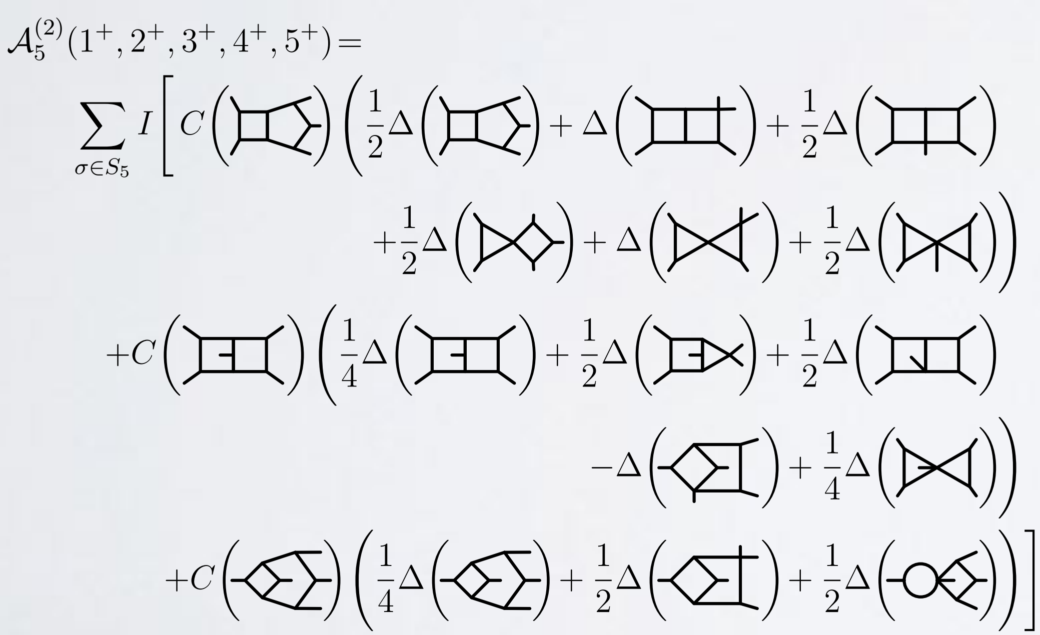
[SB, Frellesvig, Zhang (2013)]

[SB, Mogull, O'Connell, Ochirov (2015)]

[SB, Mogull, Peraro (2016)]







analytic d-dimensional integrands using six-dimensional spinor-helicity and generalised unitarity cuts

amplitudes and integrands

$$A = \int_k \sum_i \frac{\Delta_i(k,p)}{(\text{propagators})_i}$$

how can we parameterise the irreducible numerator?

constructing the integrand basis

[c.f. van Neerven, Vermaseren (1983)]

$$k_i^\mu = k_{\parallel,i}^\mu + k_{\perp,i}^\mu, \qquad k_{\perp,i} = k_{\perp,i}^{[4]} + k_{\perp,i}^{[-2\epsilon]}$$

$$k_{\parallel,i}^\mu = \sum_{j=1}^{d_\parallel} a_{ij} v_j^\mu$$

$$k_{\perp,i}^{\mu,[4]} = \sum_{j=1}^{d_{\perp,[4]}} b_{ij} w_j^\mu$$

(linear system) ISPs **k.p** (and propagators)

quadratic relations amongst the numerator ISPs remain

$$\mu_{ij} = k_i.k_j - k_{\parallel,i}.k_{\parallel,j} - k_{\perp,i}^{[4]}.k_{\perp,j}^{[4]}$$

spurious ISPs **k.w**

any integrand is a function of: $k_i \cdot p_j$, $k_i \cdot w_j$ and μ_{ij}

constructing the integrand basis

$$\Delta(k_i \cdot p_j, k_i \cdot w_j, \mu_{ij}) = \sum (\text{coefficients})(\text{monomial})$$

- updated algorithm no longer requires polynomial division
- integrand contains spurious terms

$$\int_k k_i \cdot w_j = 0$$

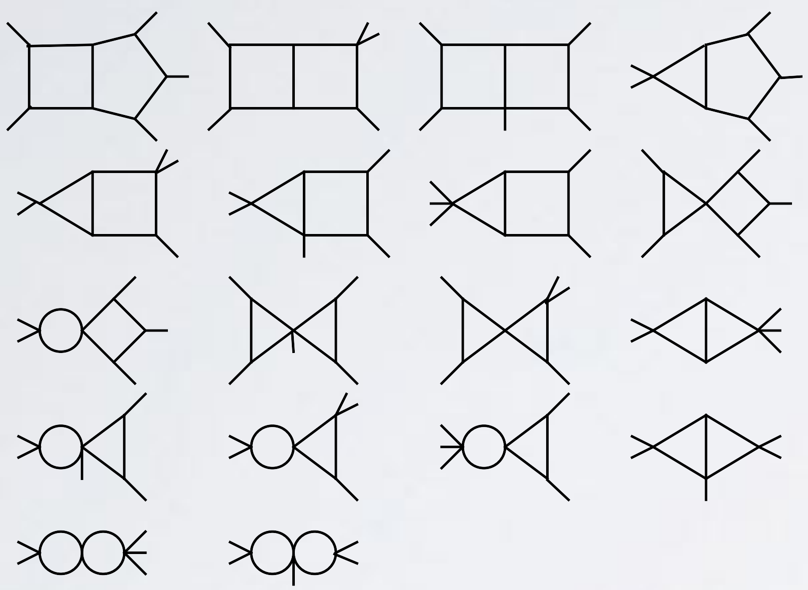
- integrand basis depends on the ordering of the possible ISP monomials
- beyond one-loop the integrals can be further reduced using integration-by-parts identities

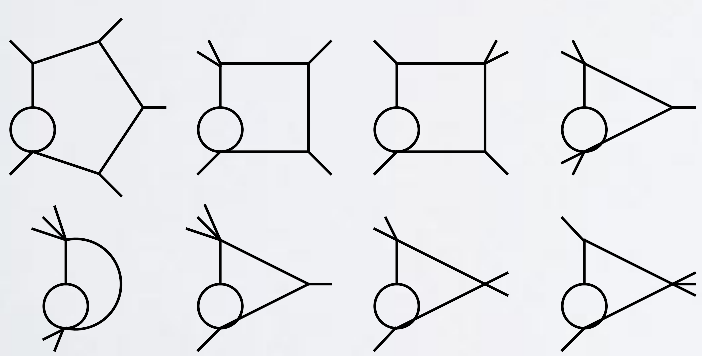
$$\int_k \frac{\partial}{\partial k_\mu} \frac{v_\mu(k,p)}{(\text{propagators})} = 0$$

