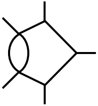
Multi-Loop Numerical Unitarity

Applications to two-loop QCD amplitudes



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Outline

Multi-scale two-loop amplitudes in QCD

- Precise measurements with few percent uncertainties at the LHC are to become common place
- Mature set of tools for computing NLO QCD and EW corrections will need to be upgraded to handle NNLO QCD corrections for a wide variety of multi-scale processes
- We develop the multi-loop numerical unitarity method to compute needed multi-parton two-loop amplitudes
- First results for 5-gluon two-loop amplitudes show potential to tackle many processes in the SM and beyond

Key building blocks for NNLO QCD corrections

- Strategy to handle and cancel IR divergences
- Two-loop matrix elements

- Many recent advances and complete calculations (e.g. $t\bar{t}$, $2j$, VV', Vj, HH, etc)
- Several well-developed approaches
  - Antenna subtraction
  - ColorfulNNLO
  - Nested soft-collinear subtractions
  - N-Jettiness slicing
  - Projection to born
  - q_T slicing
  - SecToR Improved Phase sPacE for real Radiation
  - ...
- Different degrees of automation, we might have public tools in the near future

Key building blocks for NNLO QCD corrections

- Strategy to handle and cancel IR divergences
- Two-loop matrix elements

- Great steps towards understanding mechanisms to compute multi-scale master Feynman integrals, including insights into functional forms over the last few years

- Integrand reduction techniques have shown power to tackle 5-point amplitudes [Badger, Brønnum-Hansen, Hartanto, Peraro '17]; also see recent progress with the IBP reduction approach [Boels, Jin, Luo '18], [Chawdhry, Lim, Mitov '18]. Here we focus on the numerical unitarity method

The standard approach to general two-loop amplitudes

Feynman diagrams

Tensor reduction
[Passarino, Veltman '79]
IBPs
[Tkachov, Chetyrkin '81]
↓

Sum of master integrals

$$A = \sum_{\Gamma \in \Delta} \sum_{i \in M_\Gamma} c_{\Gamma,i} I_{\Gamma,i}$$

Differential equations
[Kotikov '91; Remiddi '97; Gehrmann, Remiddi '01; Henn '13]
↓

Integrated form

General procedure, but:

- Large intermediate expressions
- Generating IBP relations is practically difficult

Two-loop numerical unitarity tries to avoid these issues by:

