Integration-by-parts reductions via algebraic geometry

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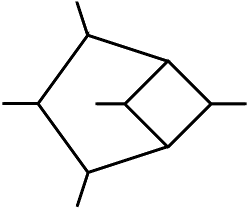

Amplitudes 2018

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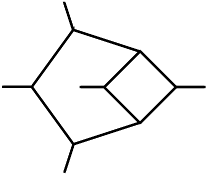
Based on PRD **93**(2016)041701, 1712.09737 and 1805.01873
with J. Böhm, A. Georgoudis, H. Schönemann, M. Schulze, Y. Zhang

Overview

1. Motivation
2. IBP identities on unitarity cuts
3. Syzygy equations and their solution
4. Main example: 

Integration-by-parts reductions

IBP identities arise from the vanishing integration of total derivatives,

[Chertyrkin, Tchakov, Nucl. Phys. B **192**, 159 (1981)]

$$\int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\pi^{D/2}} \sum_{j=1}^{L} \frac{\partial}{\partial \ell_j^\mu} \frac{v_j^\mu P}{D_1^{a_1} \cdots D_k^{a_k}} = 0$$

where P and v_j^μ are polynomials in ℓ_i, p_j, and $a_i \in \mathbb{N}$.

Role in perturbative QFT calculations:

- **Reduction.** Reduce number of contributing loop integrals by factor of $\mathcal{O}(10^2) - \mathcal{O}(10^6)$ to basis.
- **Computing master integrals.** Enable setting up differential equations for basis integrals $\mathcal{I}_j$:

  [Gehrmann and Remiddi, Nucl. Phys. B **580**, 485 (2000)]

  $$\frac{\partial}{\partial x_m} \boldsymbol{\mathcal{I}}(\boldsymbol{x}, \epsilon) = A_m(\boldsymbol{x}, \epsilon) \boldsymbol{\mathcal{I}}(\boldsymbol{x}, \epsilon)$$

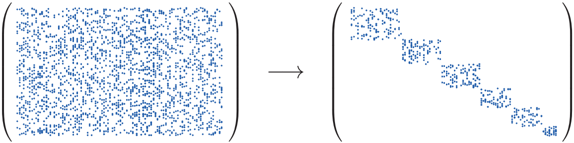
  where x_m denotes a kinematical invariant.

IBP reductions on unitarity cuts

Standard approach: enumerate all linear relations and apply
Gauss-Jordan elimination to *large* linear systems

[Laporta, Int.J.Mod.Phys. A **15** (2000) 5087-5159]

Idea here: use *unitarity cuts* to block-diagonalize system



We use the Baikov representation $\left(k = \frac{L(L+1)}{2} + LE\right)$,

$$I(N;a) \equiv \int \prod_{j=1}^{L} \frac{\mathrm{d}^D \ell_j}{i\pi^{D/2}} \frac{N}{D_1^{a_1} \cdots D_k^{a_k}} = \int \frac{\mathrm{d}z_1 \cdots \mathrm{d}z_k}{z_1^{a_1} \cdots z_k^{a_k}} \underset{(\widehat{\boldsymbol{p}},\boldsymbol{\ell})}{\mathrm{Gram}}(z)^{\frac{D-L-E-1}{2}} N$$

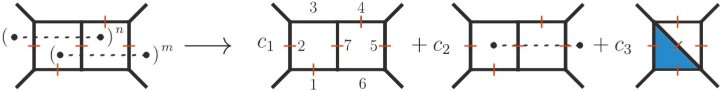
[Baikov, Phys.Lett. B **385** (1996) 404-410]

in which cuts are straightforward to apply,

$$\int \frac{\mathrm{d}z_i}{z_i^{a_i}} \xrightarrow{\text{cut}} \oint_{\Gamma_\epsilon(0)} \frac{\mathrm{d}z_i}{z_i^{a_i}} \qquad\qquad i \in \mathcal{S}_{\text{cut}}$$

Example: Zurich-flag cut

Let us construct IBP identities on the Zurich-flag cut



Define $S_{\text{cut}} = \{1, 2, 4, 5, 7\}$ and $G = \text{Gram}_{(\widehat{\boldsymbol{p}},\boldsymbol{\ell})}$.

On S_{cut}, the double-box integral takes the form

$$I_{\text{cut}}^{\text{DB}}[P] = \prod_{i \in S_{\text{cut}}} \oint_{\Gamma_\epsilon(0)} \frac{\mathrm{d}\widetilde{z}_i}{\widetilde{z}_i} \int \prod_{j \notin S_{\text{cut}}} \mathrm{d}\widetilde{z}_j \frac{G(\widetilde{\boldsymbol{z}})^{\frac{D-6}{2}}}{\widetilde{z}_3\, \widetilde{z}_6} P(\widetilde{\boldsymbol{z}})$$

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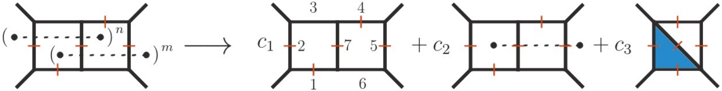

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Relabeling $z_{\{1,2,3,4\}} = \widetilde{z}_{\{3,6,8,9\}}$, this becomes

$$I^{\text{DB}}_{\text{cut}}[P] \;=\; \int \frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4}{z_1 z_2} G(\boldsymbol{z})^{\frac{D-6}{2}} P(\boldsymbol{z})$$