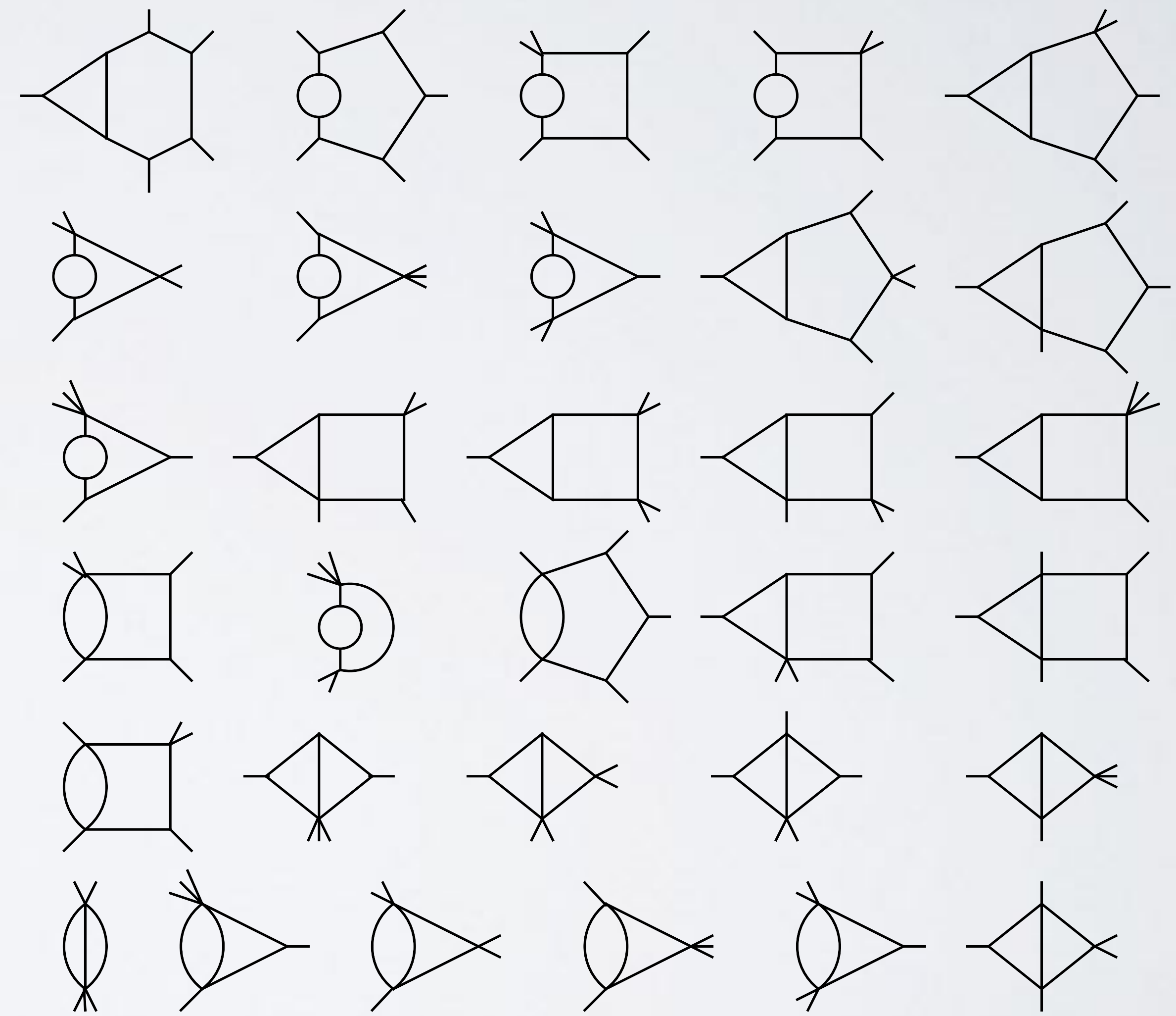
many new ideas for efficient solutions of IBP systems:
Kosower, Kajda, Gluza, Schabinger, von Manteuffel, Ita, Larsen, Zhang, Böhm, Georgoudis, Schönemann, Abreu, Page, Febres-Cordero, Zeng

[see talks by Febres-Cordero, Larsen, Zeng]

two-loop five-gluon scattering in QCD

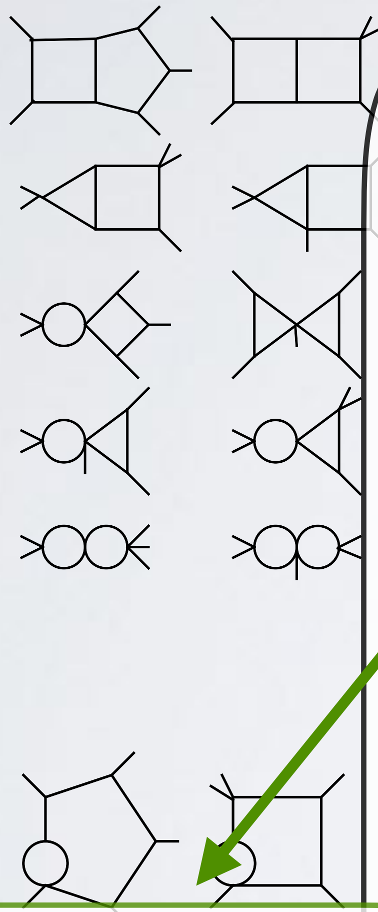






a first look at two-loop five-gluon scattering in QCD

SB, Brønnum-Hansen, Hartanto Peraro
Phys.Rev.Lett. 120 (2018) no.9, 092001



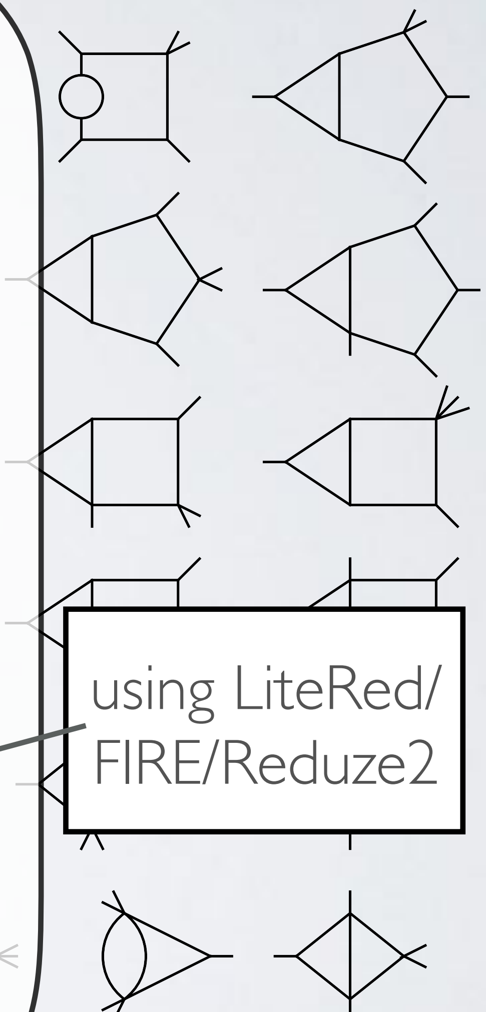
(analytic) integrand reconstruction from **finite-field evaluations** of unitarity cuts in six dimensions