- Performing reduction and evaluation simultaneously
- Working numerically

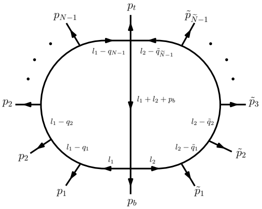
Outline

INTRODUCTION

TWO-LOOP NUMERICAL UNITARITY

RESULTS AND OUTLOOK

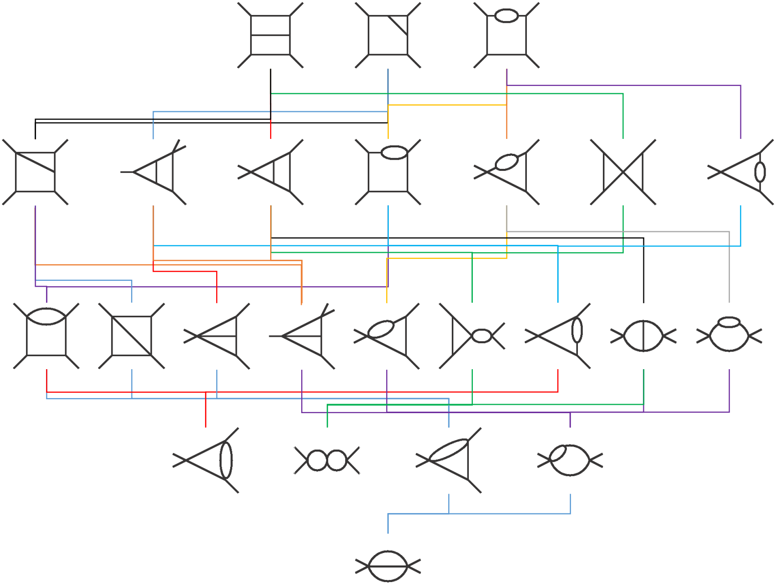
Diagrammatic decomposition



We write all diagrams Γ as members of the set Δ

- (Feynman) Diagrams are stripped of particle content and only edges that carry loop momenta are kept
- Each diagram Γ defines a propagator structure P_Γ
- We label inverse propagator sets $\{\rho_k\}$ with $k \in P_\Gamma$
- We introduce a partial ordering $\Gamma_1 > \Gamma_2$ for *ancestors* and *descendants* according to the propagator structures
- Sets of trees labelled T_Γ, according to the vertices of Γ

Example: partial ordering in planar 4-point 2-loop Δ



The master decomposition at two loops

We start with the decomposition:

$$\mathcal{A}^{2-\text{loops}} = \sum_{\Gamma\in\Delta} \sum_{i\,\in\, M_\Gamma} c_{\Gamma,i}\, I_{\Gamma,i}$$

which for a numerical unitarity approach is convenient to extend to:

$$\mathcal{A}(\ell_l) = \sum_{\Gamma\in\Delta} \frac{1}{\prod_{k\in P_\Gamma} \rho_k} \sum_{i\,\in\, M_\Gamma \cup S_\Gamma} c_{\Gamma,i}\, m_{\Gamma,i}(\ell_l)$$

with *master* and *surface* integrands:

$$\int \frac{d^D\ell_1 d^D\ell_2}{(2\pi)^{2D}} \frac{m_{\Gamma,i}(\ell_l)}{\prod_{k\in P_\Gamma} \rho_k} = \begin{cases} I_{\Gamma,i} & \text{for} \quad i \in M_\Gamma\,, \\ 0 & \text{for} \quad i \in S_\Gamma\,. \end{cases}$$

Implicit dependence on D and D_s for the coefficient functions $c_{\Gamma,i}$.

Key ingredients for a unitarity-based numerical calculation:

1. Control propagator powers in integral reductions by using IBP-vectors [Gluza,Kajda,Kosower 10]
2. parametrize 2-loop integrands (on- and off-shell) through master/surface decompositions [Ita 15] also [Badger, Larsen, Mastrolia, Zhang]

Amplitudes through generalized unitarity

Consider the decomposed integrand:

$$\sum_i c_i \frac{m_i^{\text{master}}(\ell)}{\rho_1 \cdots \rho_{n_i}} + \sum_j c_j \frac{m_j^{\text{surface}}(\ell)}{\rho_1 \cdots \rho_{n_j}} \quad = \quad \sum_i \frac{\mathcal{N}_i(\ell)}{\rho_1 \cdots \rho_{n_i}}$$

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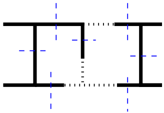
Factorization on the on-shell surfaces $\{\rho_1, \cdots, \rho_{n_l}\} = 0$:

$$\sum_i \frac{\mathcal{N}_i(\ell)}{\rho_1 \cdots \rho_{n_i}} \qquad \longrightarrow$$



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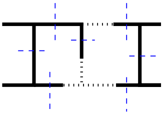
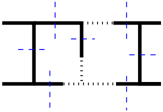


And so we get access to the set of (maximal) *cut equations*, like:

$$N(\Gamma, \ell_\Gamma) \equiv \sum_k c_k m_{\Gamma,k}(\ell_\Gamma) =$$



$$\equiv R(\Gamma, \ell_\Gamma)$$

On-shell phase space and factorization

Each diagram $\Gamma \in \Delta$ also defines an *on-shell* phase space according to:

$$\ell_l^\Gamma : \quad \ell_l \quad \text{with} \quad \rho_k = 0 \quad \text{for all} \quad k \in P_\Gamma$$