Generic total derivative on cut

Need to find IBP identities which involve

$$I_{\text{cut}}^{\text{DB}}[P] \;=\; \int \frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4}{z_1 z_2}\, G(\boldsymbol{z})^{\frac{D-6}{2}}\, P(\boldsymbol{z})$$

Total derivatives $\longrightarrow$ IBP identities. Generic total derivative on cut:

$$\begin{aligned} 0 \;&=\; \int \left[\sum_{i=1}^{4} \frac{\partial}{\partial z_i} \left(\frac{a_i(\boldsymbol{z}) G(\boldsymbol{z})^{\frac{D-6}{2}}}{z_1 z_2} \right) \right] \mathrm{d}z_1 \cdots \mathrm{d}z_4 \\ &=\; \int \left[\sum_{i=1}^{4} \left(\frac{\partial a_i}{\partial z_i} + \frac{D-6}{2G} a_i \frac{\partial G}{\partial z_i} \right) - \sum_{j=1,2} \frac{a_j}{z_j} \right] \frac{G(\boldsymbol{z})^{\frac{D-6}{2}}}{z_1 z_2} \mathrm{d}z_1 \cdots \mathrm{d}z_4 \end{aligned}$$

The red term corresponds to an integral in $(D-2)$ dimensions, and the purple term in general produces doubled propagators.

IBP identities from syzygies

To avoid dimension shifts and doubled propagators in

$$0 = \int \Big[\sum_{i=1}^{4} \Big(\frac{\partial a_i}{\partial z_i} + \frac{D-6}{2G} a_i \frac{\partial G}{\partial z_i} \Big) - \sum_{j=1,2} \frac{a_j}{z_j} \Big] \frac{G(\boldsymbol{z})^{\frac{D-6}{2}}}{z_1 z_2} \mathrm{d}z_1 \cdots \mathrm{d}z_4$$

we demand that each term is *polynomial,*

$$\sum_{i=1}^{4} a_i \frac{\partial G}{\partial z_i} + bG = 0$$

$$a_j + b_j z_j = 0$$

with a_i, b_i, b polynomials in z. Such eqs. are known as *syzygy equations*.

[Gluza, Kajda, Kosower, PRD**83**(2011)045012], [Schabinger, JHEP**01**(2012)077], [Ita, PRD**94**(2016)116015]

Obtain IBPs by plugging (a_i, b) into the top equation.
Note: (qa_i, qb) is also a solution, for polynomial q.

Strategy to solve syzygy equations

Solve syzygy equations with c cuts

$$a_j + b_j z_j = 0\,, \quad j = 1, \ldots, k-c \tag{1}$$

$$\sum_{j=1}^{m-c} a_j \frac{\partial G}{\partial z_k} + bG = 0 \tag{2}$$

as follows.

1) The generators of (1) are trivial:

$$\mathcal{M}_2 = \langle z_1 \mathbf{e}_1, \ldots, z_k \mathbf{e}_k, \mathbf{e}_{k+1}, \ldots, \mathbf{e}_m \rangle$$

2) Generators $\mathcal{M}_1 = \Big\langle (a_1, \ldots, a_m, b), \ \ldots \Big\rangle$ of (2) for the *off-shell case* $c = 0$ **can be explicitly found:**

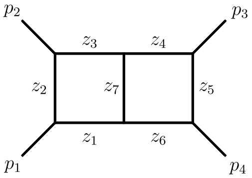
$$(a_\alpha, b) = \left(\sum_{k=1}^{E+L} (1+\delta_{ik}) x_{jk} \frac{\partial z_\alpha}{\partial x_{ik}}, \ 2\delta_{ij} \right)$$

where $x_{ij} = v_i \cdot v_j$ with $v_{i,j} \in \{p_1, \ldots, p_E, \ell_1, \ldots, \ell_L\}$.

[Böhm, Georgoudis, KJL, Schulze, Zhang, 1712.09737]

3) Take module intersection $\mathcal{M}_1\big|_{\text{cut}} \cap \mathcal{M}_2\big|_{\text{cut}}$

Example 1: syzygies of planar double box



Set $P_{12} = p_1 + p_2$ and

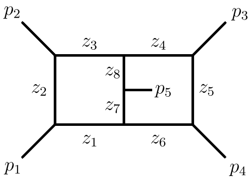
$$z_1 = \ell_1^2\,, \qquad z_2 = (\ell_1 - p_1)^2\,, \qquad z_3 = (\ell_1 - P_{12})^2$$
$$z_4 = (\ell_2 + P_{12})^2\,, \qquad z_5 = (\ell_2 - p_4)^2\,, \qquad z_6 = \ell_2^2$$
$$z_7 = (\ell_1 + \ell_2)^2\,, \qquad z_8 = (\ell_1 + p_4)^2\,, \qquad z_9 = (\ell_2 + p_1)^2$$