numerical reduction to 'master integrals'* with IBPs

(by-passing original sector decomposition, SecDec/FIESTA)

using LiteRed/ FIRE/Reduze2

application to scattering amplitudes: Peraro [1608.0192]



* Gehrmann, Henn, Lo Presti (2015), Papadopoulos, Tommasini, Wever (2015)

two-loop five-gluon scattering in QCD

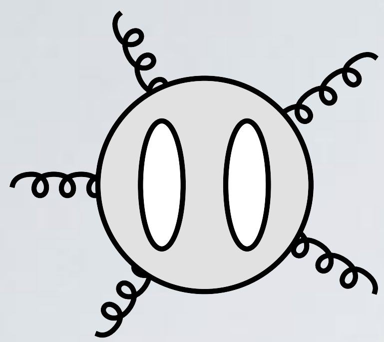
helicity	flavour	non-zero coefficients	non-spurious coefficients	contributions @ $\mathcal{O}(\epsilon^0)$
	$(d_s-2)^0$	50	50	0
$+++++$	$(d_s-2)^1$	175	165	50
	$(d_s-2)^2$	320	90	60
	$(d_s-2)^0$	1153	761	405
$-++++$	$(d_s-2)^1$	8745	4020	3436
	$(d_s-2)^2$	1037	100	68
	$(d_s-2)^0$	2234	1267	976
$--+++$	$(d_s-2)^1$	11844	5342	4659
	$(d_s-2)^2$	1641	71	48
	$(d_s-2)^0$	3137	1732	1335
$-+-++$	$(d_s-2)^1$	15282	6654	5734
	$(d_s-2)^2$	3639	47	32

TABLE I. The number of non-zero coefficients found at the integrand level both before ('non-zero') and after ('non-spurious') removing monomials which integrate to zero. Last column ('contributions @ $\mathcal{O}(\epsilon^0)$') gives the number of coefficients contributing to the finite part. Each helicity amplitude is split into the components of d_s-2.

SB, Brønnum-Hansen, Hartanto Peraro
Phys.Rev.Lett. 120 (2018) no.9, 092001

$$\mathcal{A}^{(L)}(1,2,3,4,5) = n^L g_s^3 \sum_{\sigma\in S_5/Z_5} \mathrm{tr}\left(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(5)}}\right) \times A^{(L)}\left(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5)\right), \quad (1)$$

$$A^{(2)}(1,2,3,4,5) = \int [dk_1][dk_2] \sum_T \frac{\Delta_T(\{k\},\{p\})}{\prod_{\alpha\in T} D_\alpha}$$



numerical evaluation in the Euclidean region

$$x_1 = -1, \quad x_2 = \frac{79}{90}, \quad x_3 = \frac{16}{61}, \quad x_4 = \frac{37}{78}, \quad x_5 = \frac{83}{102}.$$

$$s_{12} = -1, \quad s_{23} = -\frac{37}{78}, \quad s_{34} = -\frac{2023381}{3194997}, \quad s_{45} = -\frac{83}{102}, \quad s_{15} = -\frac{193672}{606645}.$$

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$\widehat{A}^{(2),[0]}_{--+++}$	12.5	27.7526	-23.7728	-168.1162	-175.2103
$P^{(2),[0]}_{--+++}$	12.5	27.7526	-23.7728	-168.1163	—
$\widehat{A}^{(2),[0]}_{-+-++}$	12.5	27.7526	2.5028	-35.8084	69.6695
$P^{(2),[0]}_{-+-++}$	12.5	27.7526	2.5028	-35.8086	—

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$\widehat{A}^{(2),[1]}_{+++++}$	0	0	-2.5	-6.4324	-5.3107
$P^{(2),[1]}_{+++++}$	0	0	-2.5	-6.4324	—
$\widehat{A}^{(2),[1]}_{-++++}$	0	0	-2.5	-12.7492	-22.0981
$P^{(2),[1]}_{-++++}$	0	0	-2.5	-12.7492	—
$\widehat{A}^{(2),[1]}_{--+++}$	0	-0.625	-1.8175	-0.4869	3.1270
$P^{(2),[1]}_{--+++}$	0	-0.625	-1.8175	-0.4869	—
$\widehat{A}^{(2),[1]}_{-+-++}$	0	-0.625	-2.7759	-5.0018	0.1807
$P^{(2),[1]}_{-+-++}$	0	-0.625	-2.7759	-5.0018	—

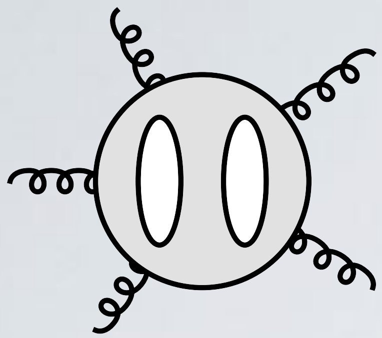
	$\widehat{A}^{(2),[2]}_{+++++}$	$\widehat{A}^{(2),[2]}_{-++++}$	$\widehat{A}^{(2),[2]}_{--+++}$	$\widehat{A}^{(2),[2]}_{-+-++}$
ϵ^{0}	3.6255	-0.0664	0.2056	0.0269

universal poles

$$P^{(2)} = I^{(2)} A^{(0)} + I^{(1)} A^{(1)}$$

[Catani] [Becher,Neubert]
[Gnendiger, Signer, Stockinger]

evaluation in the physical region



reduction to MI of Gehrmann, Henn, Lo Presti (or alternatively Papadopoulos, Tommasini, Wever)

d_s=2 fully analytic
full d_s dep. partially numerical

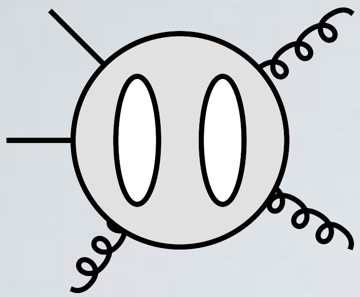
$$x_1 = \frac{113}{7}, \quad x_2 = -\frac{2}{9} - \frac{i}{19}, \quad x_3 = -\frac{1}{7} - \frac{i}{5}, \quad x_4 = \frac{1351150}{13847751}, \quad x_5 = -\frac{91971}{566867}.$$

$$s_{12} = \frac{113}{7}, \quad s_{23} = -\frac{152679950}{96934257}, \quad s_{34} = \frac{1023105842}{138882415}, \quad s_{45} = \frac{10392723}{3968069}, \quad s_{15} = -\frac{8362}{32585}.$$