One-loop numerical unitarity builds on the fact that on each on-shell phase space, $\mathcal{A}(\ell_l)$ factorizes as a product of trees

Starting at two-loops subleading poles limit this 1-to-1 correspondence

(a) (b)

The subset $\Delta' \subset \Delta$ is employed to label the diagrams Γ which lead to a well defined product of trees! $\longrightarrow \Gamma^{(\mathrm{b})} \notin \Delta'$

Leading poles in multi-loop amplitudes

When approaching an on-shell phase space, for each $\Gamma \in \Delta'$

$$\lim_{\ell_l \to \ell_l^\Gamma} \mathcal{A}(\ell_l) = \frac{1}{\prod_{k \in P_\Gamma} \rho_k} \left(R(\Gamma, \ell_l^\Gamma) + \mathcal{O}(\rho_{k \in P_\Gamma}) \right)$$

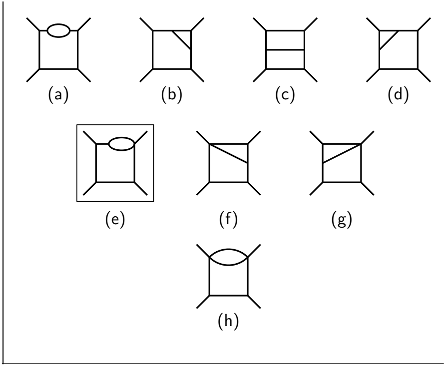
and in this limit $R(\Gamma, \ell_l^\Gamma)$ is given as a product of trees

$$R(\Gamma, \ell_l^\Gamma) = \sum_{\text{states}} \prod_{k \in T_\Gamma} \mathcal{A}_k^{\text{tree}}(\ell_l^\Gamma)$$

Notice that the integrand *ansatz* also diverges in the on-shell limit, and this allows a triangular approach to find all $\{c_{\Gamma,i}\}$

When a $\Gamma^* \notin \Delta'$ a non-factorizing contribution appears...

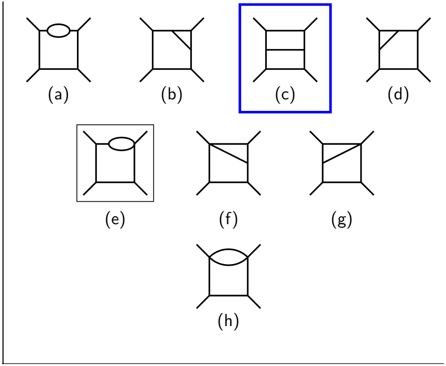
Subleading poles: and example, the *bubble-box*



Subleading poles: and example, the *bubble-box*

For a maximal:

N ( , ℓ_l^{c}) $=$ R ( , ℓ_l^{c})



Subleading poles: and example, the *bubble-box*

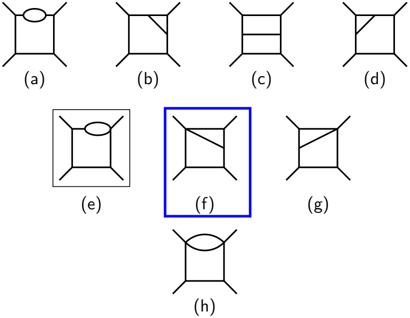
For a maximal:

$N\Big($  $, \ell_l^{\mathrm{c}}\Big) \quad = \quad R\Big($  $, \ell_l^{\mathrm{c}}\Big)$

For a next-to-maximal:

$N\Big($  $, \ell_l^{\mathrm{f}}\Big) = R\Big($  $, \ell_l^{\mathrm{f}}\Big)$

$- \frac{1}{\rho_{\mathrm{fb}}} N\Big($  $, \ell_1^{\mathrm{f}}\Big) - \frac{1}{\rho_{\mathrm{fc}}} N\Big($  $, \ell_l^{\mathrm{f}}\Big)$