Only need to find explicit relation $z = Ax$. Here $A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ -2 & 0 & -2 & 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Set $t_{i,j} = (a_\alpha, b)$. The syzygy generators are *linear* in the z_k

$$t_{4,1} = (z_1-z_2,\ z_1-z_2,\ -s+z_1-z_2,\ 0,\ 0,\ 0,\ z_1-z_2-z_6+z_9,\ t+z_1-z_2,\ 0,\ 0)$$
$$t_{4,2} = (s+z_2-z_3,\ z_2-z_3,\ z_2-z_3,\ 0,\ 0,\ 0,\ z_2-z_3+z_4-z_9,\ -t+z_2-z_3,\ 0,\ 0)$$
$$t_{4,3} = (-s+z_3-z_8,\ t+z_3-z_8,\ z_3-z_8,\ 0,\ 0,\ 0,\ z_3-z_4+z_5-z_8,\ z_3-z_8,\ 0,\ 0)$$
$$t_{4,4} = (2z_1,\ z_1+z_2,\ -s+z_1+z_3,\ 0,\ 0,\ 0,\ z_1-z_6+z_7,\ z_1+z_8,\ 0,\ -2)$$
$$t_{4,5} = (-z_1-z_6+z_7,\ -z_1+z_7-z_9,\ s-z_1-z_4+z_7, 0,\ 0,\ 0,\ -z_1+z_6+z_7,\ -z_1-z_5+z_7,\ 0,\ 0)$$
$$t_{5,1} = (0,\ 0,\ 0,\ s-z_6+z_9,\ -t-z_6+z_9,\ z_9-z_6,\ z_1-z_2-z_6+z_9,\ 0,\ z_9-z_6,\ 0)$$
$$t_{5,2} = (0,\ 0,\ 0,\ z_4-z_9,\ t+z_4-z_9,\ -s+z_4-z_9,\ z_2-z_3+z_4-z_9,\ 0,\ z_4-z_9,\ 0)$$
$$t_{5,3} = (0,\ 0,\ 0,\ z_5-z_4,\ z_5-z_4,\ s-z_4+z_5,\ z_3-z_4+z_5-z_8,\ 0,\ -t-z_4+z_5,\ 0)$$
$$t_{5,4} = (0,\ 0,\ 0,\ s-z_3-z_6+z_7,\ -z_6+z_7-z_8,\ -z_1-z_6+z_7,\ z_1-z_6+z_7,\ 0,\ -z_2-z_6+z_7,\ 0)$$
$$t_{5,5} = (0,\ 0,\ 0,\ -s+z_4+z_6,\ z_5+z_6,\ 2z_6,\ -z_1+z_6+z_7,\ 0,\ z_6+z_9,\ -2)$$

Example 2: syzygies of non-planar double pentagon



Set $P_{i,j} \equiv p_i + p_j$ and

$$
\begin{aligned}
z_1 &= \ell_1^2\,, & z_2 &= (\ell_1 - p_1)^2\,, & z_3 &= (\ell_1 - P_{1,2})^2\,,\\
z_4 &= (\ell_2 - P_{3,4})^2\,, & z_5 &= (\ell_2 - p_4)^2\,, & z_6 &= \ell_2^2\,,\\
z_7 &= (\ell_1 + \ell_2)^2\,, & z_8 &= (\ell_1 + \ell_2 + p_5)^2\,, & z_9 &= (\ell_1 + p_3)^2\,,\\
z_{10} &= (\ell_1 + p_4)^2\,, & z_{11} &= (\ell_2 + p_1)^2 & &
\end{aligned}
$$

Here $z = Ax$ with

$$
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
-2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -2 & 0 & -2 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1\\
-2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & 1 & 2 & 1\\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0\\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

and the syzygy generators are again compact:

$$
\begin{aligned}
t_{5,1} = (&z_1-z_2,\, z_1-z_2,\, -s_{1,2}+z_1-z_2,\, 0,\, 0,\, 0,\\
&z_1-z_2-z_6+z_{11},\, -s_{1,2}-s_{1,3}-s_{1,4}+z_1-z_2-z_6+z_{11},\\
&s_{1,3}+z_1-z_2,\, s_{1,4}+z_1-z_2,\, 0,\, 0)\,,\\
t_{5,2} = (&s_{1,2}+z_2-z_3,\, z_2-z_3,\, z_2-z_3,\, 0,\, 0,\, 0,\\
&-s_{3,4}+z_1+z_2+z_4+z_7-z_8-z_9-z_{10}-z_{11},\\
&s_{1,3}+s_{1,4}+z_1+z_2+z_4+z_7-z_8-z_9-z_{10}-z_{11},\\
&s_{1,2}+s_{2,3}+z_2-z_3,\, -s_{1,3}-s_{1,4}-s_{2,3}-s_{3,4}+z_2-z_3,\, 0,\, 0)\,,\\
t_{5,3} = (&z_9-z_1,\, -s_{1,3}-z_1+z_9,\, -s_{1,3}-s_{2,3}-z_1+z_9,\, 0,\, 0,\, 0,\\
&s_{3,4}-z_1-z_4+z_5+z_9,\, -s_{1,3}-s_{2,3}-z_1-z_4+z_5+z_9,\\
&z_9-z_1,\, s_{3,4}-z_1+z_9,\, 0,\, 0)\,,\\
t_{5,4} = (&z_{10}-z_1,\, -s_{1,4}-z_1+z_{10},\, -s_{1,4}-s_{2,4}-z_1+z_{10},\\
&0,\, 0,\, 0,\, -z_1-z_5+z_6+z_{10},\\
&s_{1,2}+s_{1,3}+s_{2,3}-z_1-z_5+z_6+z_{10},\\
&s_{3,4}-z_1+z_{10},\, z_{10}-z_1,\, 0,\, 0)\,,\\
t_{5,5} = (&2z_1,\, z_1+z_2,\, -s_{1,2}+z_1+z_3,\, 0,\, 0,\, 0,\\
&z_1-z_6+z_7,\, -s_{1,2}+2z_1+z_3-z_6+z_7-z_9-z_{10},\\
&z_1+z_9,\, z_1+z_{10},\, 0,\, -2)\,,\\
t_{5,6} = (&-z_1-z_6+z_7,\, -z_1+z_7-z_{11},\\
&s_{1,2}+s_{3,4}-2z_1-z_3-z_4+z_8+z_9+z_{10},\, 0,\, 0,\, 0,\\
&-z_1+z_6+z_7,\, s_{1,2}-2z_1-z_3+z_6+z_8+z_9+z_{10},\\
&s_{3,4}-z_1-z_4+z_5-z_6+z_7,\, -z_1-z_5+z_7,\, 0,\, 0)\,, \qquad (6.11)\\
t_{6,1} = (&0,\, 0,\, 0,\, -s_{1,3}-s_{1,4}-z_6+z_{11},\, -s_{1,4}-z_6+z_{11},\\
&z_{11}-z_6,\, z_1-z_2-z_6+z_{11},\\
&-s_{1,2}-s_{1,3}-s_{1,4}+z_1-z_2-z_6+z_{11},\, 0,\, 0,\, z_{11}-z_6,\, 0)\,,\\
t_{6,2} = (&0, 0, 0, s_{1,3}+s_{1,4}+z_1+z_3+z_4+z_7-z_8-z_9-z_{10}-z_{11},\\
&s_{1,3}+s_{1,4}+s_{2,3}+z_1+z_3+z_4+z_7-z_8-z_9-z_{10}-z_{11},\\
&-s_{1,2}-s_{3,4}+z_1+z_3+z_4+z_7-z_8-z_9-z_{10}-z_{11},\\
&-s_{3,4}+z_1+z_2+z_4+z_7-z_8-z_9-z_{10}-z_{11},\\
&s_{1,3}+s_{1,4}+z_1+z_2+z_4+z_7-z_8-z_9-z_{10}-z_{11},\, 0,\, 0,\\
&-s_{3,4}+z_1+z_3+z_4+z_7-z_8-z_9-z_{10}-z_{11},\, 0)\,,\\
t_{6,3} = (&0,\, 0,\, 0,\, z_5-z_4,\, z_5-z_4,\, s_{3,4}-z_4+z_5,\\
&s_{3,4}-z_1-z_4+z_5+z_9,\, -s_{1,3}-s_{2,3}-z_1-z_4+z_5+z_9,\, 0,\, 0,\\
&s_{1,3}+s_{3,4}-z_4+z_5,\, 0)\,,\\
t_{6,4} = (&0,\, 0,\, 0,\, -s_{3,4}-z_5+z_6,\, z_6-z_5,\, z_6-z_5,\\
&-z_1-z_5+z_6+z_{10},\, s_{1,2}+s_{1,3}+s_{2,3}-z_1-z_5+z_6+z_{10},\, 0,\, 0,\\
&s_{1,4}-z_5+z_6,\, 0)\,,\\
t_{6,5} = (&0,\, 0,\, 0,\, z_1-z_6+z_7-z_9-z_{10},\, -z_6+z_7-z_{10},\\
&-z_1-z_6+z_7,\, z_1-z_6+z_7,\, -s_{1,2}+2z_1+z_3-z_6+z_7-z_9-z_{10},\\
&0,\, 0,\, -z_2-z_6+z_7,\, 0)\,,\\
t_{6,6} = (&0,\, 0,\, 0,\, -s_{3,4}+z_4+z_6,\, z_5+z_6,\, 2z_6,\, -z_1+z_6+z_7,\\
&s_{1,2}-2z_1-z_3+z_6+z_8+z_9+z_{10},\, 0,\, 0,\, z_6+z_{11},\, -2)\,.
\end{aligned}
$$

Computing module intersections

Given $\mathcal{M}_1 = \langle v_1, \ldots, v_p \rangle$ and $\mathcal{M}_2 = \langle w_1, \ldots, w_q \rangle$ with v_i, w_j m-tuples of polynomials. Let Q denote the $m \times (p+q)$ matrix

$$Q = \begin{pmatrix} \vdots & & \vdots & \vdots & & \vdots \\ v_1 & \cdots & v_p & w_1 & \cdots & w_q \\ \vdots & & \vdots & \vdots & & \vdots \end{pmatrix}$$

Then compute wrt. POT and variable order $[z_1, \ldots, z_m] \succ [s_{ij}]$

$$\langle h_1, \ldots, h_t \rangle \equiv \text{Gröbner basis of column space of } \begin{pmatrix} & Q & \\ \hline 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Selecting $h_i = (\overbrace{0, \ldots, 0}^{m}, x_1, \ldots, x_p, y_1, \ldots, y_q)$, we have

$$0 = \sum_{j=1}^{p} x_j v_j + \sum_{k=1}^{q} y_k w_k$$

Computing module intersections

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$$Q = \begin{pmatrix} \vdots & & \vdots & \vdots & & \vdots \\ v_1 & \cdots & v_p & w_1 & \cdots & w_q \\ \vdots & & \vdots & \vdots & & \vdots \end{pmatrix}$$

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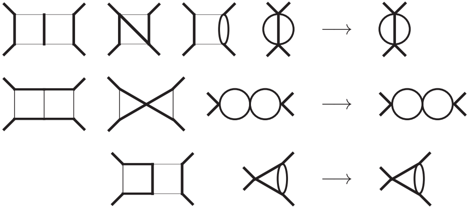
Selecting $h_i = (\overbrace{0, \ldots, 0}^{m}, x_1, \ldots, x_p, y_1, \ldots, y_q)$, we have

$$0 = \sum_{j=1}^{p} x_j v_j + \sum_{k=1}^{q} y_k w_k \quad \Longrightarrow \quad \sum_{j=1}^{p} x_j v_j = -\sum_{k=1}^{q} y_k w_k \in \mathcal{M}_1 \cap \mathcal{M}_2$$

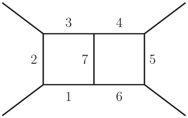
Hence $\sum_{j=1}^{p} x_j v_j$ generate $\mathcal{M}_1 \cap \mathcal{M}_2$, taking $(x_1, \ldots, x_p)$ from each h_i.