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$\widehat{A}^{(2),[0]}_{--+++}$	12.5	-9.17716 + 47.12389 i	-107.40046 - 25.96698 i	17.24014 - 221.41370 i	388.44694 - 167.45494 i
$P^{(2),[0]}_{--+++}$	12.5	-9.17716 + 47.12389 i	-107.40046 - 25.96698 i	17.24013 - 221.41373 i	—
$\widehat{A}^{(2),[0]}_{-+-++}$	12.5	-9.17716 + 47.12389 i	-111.02853 - 12.85282 i	-39.80016 - 216.36601 i	342.75366 - 309.25531 i
$P^{(2),[0]}_{-+-++}$	12.5	-9.17716 + 47.12389 i	-111.02853 - 12.85282 i	-39.80018 - 216.36604 i	—

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$\widehat{A}^{(2),[1]}_{+++++}$	0	0	-2.5	0.60532 - 12.48936 i	35.03354 + 9.27449 i
$P^{(2),[1]}_{+++++}$	0	0	-2.5	0.60532 - 12.48936 i	—
$\widehat{A}^{(2),[1]}_{-++++}$	0	0	-2.5	-7.59409 - 2.99885 i	-0.44360 - 20.85875 i
$P^{(2),[1]}_{-++++}$	0	0	-2.5	-7.59408 - 2.99885 i	—
$\widehat{A}^{(2),[1]}_{--+++}$	0	-0.625	-0.65676 - 0.42849 i	-1.02853 + 0.30760 i	-0.55509 - 6.22641 i
$P^{(2),[1]}_{--+++}$	0	-0.625	-0.65676 - 0.42849 i	-1.02853 + 0.30760 i	—
$\widehat{A}^{(2),[1]}_{-+-++}$	0	-0.625	-0.45984 - 0.97559 i	1.44962 + 0.53917 i	-0.62978 + 2.07080 i
$P^{(2),[1]}_{-+-++}$	0	-0.625	-0.45984 - 0.97559 i	1.44962 + 0.53917 i	—

	$\widehat{A}^{(2),[2]}_{+++++}$	$\widehat{A}^{(2),[2]}_{-++++}$	$\widehat{A}^{(2),[2]}_{--+++}$	$\widehat{A}^{(2),[2]}_{-+-++}$
ϵ^{0}	0.60217 - 0.01985 i	-0.10910 - 0.01807 i	-0.06306 - 0.01305 i	-0.03481 - 0.00699 i



fermion channels

preliminary!

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Leading-colour amplitude:

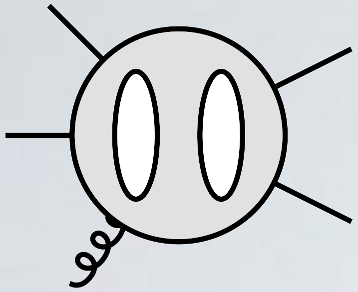
$$\mathcal{A}^{(L)}(1_q, 2_g, 3_g, 4_g, 5_{\bar{q}}) = n^L g_s^3 \sum_{\sigma \in S_3} (T^{a_{\sigma(2)}} T^{a_{\sigma(3)}} T^{a_{\sigma(4)}})_{i_1}^{\bar{i}_5} A^{(L)}(1_q, \sigma(2), \sigma(3), \sigma(4), 5_{\bar{q}})$$

$$n = m_\epsilon N_c \alpha_s/(4\pi), \quad \alpha_s = g_s^2/(4\pi), \quad m_\epsilon = i(4\pi)^\epsilon e^{-\epsilon\gamma_E}$$

checks against poles in CDR

$$\widehat{A}^{(2)}_{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5} = \frac{A^{(2)}(1_q^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}, 5_{\bar{q}}^{\lambda_5})}{A^{(0)}(1_q^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}, 5_{\bar{q}}^{\lambda_5})}$$

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$\widehat{A}^{(2)}_{++++-}$	0	0	-4	-13.5322768031	6.048656403
$P^{(2)}_{++++-}$	0	0	-4	-13.5322768028	—
$\widehat{A}^{(2)}_{+++--}$	8	7.9682909085	-52.3927085027	-140.1563714534	47.5687220127
$P^{(2)}_{+++--}$	8	7.9682909085	-52.3927085034	-140.1563714829	—
$\widehat{A}^{(2)}_{++-+-}$	8	7.9682909085	-32.2213536407	-47.9234973502	145.9720111187
$P^{(2)}_{++-+-}$	8	7.9682909085	-32.2213536403	-47.9234973889	—
$\widehat{A}^{(2)}_{+-++-}$	8	7.9682909084	-40.8851109385	-87.0299398048	101.2329971544
$P^{(2)}_{+-++-}$	8	7.9682909085	-40.8851109386	-87.0299398374	—



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Leading-colour amplitude:

$$\mathcal{A}^{(L)}(1_q, 2_{\bar{q}}, 3_g, 4_Q, 5_{\bar{Q}}) = n^L g_s^3 \Big[(T^{a_3})_{i_4}^{\;\bar{i}_2} \, \delta_{i_1}^{\;\bar{i}_5} A^{(L)}(1_q, 2_{\bar{q}}, 3_g, 4_Q, 5_{\bar{Q}}) + (1 \leftrightarrow 4, 2 \leftrightarrow 5) \Big]$$

$$n = m_\epsilon N_c \alpha_s/(4\pi), \quad \alpha_s = g_s^2/(4\pi), \quad m_\epsilon = i(4\pi)^\epsilon e^{-\epsilon\gamma_E}$$

checks against poles in CDR

$$\widehat{A}^{(2)}_{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5} = \frac{A^{(2)}(1_q^{\lambda_1}, 2_{\bar{q}}^{\lambda_2}, 3_g^{\lambda_3}, 4_Q^{\lambda_4}, 5_{\bar{Q}}^{\lambda_5})}{A^{(0)}(1_q^{\lambda_1}, 2_{\bar{q}}^{\lambda_2}, 3_g^{\lambda_3}, 4_Q^{\lambda_4}, 5_{\bar{Q}}^{\lambda_5})}$$