Subleading poles: and example, the *bubble-box*

For a maximal:

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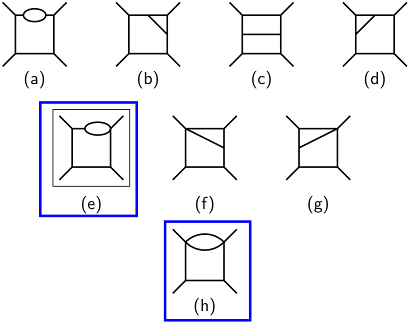
For a next-to-maximal:

$N\Big($  $, \ell_l^{\mathrm{f}}\Big) = R\Big($  $, \ell_l^{\mathrm{f}}\Big)$

$-\frac{1}{\rho_{\mathrm{fb}}} N\Big($  $, \ell_1^{\mathrm{f}}\Big) - \frac{1}{\rho_{\mathrm{fc}}} N\Big($  $, \ell_l^{\mathrm{f}}\Big)$



And for the combined single-pole diagram an bubble-box:

$N\Big($  $, \ell_l^{\mathrm{h}}\Big) + \frac{1}{\rho_{\mathrm{he}}} N\Big($  $, \ell_l^{\mathrm{h}}\Big) = R\Big($  $, \ell_l^{\mathrm{h}}\Big)$

$- \frac{1}{\rho_{\mathrm{hf}}} N\Big($  $, \ell_l^{\mathrm{h}}\Big) - \frac{1}{\rho_{\mathrm{hg}}} N\Big($  $, \ell_l^{\mathrm{h}}\Big) - \frac{1}{(\rho_{\mathrm{he}})^2} N\Big($  $, \ell_l^{\mathrm{h}}\Big)$

$- \frac{1}{\rho_{\mathrm{hf}}\rho_{\mathrm{fb}}} N\Big($  $, \ell_l^{\mathrm{h}}\Big) - \frac{1}{\rho_{\mathrm{hf}}\rho_{\mathrm{fc}}} N\Big($  $, \ell_l^{\mathrm{h}}\Big) - \frac{1}{\rho_{\mathrm{hg}}\rho_{\mathrm{gd}}} N\Big($  $, \ell_l^{\mathrm{h}}\Big)$

Master/surface decompositions

[Ita '15; Abreu, FFC, Ita, Page, Zeng '17]

Consider the IBP relation on Γ

$$0 = \int \prod_i d^D\ell_i \ \frac{\partial}{\partial \ell_j^\nu} \left[\frac{u_j^\nu}{\prod_{k\in P_\Gamma} \rho_k} \right]$$

while controlling the propagator structure [Gluza, Kadja, Kosower '11]

$$u_j^\nu \frac{\partial}{\partial \ell_j^\nu} \rho_k = f_k \rho_k$$

Write ansatz for u_j^ν expanded in external and loop momenta, and find solution to the polynomial equations using SINGULAR

Build a full set of surface terms and fill the rest of the space with master integrands

Related [Georgoudis, Larsen, Zhang '16]

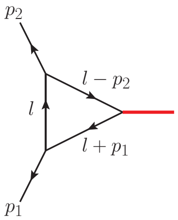
A simple example for surface terms: Part 1

Consider the 1-loop 1-mass triangle with

$$\rho_1 = (\ell + p_1)^2, \quad \rho_2 = \ell^2, \quad \rho_3 = (\ell - p_2)^2$$

and we construct $u^\nu \partial/\partial \ell^v$ by parametrizing

$$u^\nu = u_1^{\text{ext}} p_1^\nu + u_2^{\text{ext}} p_2^\nu + u^{\text{loop}} \ell^\nu$$