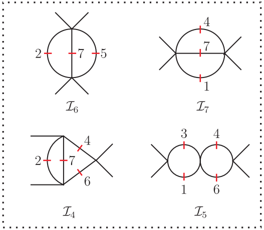
Spanning set of cuts for IBPs

To find the complete IBP reduction, we must consider the cuts associated with "uncollapsible" masters:



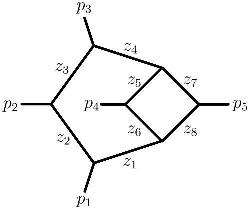
A bit more explicitly, the cuts we need to consider are



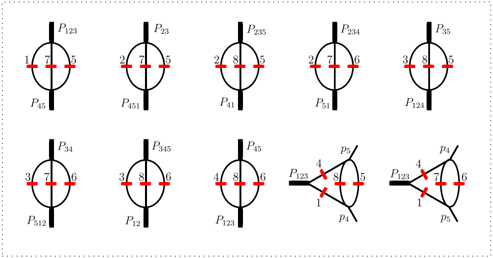


Task: IBP reduce non-planar hexagon box with numerator insertions of degree four in the z_i

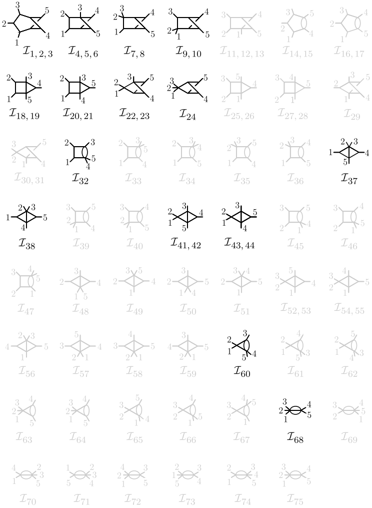
[Chicherin, Henn, Mitev JHEP **05**(2018)164]
[Badger, Brønnum-Hansen, Hartanto, Peraro, PRL **120**(2018)092001]
[Abreu, Cordero, Ita, Page, Zeng, 1712.03946]
[Chawdhry, Lim, Mitov, 1805.09182]



There are 10 cuts to consider:

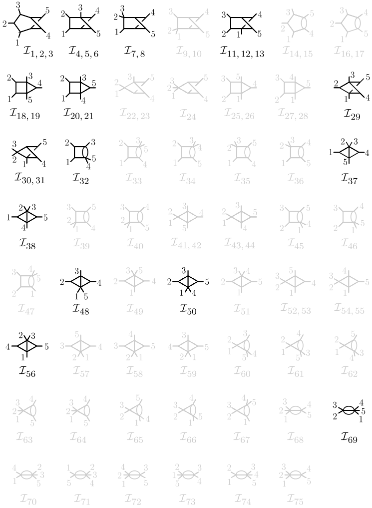


Construct and solve IBP identities on a spanning set of cuts.



Cut $\{1,5,7\}$

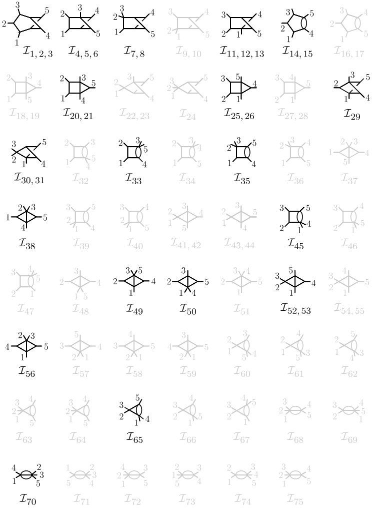
Construct and solve IBP identities on a spanning set of cuts.



Cut $\{2, 5, 7\}$

Non-planar hexagon box: spanning set of cuts

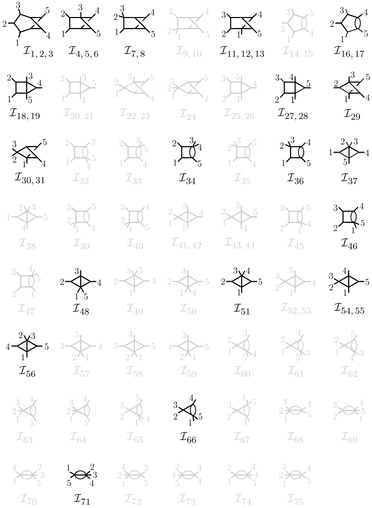
Construct and solve IBP identities on a spanning set of cuts.



Cut $\{2, 5, 8\}$

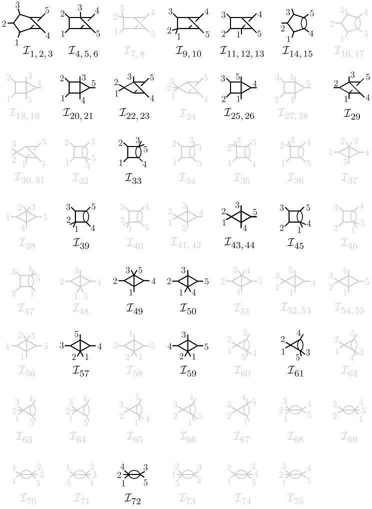
Non-planar hexagon box: spanning set of cuts

Construct and solve IBP identities on a spanning set of cuts.