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$\widehat{A}^{(2)}_{+-++-}$	4.5	2.2831548748	-32.0984879203	-41.3935016796	149.3305056672
$P^{(2)}_{+-++-}$	4.5	2.2831548747	-32.0984879200	-41.3935017002	—
$\widehat{A}^{(2)}_{+--+-}$	4.5	2.2831548747	-4.6165799244	-6.3236903564	-32.0327897453
$P^{(2)}_{+--+-}$	4.5	2.2831548747	-4.6165799294	-6.3236903481	—
$\widehat{A}^{(2)}_{+-+-+}$	4.5	2.2831548747	-38.294786128	-43.5232972386	-56.7196838792
$P^{(2)}_{+-+-+}$	4.5	2.2831548747	-38.294786128	-43.5232972694	—
$\widehat{A}^{(2)}_{+---+}$	4.5	2.2831548747	-26.7131604905	-69.7580529817	22.2365337061
$P^{(2)}_{+---+}$	4.5	2.2831548747	-26.7131604895	-69.758052969	—

can we use this for phenomenology? $\int d\Phi_3 \sum_{\text{colour,helicity}} 2\text{Re}(A^{(2)}A^{(0)})$

the analytic integrands are very large…

proceed numerically?

this was a good option at NLO (BLACKHAT, NJET, GOSAM, OPENLOOPS,…)

also works at two loops

see talk from Febres-Cordero

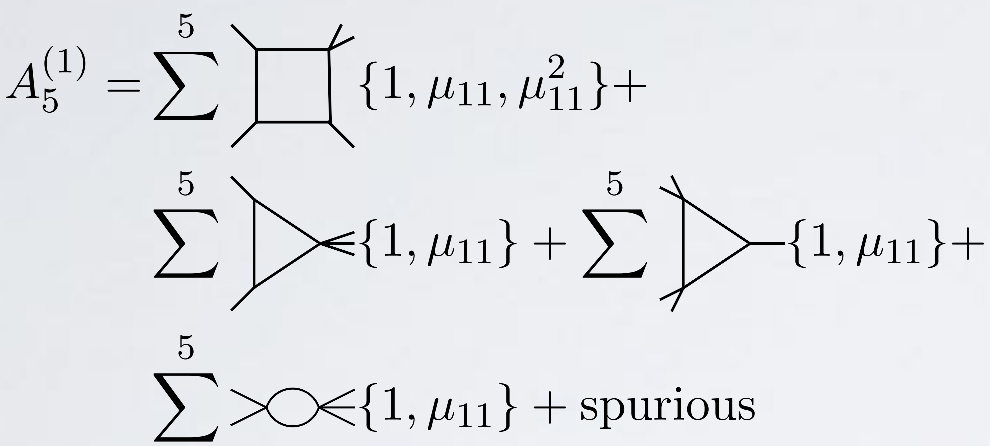
- universal IR and UV poles not manifest
- large cancellations between topologies
- high rank tensor integrals lead to extremely difficult IBP systems (IBPs are d-dim.)
- need to understand analytic structure better: what is a 'good' basis?

related questions:
(quasi) finite integrals
local integrals, pure functions…

[von Mantueffel, Schabinger, Panzer]

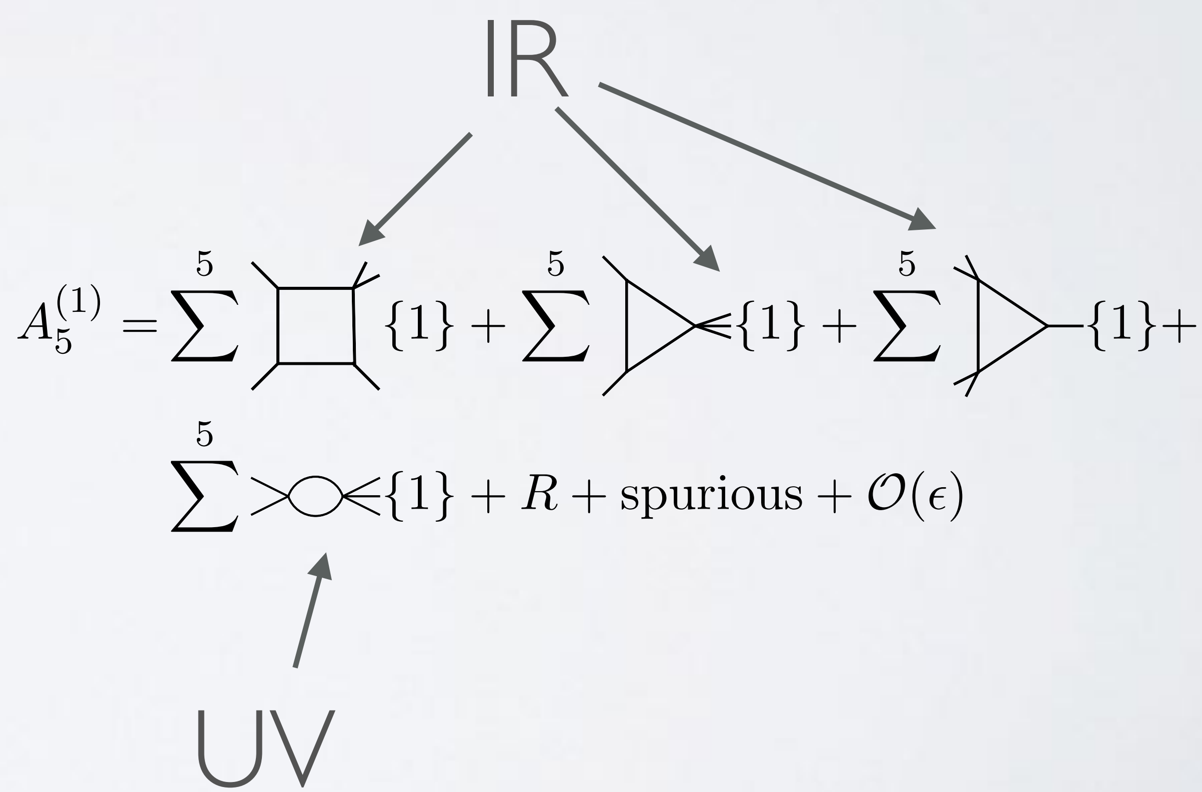
[Arkani-Hamed, Bourjaily, Cachazo, Trnka]

back to one-loop



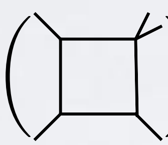
d-dimensional basis
(EGKM, OPP etc.)