By constraining the propagator structure, we get the polynomial equation:

$$\left(u_1^{\text{ext}} p_1^\nu + u_2^{\text{ext}} p_2^\nu + u^{\text{loop}} \ell^\nu\right) \frac{\partial}{\partial \ell^\nu} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} - \begin{pmatrix} f_1 \rho_1 \\ f_2 \rho_2 \\ f_3 \rho_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We can then show that we have an IBP-generating vector, with constrained propagator structure:

$$u^\nu \frac{\partial}{\partial \ell^\nu} = \left[(\rho_3 - \rho_2) p_1^\nu + (\rho_1 + \rho_2) p_2^\nu + (-s + 2\rho_3 - 2\rho_2)\ell^\nu\right] \frac{\partial}{\partial \ell^\nu}$$

A simple example for surface terms: Part 2

Now we have the surface term:

$$0 = \int d^D \ell \frac{\partial}{\partial l^\nu} \frac{u^\nu}{\rho_1 \rho_2 \rho_3} = \int d^D l \frac{1}{\rho_1 \rho_2 \rho_3} \left[-(D-4) - 2(D-3)\rho_2 + 2(D-3)\rho_3 \right]$$

The scalar 1-loop triangle integrand on-shell could be replaced by a surface term, though commonly it is kept as a master integral.

The IBP relation between the triangle and the $s = (p_1 + p_2)^2$ bubble is:

$$-(D-4) I_{\text{tri}} - 2(D-3) I_{\textsf{s-bub}} = 0$$

Similar manipulations can be carried out at two loops. More complicated polynomial relations (*syzygy* equations) need to be solved → Singular. Surface terms appear as relatively compact

On-shell phase spaces and finite fields

- On-shell parametrization requires solving quadratic equations over the number field
- Avoid in $\mathbb{F}_p$ with good basis choice

$$\ell_1^2 = (\ell_1 - q_i)^2 \qquad = \cdots = 0$$
$$\ell_2^2 = (\ell_2 - q_j)^2, \qquad = \cdots = 0$$
$$(\ell_1 + \ell_2)^2 = (\ell_1 + \ell_2 - q_k)^2 = \cdots = 0$$

- Aim: Algebraic momenta in controlled fashion
- Use adapted coordinates
- Take μ_k as basis vectors - coefficients non-algebraic
- Affects scalar product and state sums

$$\ell_l^\mu \to (\rho_i, \alpha_j, \mu_{ij})$$
$$(\mu_l)^2 = \rho_{l0} - \sum_{v=0}^{3} \ell_l^\nu \ell_{l\,\nu}$$
$$\ell_l^{(D-4)} = w_{l,1}\mu_1 + w_{l,2}\mu_2$$
$$\ell_r \cdot \ell_s = \ell_r^4 \cdot \ell_s^4 - \sum_{i,j=1}^{2} w_i^r w_j^s \mu_{ij}$$

Related work: [Peraro '16]

Other challenges in multi-loop numerical unitarity

- Efficient algorithm to color decompose the amplitude's integrand [Ochirov, Page '16]
- On-the-fly reconstruction of functional dependence on regulators [Giele, Kunszt, Melnikov '08], [Peraro '16]
- Fast implementation of multi-dimensional cuts (through Berends-Giele off-shell recursions)
- Ensure numerical stability of calculation (through high-precision arithmetics and exploiting exact kinematics [von Manteuffel, Schabinger '14], [Peraro '16])
- Availability of master integrals (analytic expressions, for 5-pt examples see [Papadopoulos, Tommasini, Wever '15], [Gehrmann, Henn, lo Presti '15], or employing numerical tools, e.g. SecDec, Fiesta)

Modular library

We are constructing a C++ framework for D-dimensional multi-loop numerical unitarity, with a highly modular structure

- Hierarchical relations between propagator structures
- Decompositions of numerator functions into master/surface terms
- Color handling with interaction to algebraic libraries
- Automated construction of cut equations handling subleading poles, enable with powerful linear system solvers
- Engine to solve off-shell recursions to compute trees and multi-loop cuts
- Toolkit to handle kinematic structures using high-precision and exact arithmetics
- D-dimensional on-shell phase spaces generator
- Machinery for univariate functional reconstruction
- Integral library

