Exact scattering amplitudes in $\gamma-$deformed $\mathcal{N}=4$ SYM / fishnet theory

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Why to deform $\mathcal{N}=4$ SYM?

✔ Scattering amplitudes have remarkable properties in $\mathcal{N}=4$ SYM ... however

- ✗ Some of symmetries are broken by IR divergences
- ✗ Difficult to obtain analytical expressions valid for any $\lambda = g^2_{\rm YM} N_c$
- ✗ The origin of integrability remains obscure

✔ Deform $\mathcal{N}=4$ SYM by preserving integrability in such a way that

- ✗ Scattering amplitude are IR finite
- ✗ Can be computed analytically for any 't Hooft coupling

✔ Simplified model: strongly twisted $\mathcal{N}=4$ SYM / fishnet theory

- ✗ A non-unitary CFT dominated by fishnet graphs
- ✗ Integrable in the planar limit, related to conformal $SU(2,2)$ spin chain

This talk: *compute exactly 4-particle scattering amplitudes in this theory*

Strongly twisted $\mathcal{N}=4$ SYM

$$L = -\frac{1}{4}F_{\mu\nu}^2 + D^\mu\phi_i^\dagger D_\mu\phi^i + i\bar\psi_A D\!\!\!/\psi^A + L_{\rm int}$$ [Leigh,Strassler][Frolov]

$$L_{\rm int} = g^2\Big(\frac{1}{4}\{\phi_i^\dagger,\phi^i\}\{\phi_j^\dagger,\phi^j\} - {\rm e}^{-i\epsilon^{ijk}\gamma_k}\,\phi_i^\dagger\phi_j^\dagger\phi^i\phi^j$$
$$- \,{\rm e}^{-\frac{i}{2}\gamma_j^-}\,\bar\psi_j\phi^j\bar\psi_4 + {\rm e}^{\frac{i}{2}\gamma_j^-}\,\bar\psi_4\phi^j\bar\psi_j + i\epsilon_{ijk}\,{\rm e}^{\frac{i}{2}\epsilon_{jkm}\gamma_m^+}\,\bar\psi^k\phi^i\bar\psi^j + \text{c.c.}\Big)$$

Deformation parameters $\gamma_1, \gamma_2, \gamma_3$

- ✔ Is expected to be integrable in the planar limit
- ✔ Double scaling limit: strong twist $+$ weak coupling [Gurdogan,Kazakov]

$$\gamma_{1,2} = \text{fixed}\,, \qquad \gamma_3 \to +i\,\infty\,, \qquad \xi^2 = g^2\,{\rm e}^{-i\gamma_3} = \text{fixed} \quad + \quad g^2 \to 0$$

Gauge field, fermions and one scalar decouple

$$L = {\rm tr}\left[\partial^\mu\phi_1^\dagger\partial_\mu\phi_1 + \partial^\mu\phi_2^\dagger\partial_\mu\phi_2 + \xi^2\phi_1^\dagger\phi_2^\dagger\phi_1\phi_2\right]$$

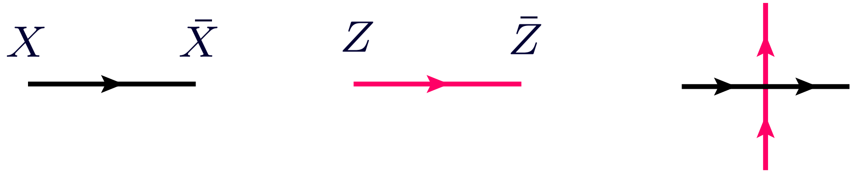
- ✔ Supersymmetry and $R-$symmetry are broken $PSU(2,2|4) \to SU(2,2)\times U(1)\times U(1)$