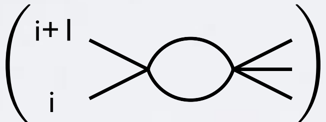
expansion around d=4
rational terms come from
UV poles



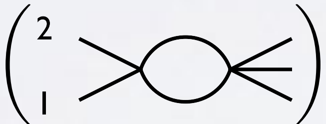
back to one-loop

local numerators regulate IR
[Arkani-Hamed, Bourjaily, Cachazo, Trnka (2012)]

$$A_5^{(1)} - I^{(1)} A_5^{(0)} = \Delta \Bigg($$

$$\Bigg) \{(k-k_1^*),(k-k_2^*)\} +$$

$$\Delta \Bigg($$


$$\Bigg) \{1 - \frac{(k)^2 (k-p_{i,i+1})^2}{(k+p_{1,i-1})^2 (k+p_{1,i-1}-p_{1,2})^2}\} +$$

$$\Delta \Bigg($$


$$\Bigg) \{\mu_{11}\} + \text{spurious} + \mathcal{O}(\epsilon)$$

universal IR poles

UV counter-terms to
push all rational terms into
bubble numerator

[still not quite as good as BDK '93…additional spurious pole cancellations between rational term and bubble coefficients]

an attempt at 2-loops

the pentagon-box sector of our planar integrand has the most complicated integrals

standard gauge theory power counting: 76 integrand coefficients

old version

$\{k_1.p_1, k_2.p_2, k_2.p_3, \mu_{11}, \mu_{12}, \mu_{22}\}$

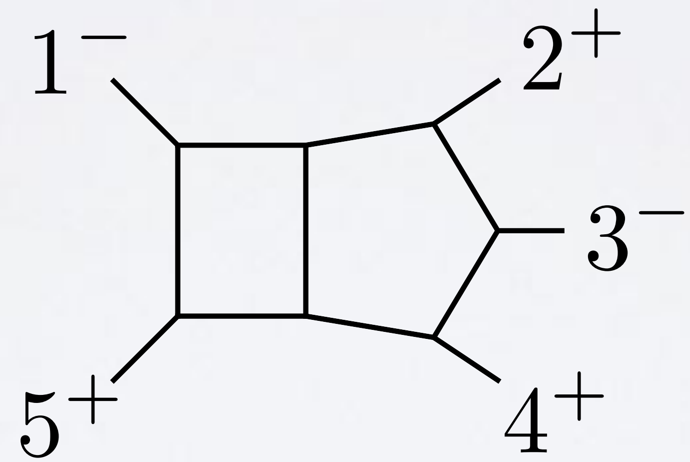
54 non-zero integrand coeffs.

28 non-zero coeffs. at $\mathcal{O}(\epsilon)$

worst case:

$(k_2.p_2)^4 \mu_{11}$

(rank 4, 1 dot)



$k_2.n_2 = \langle 5 k_2 1]$

$k_1.n_1 \sim (k_1 - n_1)^2 \propto \mathrm{tr}_+(p_2 k_1 (k_1 - p_{23}) p_4)$

new version

local numerators

$\{k_1.n_1, k_1.n_1^*, k_1.n_2, k_2.n_2^*, \mu_{11}, \mu_{12}, \mu_{22}\}$

(over complete set)

55 non-zero integrand coeffs.

4 non-zero coeffs. at $\mathcal{O}(\epsilon)$

worst case:

$k_2.n_2 k_1.n_1$

(rank 2)

summary

- two-loop amplitudes from on-shell building blocks:
    - generalised unitarity cuts and integrand reduction in d-dimensions
    - reconstruction of rational functions over finite fields
    - first results for realistic processes. Lots more to do for NNLO

a local integrand **basis**?

['prescriptive unitarity' Bourjaily, Herrmann, Trnka (2017)]

non-planar?

[Arkani-Hamed Bourjaily, Cachazo, Postnikov, Trnka (2015)]
[Bern, Herrmann, Litsey, Stankowicz, Trnka (2016)]
[Bern, Enciso, Ita, Zeng (2017)]

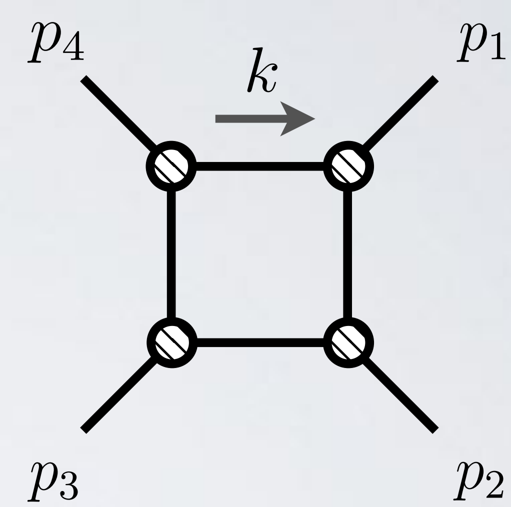
backup

one-loop box example

propagators

$$P = \langle x_{14}^2 - \mu_{11} - stu, x_{11}, x_{12}, x_{13} \rangle$$

scalar products



irreducible numerator

$$\Delta_4 = c_0 + c_1 x_{14} + c_2 \mu_{11} + c_3 \mu_{11} x_{14} + c_4 \mu_{11}^2$$

on-shell solution

$$\bar{k}^\mu = \frac{s(1+\tau)}{4\langle 4|2|1]} \langle 4|\gamma^\mu|1] + \frac{s(1-\tau)}{4\langle 1|2|4]} \langle 1|\gamma^\mu|4]$$

$$x_{14} = \frac{st}{2}\tau \qquad \mu_{11} = -\frac{st}{4u}(1-\tau^2)$$

tree-level "data"

$$\Delta_4(k(\tau)) = \sum_{i=0}^{4} d_i \tau^i$$

$$\begin{pmatrix} 1 & -\frac{t}{2} & 0 & 0 & 0 \\ 0 & t & -\frac{st}{u} & \frac{st^2}{2u} & 0 \\ 0 & 0 & \frac{st}{u} & -\frac{3st^2}{2u} & \frac{s^2t^2}{u^2} \\ 0 & 0 & 0 & \frac{st^2}{u} & -\frac{2s^2t^2}{u^2} \\ 0 & 0 & 0 & 0 & \frac{s^2t^2}{u^2} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

$$x_{ij} = k_i \cdot v_j \qquad k_i^{[-2\epsilon]} \cdot k_j^{[-2\epsilon]} = -\mu_{ij}$$