Dependencies like `GiNaC, Givaro, GMP, Lapack, MPACK, QD`

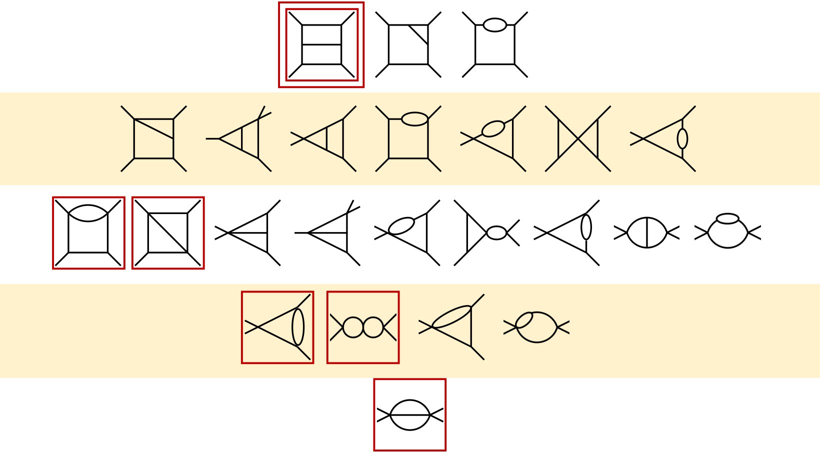
Outline

INTRODUCTION

TWO-LOOP NUMERICAL UNITARITY

RESULTS AND OUTLOOK

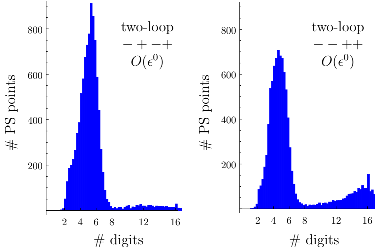
The planar two-loop four-point hierarchy

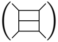


In total there are 14 *master* integral coefficients

Two-loop four-gluon helicity amplitudes

- Originally computed by [Glover, Oleari, Tejeda-Yeomans '01] and [Bern, de Freitas, Dixon '02]
- We reproduce results from analytics
- Floating point 4-pt calculation shows large cancellations
- Univariate reconstruction exploited to extract (known) analytic results