Cut $\{2, 6, 7\}$

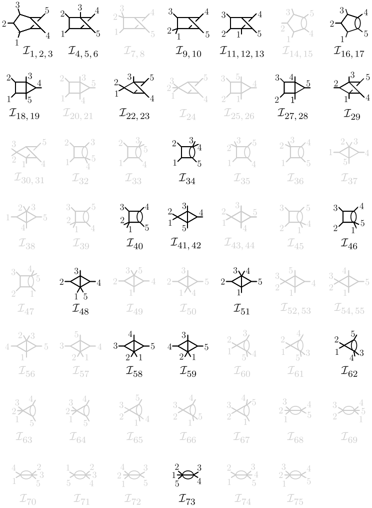
Construct and solve IBP identities on a spanning set of cuts.



Cut $\{3, 5, 8\}$

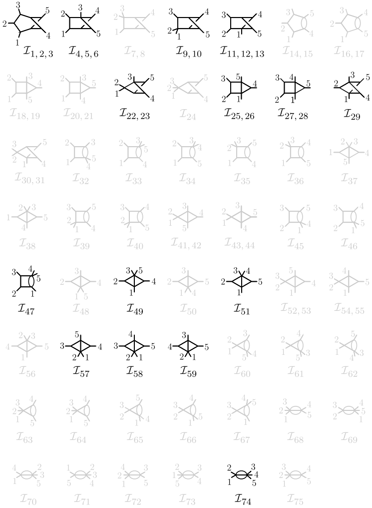
Non-planar hexagon box: spanning set of cuts

Construct and solve IBP identities on a spanning set of cuts.



Cut {3, 6, 7}

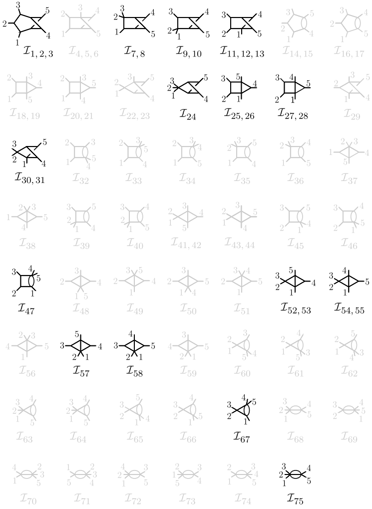
Construct and solve IBP identities on a spanning set of cuts.



Cut $\{3, 6, 8\}$

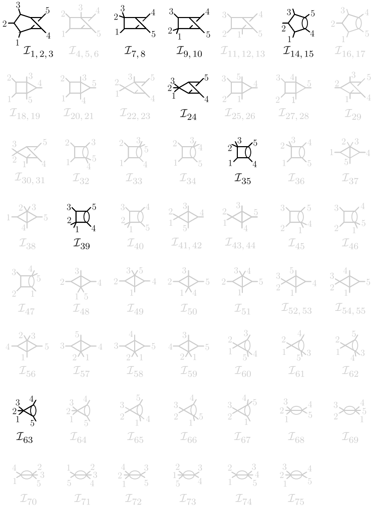
Non-planar hexagon box: spanning set of cuts

Construct and solve IBP identities on a spanning set of cuts.



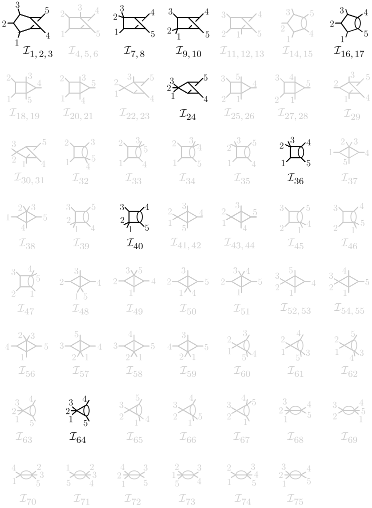
Cut $\{4, 6, 8\}$

Construct and solve IBP identities on a spanning set of cuts.



Cut $\{1, 4, 5, 8\}$

Construct and solve IBP identities on a spanning set of cuts.



Cut $\{1, 4, 6, 7\}$

Syzygies for the non-planar hexagon box

Syzygies for ensuring D-dimensionality:

$$
\begin{aligned}
M_1 = \langle &(z_1 - z_2, z_1 - z_2, -s_{12} + z_1 - z_2, -s_{12} - s_{13} + z_1 - z_2, s_{14} + z_1 - z_2 - z_8 + z_{10}, z_1 - z_2 - z_8 + z_{10}, 0, 0, -s_{12} - s_{13} - s_{14} + z_1 - z_2, 0, 0) \\
&(0,0,0,0, s_{14} + z_1 - z_2 - z_8 + z_{10}, z_1 - z_2 - z_8 + z_{10}, s_{12} + s_{13} + s_{14} - z_8 + z_{10}, z_{10} - z_8, 0, z_{10} - z_8, s_{12} - z_8 + z_{10}) \\
&(s_{12} + z_2 - z_3, z_2 - z_3, z_2 - z_3, -s_{23} + z_2 - z_3, s_{12} + s_{24} + z_2 - z_3 - z_8 + z_{11}, s_{12} + z_2 - z_3 - z_8 + z_{11}, 0, 0, -s_{23} - s_{24} + z_2 - z_3, 0, 0) \\
&(0,0,0,0, s_{12} + s_{24} + z_2 - z_3 - z_8 + z_{11}, s_{12} + z_2 - z_3 - z_8 + z_{11}, s_{12} + s_{23} + s_{24} - z_8 + z_{11}, z_{11} - z_8, 0, s_{12} - z_8 + z_{11}, z_{11} - z_8) \\
&(s_{13} + s_{23} + z_3 - z_4, s_{23} + z_3 - z_4, z_3 - z_4, z_3 - z_4, -2s_{12} - s_{13} - s_{14} - s_{23} - s_{24} + z_3 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, \\
&\quad -s_{12} + z_3 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, 0, 0, s_{12} + s_{13} + s_{14} + s_{23} + s_{24} + z_3 - z_4, 0, 0) \\
&(0,0,0,0, -2s_{12} - s_{13} - s_{14} - s_{23} - s_{24} + z_3 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, -s_{12} + z_3 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, \\
&\quad -2s_{12} - s_{13} - s_{14} - s_{23} - s_{24} + z_4 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, -s_{12} - s_{13} - s_{23} + z_4 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, \\
&\quad 0, -s_{12} - s_{23} + z_4 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, -s_{12} - s_{13} + z_4 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}) \\
&(-s_{12} - s_{13} - s_{23} + z_4 - z_9, -s_{12} - s_{13} - s_{14} - s_{23} + z_4 - z_9, -s_{12} - s_{13} - s_{14} - s_{23} - s_{24} + z_4 - z_9, z_4 - z_9, z_5 - z_6, z_5 - z_6, 0, 0, z_4 - z_9, 0, 0) \\
&(0,0,0,0, z_5 - z_6, z_5 - z_6, -z_4 + z_5 - z_6 + z_9, s_{12} + s_{13} + s_{23} - z_4 + z_5 - z_6 + z_9, \\
&\quad 0, s_{12} + s_{13} + s_{14} + s_{23} - z_4 + z_5 - z_6 + z_9, s_{12} + s_{13} + s_{23} + s_{24} - z_4 + z_5 - z_6 + z_9) \\
&(2z_1, z_1 + z_2, -s_{12} + z_1 + z_3, -s_{12} - s_{13} - s_{23} + z_1 + z_4, -s_{12} - s_{13} - s_{23} + z_1 + z_4 + z_6 - z_8 - z_9, z_1 + z_6 - z_8, 0, 0, z_1 + z_9, 0, 0) \\
&(0,0,0,0, -s_{12} - s_{13} - s_{23} + z_1 + z_4 + z_6 - z_8 - z_9, z_1 + z_6 - z_8, z_6 - z_8 - z_9, -z_1 + z_6 - z_8, 0, -z_2 + z_6 - z_8, s_{12} - z_1 + z_2 - z_3 + z_6 - z_8) \\
&(-z_1 + z_6 - z_8, -z_1 + z_6 - z_{10}, -z_1 + z_6 + z_8 - z_{10} - z_{11}, s_{12} + s_{13} + s_{23} - z_1 - z_4 + z_5 - z_7 + z_9, \\
&\quad s_{12} + s_{13} + s_{23} - z_1 - z_4 + z_5 + z_8 + z_9, -z_1 + z_6 + z_8, 0, 0, -z_1 + z_6 - z_7, 0, 0) \\
&(0,0,0,0, s_{12} + s_{13} + s_{23} - z_1 - z_4 + z_5 + z_8 + z_9, -z_1 + z_6 + z_8, z_7 + z_8, 2z_8, 0, z_8 + z_{10}, z_8 + z_{11}) \rangle
\end{aligned}
\tag{5.9}
$$

Syzygies for ensuring no doubled propagators:

$$
\begin{aligned}
M_2 = \langle &(z_1,0,0,0,0,0,0,0,0,0,0), (0,z_2,0,0,0,0,0,0,0,0,0) \\
&(0,0,z_3,0,0,0,0,0,0,0,0), (0,0,0,z_4,0,0,0,0,0,0,0) \\
&(0,0,0,0,z_5,0,0,0,0,0,0), (0,0,0,0,0,z_6,0,0,0,0,0) \\
&(0,0,0,0,0,0,z_7,0,0,0,0), (0,0,0,0,0,0,0,z_8,0,0,0) \\
&(0,0,0,0,0,0,0,0,1,0,0), (0,0,0,0,0,0,0,0,0,1,0) \\
&(0,0,0,0,0,0,0,0,0,0,1) \rangle
\end{aligned}
$$

Compute intersection of $M_1\big|_{\text{cut}} \cap M_2\big|_{\text{cut}}$ on each of the 10 cuts.

Complexity of IBP systems

- Timing to compute $M_1\big|_{\text{cut}} \cap M_2\big|_{\text{cut}}$: **25-800 s**
  (on 24 cores, 3.40 GHz)

- Size of generating systems after trimming: **1.5-10 MB**

  Plug resulting generators into ansatz for total derivative:

$$0 = \int \left[\sum_{i=1}^{m-c} \left(\frac{\partial a_{r_i}}{\partial z_{r_i}} + \frac{D-L-E-1}{2G(z)} a_{r_i} \frac{\partial G}{\partial z_{r_i}} \right) - \sum_{i=1}^{k-c} \frac{a_{r_i}}{z_{r_i}} \right] \frac{G(z)^{\frac{D-L-E-1}{2}}}{z_{r_1} \cdots z_{r_{k-c}}} \mathrm{d}z_{r_1} \cdots \mathrm{d}z_{r_{m-c}}$$

- Resulting linear systems to solve:
  700-1200 equations, size **1 MB**, density **1.5%**

Gauss-Jordan elimination of IBP systems

To find the IBP reductions, Gauss-Jordan eliminate IBP systems.