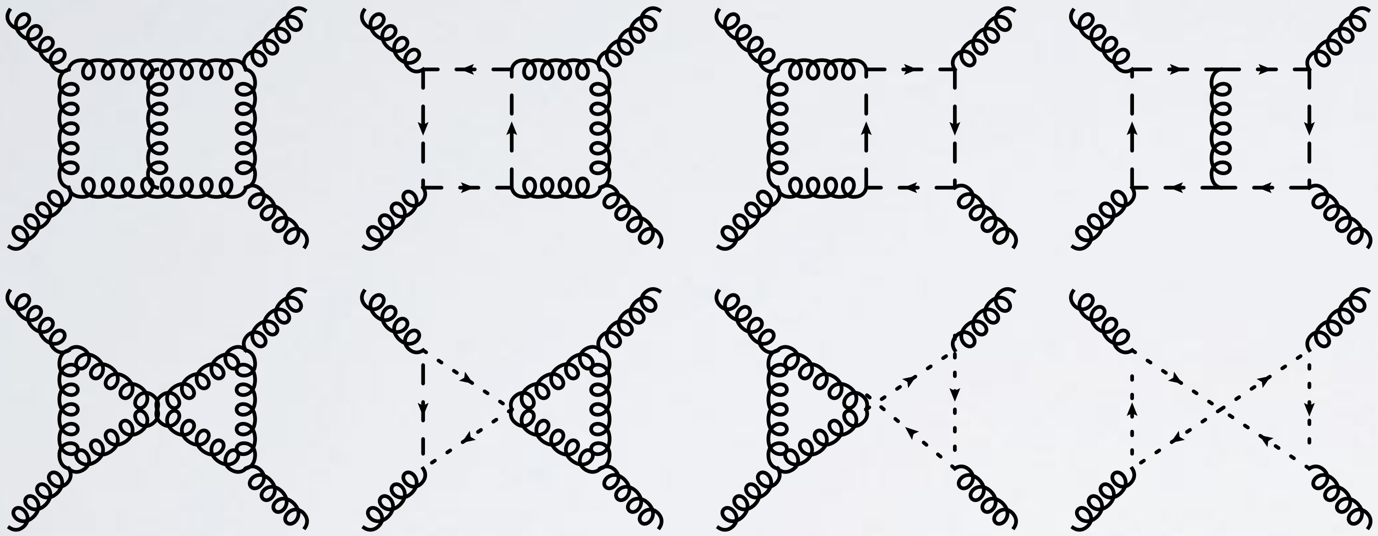
continue reduction with subtractions

$$\Delta_{3;123}(k(\tau_1,\tau_2)) = N(k(\tau_1,\tau_2), p_1, p_2, p_3, p_4) - \frac{\Delta_4(k(\tau_1,\tau_2))}{(k(\tau_1,\tau_2)+p_4)^2}$$

numerator construction

FDH scheme at two-loops

[Bern, De Freitas, Dixon, Wong (2002)]



$$g^{\mu}{}_{\mu} = d_s$$

c.f. Feynman rules + Feynman gauge and ghosts (scalars)

Tree-amplitudes using **six-dimensional** helicity method

need to capture $\mu_{11},\ \mu_{22},\ \mu_{12}$

[Cheung, O'Connell (2009)]

[Bern, Carrasco, Dennen, Huang, Ita (2011)]

[Davies (2012)]

use **momentum twistors** to deal with the complicated kinematics at 2⟶3

[Hodges (2009)]

momentum twistors

[Hodges (2009)]

recall: spinor-helicity SU(2)xSU(2) ~ $p_i^\mu \leftrightarrow (\lambda_{\alpha i}, \tilde{\lambda}_i^{\dot{\alpha}})$

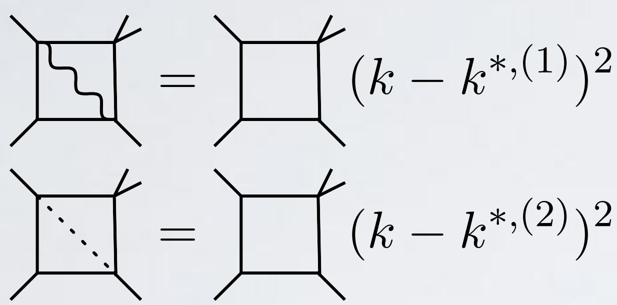
$$Z_{iA} = (\lambda_\alpha(i), \mu^{\dot{\alpha}}(i))$$

kinematic variables with manifest momentum conservation
or
a rational phase space generator

$$W_i^A = (\tilde{\mu}_\alpha(i), \tilde{\lambda}^{\dot{\alpha}}(i)) = \frac{\varepsilon^{ABCD} Z_{(i-1)B} Z_{iC} Z_{(i+1)D}}{\langle i-1 i\rangle\langle i i+1\rangle} \Rightarrow \tilde{\lambda}(i)^{\dot{\alpha}} = \frac{\langle i-1 i\rangle \mu^{\dot{\alpha}}(i+1) + \langle i+1 i-1\rangle \mu^{\dot{\alpha}}(i) + \langle i i+1\rangle \mu^{\dot{\alpha}}(i-1)}{\langle i-1 i\rangle\langle i i+1\rangle}$$

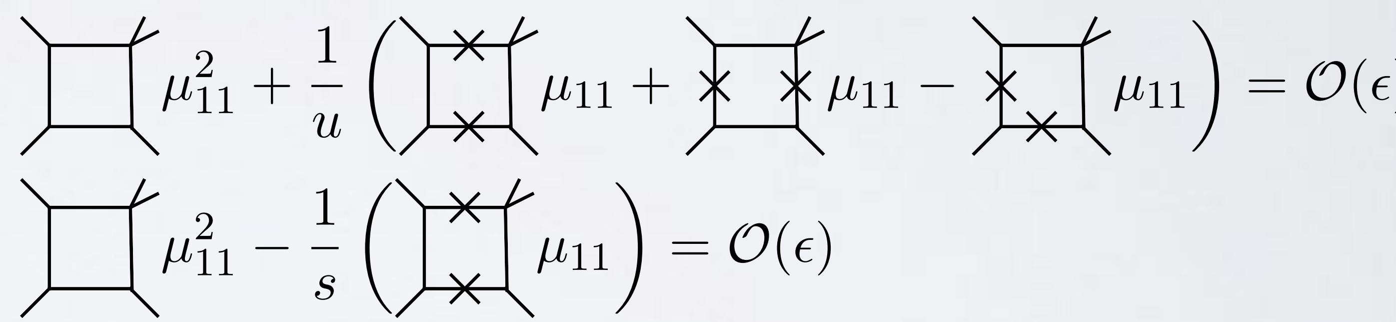
$$\Rightarrow \sum_{i=1}^{n} \lambda_\alpha(i)\tilde{\lambda}_{\dot{\alpha}}(i) = 0_{\alpha\dot{\alpha}}$$