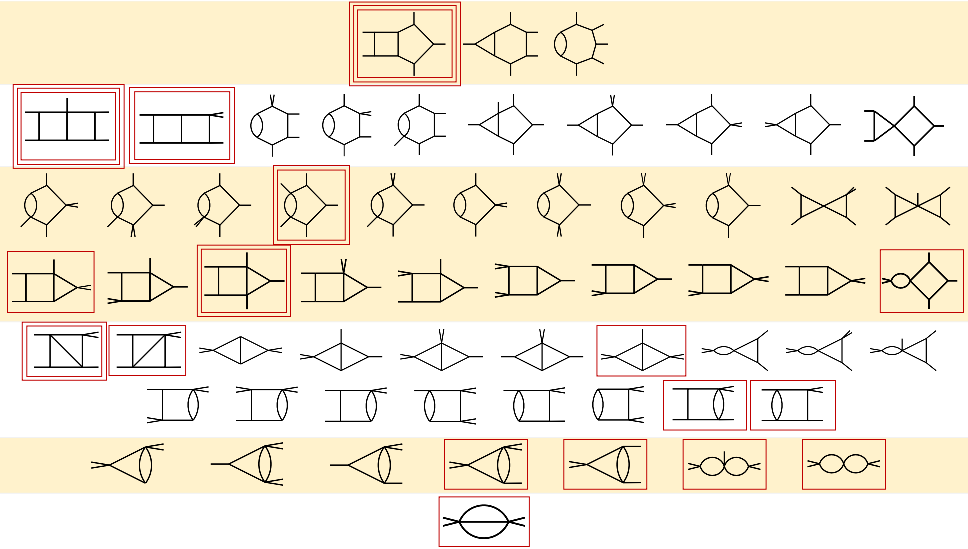




$$c_1\left(\;\;\right) = \frac{9x + \frac{\epsilon\left(-x^3 - \frac{32x^2}{11} - \frac{97x}{44} - \frac{5}{22}\right)}{\frac{x^2}{33} + \frac{2x}{33} + \frac{1}{33}} + \cdots}{27 - 81(4-2\epsilon) + 90(4-2\epsilon)^2 + \cdots}$$

$\mathcal{A}/(\mathcal{A}_0 N_c^2)(4\pi)^4$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$(1_g^-, 2_g^+, 3_g^-, 4_g^+)$	8.00000	55.6527	176.009	332.296	486.502
$(1_g^-, 2_g^-, 3_g^+, 4_g^+)$	8.00000	55.6527	164.642	222.327	-8.39044

The planar two-loop five-point hierarchy



In total there are 155 *master* integral coefficients

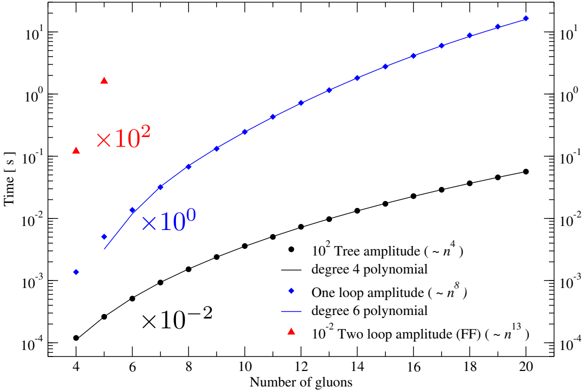
Two-loop five-gluon helicity amplitudes

- Full set of IBP-generating vectors produced to parametrize planar five-point massless integrands. Computing most complicated vector takes under a second
- Master integral coefficients in finite field, then promoted to $\mathbb{Q}$
- 5-point integrals from [Papadopoulos, Tommasini, Wever '15] ancillary. Analytic lower point integrals [Gehrmann, Remiddi '00]
- We reproduce the expected IR structure of the amplitudes
- Reproduce all-plus [Badger et al '13], [Gehrmann et al '15], [Dunbar et al '16]
- Validated concurrent calculation of [Badger, Brønnum-Hansen, Hartanto, Peraro '17]
- Single-threaded calculation in a final field in about 2.5 minutes

$$s_{ij} = \{-1, -8, -10, -7, -3\}$$

$\mathcal{A}^{(2)}/\mathcal{A}^{(0)}$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$(1^-, 2^-, 3^+, 4^+, 5^+)$	12.5000000	25.46246919	-1152.843107	-4072.938337	-3637.249566
$(1^-, 2^+, 3^-, 4^+, 5^+)$	12.5000000	25.46246919	-6.121629624	-90.22184214	-115.7836685

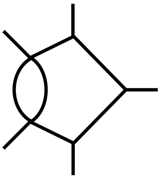
Scaling properties of gluon amplitudes



- Polynomial complexity to compute color-ordered amplitudes
- Asymptotic regime only for very large n at 1 and 2 loops
- Good initial benchmarking for two-loop five-point amplitudes

Outlook

- Multi-loop numerical unitarity appears as a robust and flexible method to tackle two-loop calculations relevant for phenomenology
- We presented results for 4- and 5-gluon helicity amplitudes
- As numerical unitarity methods are less sensitive to the presence of multiple scales, we expect to study more generic 5-point amplitudes and beyond
- Having exact numerical results can allow the study of analytic properties of the amplitudes



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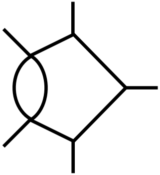

Thanks!