Some remarks:

- To preserve sparsity, use a *total pivoting* strategy (i.e., allow column swaps)
- For cut $\{1, 4, 6, 7\}$, the RREF can be performed fully analytically, requiring 31 minutes on one core and 1.5 GB RAM.

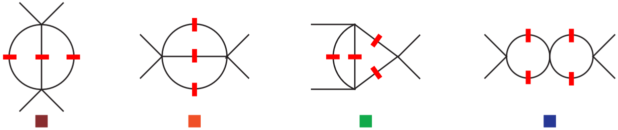
  ⋮

- For $\{3, 6, 7\}$, assigned numerical values to two s_{ij}.
  Ran 440 points on cluster (2.5 h and 1.8 GB RAM per job).
  Used interpolation code to get analytical results (23 min and 15 GB RAM on one core).

[von Manteuffel and Schabinger, PLB **744**(2015)101]
[Peraro, JHEP**12**(2016)030]

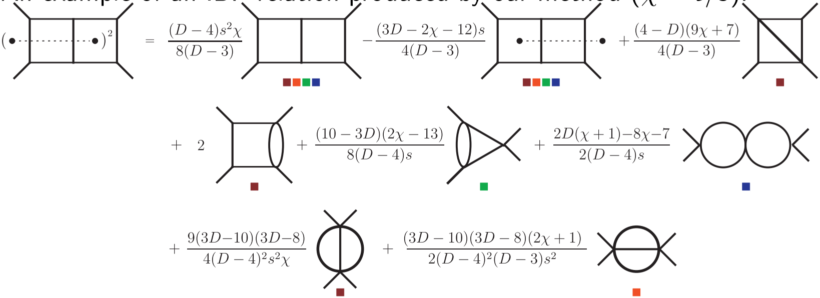
Merging on-shell IBP reductions

By solving the IBP identities on the following cuts



we reconstruct the *complete IBP reductions* by merging the partial results.

An example of an IBP relation produced by our method ($\chi \equiv t/s$):



Results for IBP reductions

- Fully analytic IBP reductions of the 32 hexagon boxes

$$
\begin{array}{lll}
\{I(1,1,1,1,1,1,1,1,0,0,-4), & I(1,1,1,1,1,1,1,1,0,-1,-3), & I(1,1,1,1,1,1,1,1,0,-2,-2) \\
I(1,1,1,1,1,1,1,1,0,-3,-1), & I(1,1,1,1,1,1,1,1,0,-4,0), & I(1,1,1,1,1,1,1,1,-1,0,-3) \\
I(1,1,1,1,1,1,1,1,-1,-1,-2), & I(1,1,1,1,1,1,1,1,-1,-2,-1), & I(1,1,1,1,1,1,1,1,-1,-3,0) \\
I(1,1,1,1,1,1,1,1,-2,0,-2), & I(1,1,1,1,1,1,1,1,-2,-1,-1), & I(1,1,1,1,1,1,1,1,-2,-2,0) \\
I(1,1,1,1,1,1,1,1,-3,0,-1), & I(1,1,1,1,1,1,1,1,-3,-1,0), & I(1,1,1,1,1,1,1,1,-4,0,0) \\
I(1,1,1,1,1,1,1,1,0,0,-3), & I(1,1,1,1,1,1,1,1,0,-1,-2), & I(1,1,1,1,1,1,1,1,0,-2,-1) \\
I(1,1,1,1,1,1,1,1,0,-3,0), & I(1,1,1,1,1,1,1,1,-1,0,-2), & I(1,1,1,1,1,1,1,1,-1,-1,-1) \\
I(1,1,1,1,1,1,1,1,-1,-2,0), & I(1,1,1,1,1,1,1,1,-2,0,-1), & I(1,1,1,1,1,1,1,1,-2,-1,0) \\
I(1,1,1,1,1,1,1,1,-3,0,0), & I(1,1,1,1,1,1,1,1,0,0,-2), & I(1,1,1,1,1,1,1,1,0,-1,-1) \\
I(1,1,1,1,1,1,1,1,0,-2,0), & I(1,1,1,1,1,1,1,1,-1,0,-1), & I(1,1,1,1,1,1,1,1,-1,-1,0) \\
I(1,1,1,1,1,1,1,1,0,0,-1), & I(1,1,1,1,1,1,1,1,0,-1,0)\} &
\end{array}
$$

can be downloaded from (268 MB compressed / 790 MB uncompressed)

https://github.com/yzhphy/hexagonbox_reduction/releases/download/1.0.0/hexagon_box_degree_4_Final.zip

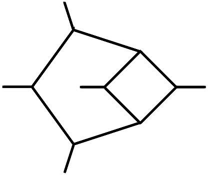
- Our results agree with fully numerical results from FIRE5 C++ (6 hours per point).

[A. Smirnov, CPC **189**(2015)182]

Conclusions

- New formalism for IBP reductions. Main ideas: cuts, IBP identities from syzygies, total pivoting, rational reconstruction

- Obtained the **fully analytic** IBP reductions of



  with numerator insertions up to degree 4 in the z_i.

- Powerful framework. IBP reductions for further $2 \to 3$ two-loop processes **seem well within reach**.