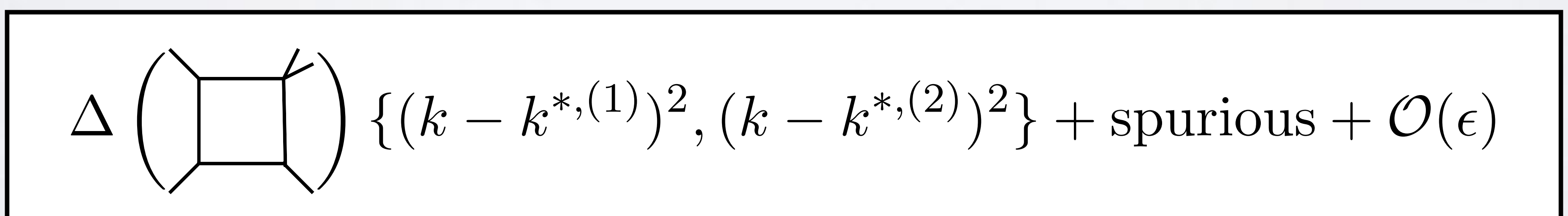
manifest UV and IR poles at the integrand level



local numerators manage IR divergences
[Arkani-Hamed, Bourjaily, Cachazo, Trnka (2012)]

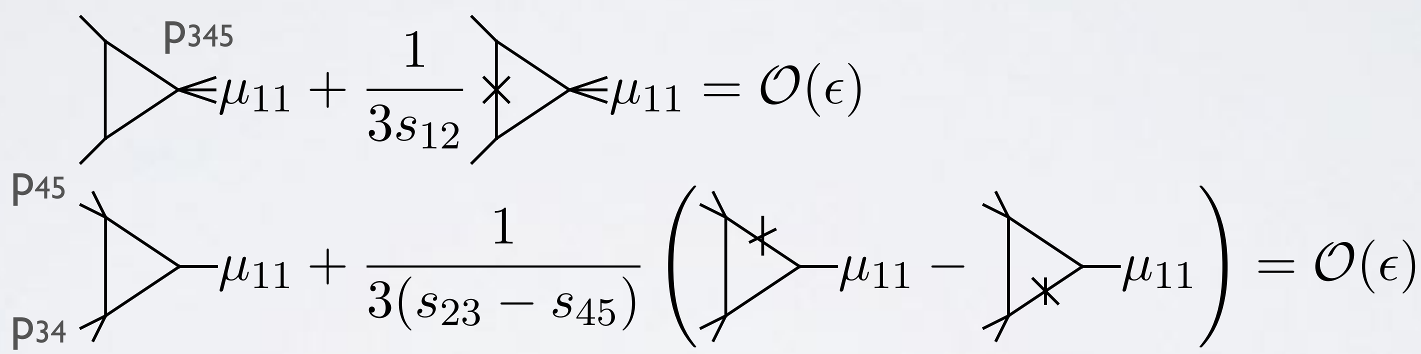
remove d-dimensional integrals with UV counter-terms

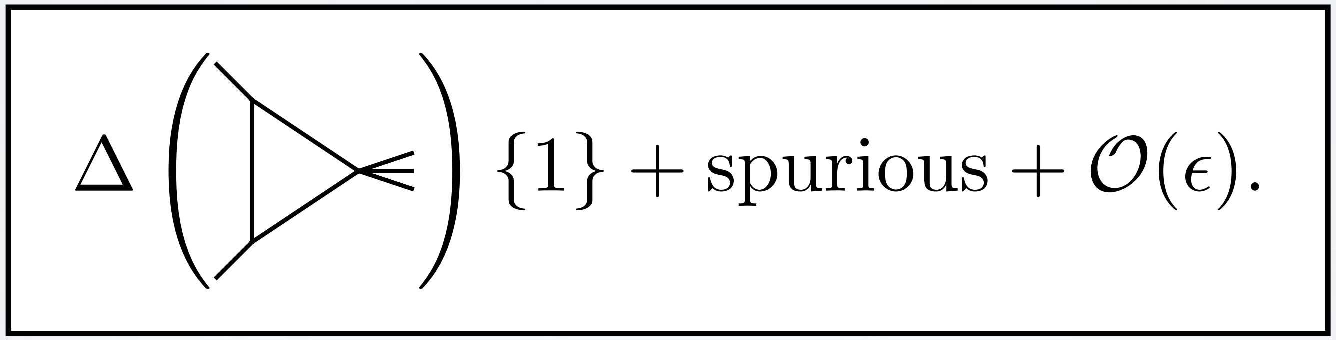




manifest UV and IR poles at the integrand level

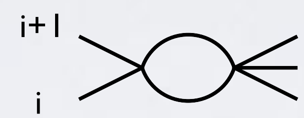
also find counter-terms for the triangle topologies

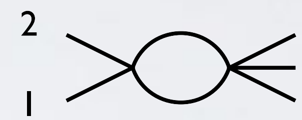




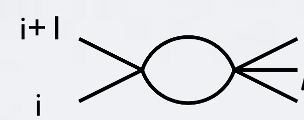
manifest UV and IR poles at the integrand level

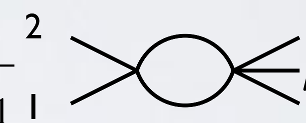
- UV counter-terms for both 4d and 6d bubbles
- Use p_{12} bubble - p_{ii+1} cuts become subtractions for the p_{12} cut



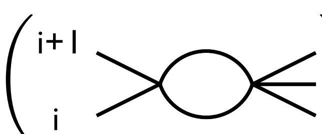
$-$


$= \mathcal{O}(\epsilon^0)$

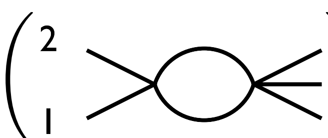


$\mu_{11} - \frac{s_{12}}{s_{i\,i+1}}$


$\mu_{11} = \mathcal{O}(\epsilon)$

$\Delta\Big($


$\Big)\{1 - \frac{(k)^2(k-p_{i,i+1})^2}{(k+p_{1,i-1})^2(k+p_{1,i-1}-p_{1,2})^2}\} + \text{spurious} + \mathcal{O}(\epsilon).$

$\Delta\Big($


$\Big)\{1, \mu_{11}\} + \text{spurious} + \mathcal{O}(\epsilon).$