Fishnet theory

- Bi-scalar theory

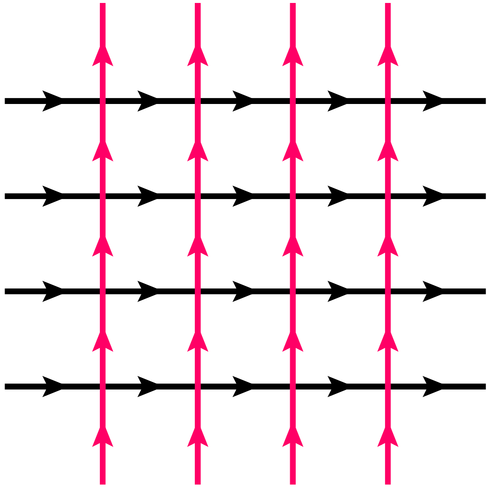
$$L = N_c \left[\partial^\mu \bar{X} \partial_\mu X + \partial^\mu \bar{Z} \partial_\mu Z + (4\pi)^2 \xi^2 \bar{X} \bar{Z} X Z \right]$$

Feynman rules:



- Non-unitary theory, chiral vertex
- Scattering amplitudes in the planar limit

$\mathcal{A}_{2n \to 2n} =$



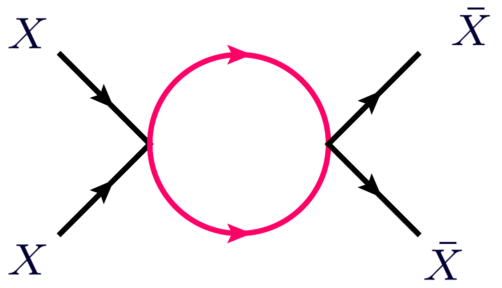
$=$ Fishnet graphs

- Amplitudes are UV + IR finite, (unbroken) dual conformal + Yangian symmetry [Chicherin et al]

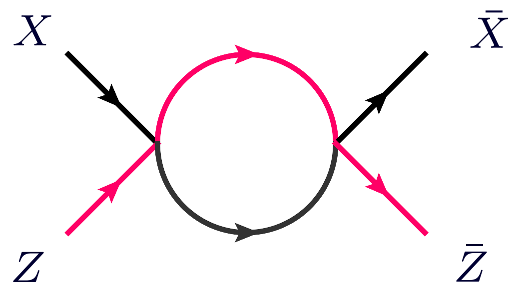
Four-particle amplitudes

$$\mathcal{A}_{2\to 2} = \delta^{(4)}\Big(\sum_i p_i\Big)\Big[N_c \underbrace{\mathrm{tr}\,(T^{a_1}T^{a_2}T^{a_3}T^{a_4})}_{\text{single trace}} A_{\rm st} + \underbrace{\mathrm{tr}(T^{a_1}T^{a_2})\,\mathrm{tr}(T^{a_3}T^{a_4})}_{\text{double trace}} A_{\rm dt}\Big] + \text{perm.}$$

- ✔ Leading color (single trace) amplitudes are protected $A_{\rm st} = A_{\rm Born}$
- ✔ Subleading (double trace) amplitudes are UV divergent



$$\sim \frac{1}{\epsilon}\,\mathrm{tr}(X^2)\,\mathrm{tr}(\bar{X}^2)\,,$$



$$\sim \frac{1}{\epsilon}\,\mathrm{tr}(XZ)\,\mathrm{tr}(\bar{X}\bar{Z})$$

The theory is not complete at the quantum level [Fokken,Sieg,Wilhelm]

- ✔ Quantum corrections induce new interaction vertices

$$\mathcal{L}_{\rm dt}/(4\pi)^2 = -\alpha_1^2\,\big[\mathrm{tr}(XZ)\,\mathrm{tr}(\bar{X}\bar{Z}) + \mathrm{tr}(X\bar{Z})\,\mathrm{tr}(\bar{X}Z)\big] + \alpha_2^2\,\big[\mathrm{tr}(X^2)\,\mathrm{tr}(\bar{X}^2) + \mathrm{tr}(Z^2)\,\mathrm{tr}(\bar{Z}^2)\big]$$

- ✔ ξ^2 does not run in the planar limit, but new couplings do run – conformal anomaly!

Beta functions

- Double-trace couplings develop beta-functions $\beta_i = d\alpha_i^2/d\ln\mu \neq 0$

$$\beta_1 = 2\left(\alpha_1^2 - \xi^2\right)^2, \qquad \beta_2 = a(\xi) + \alpha_2^2\, b(\xi) + \alpha_2^4\, c(\xi)$$

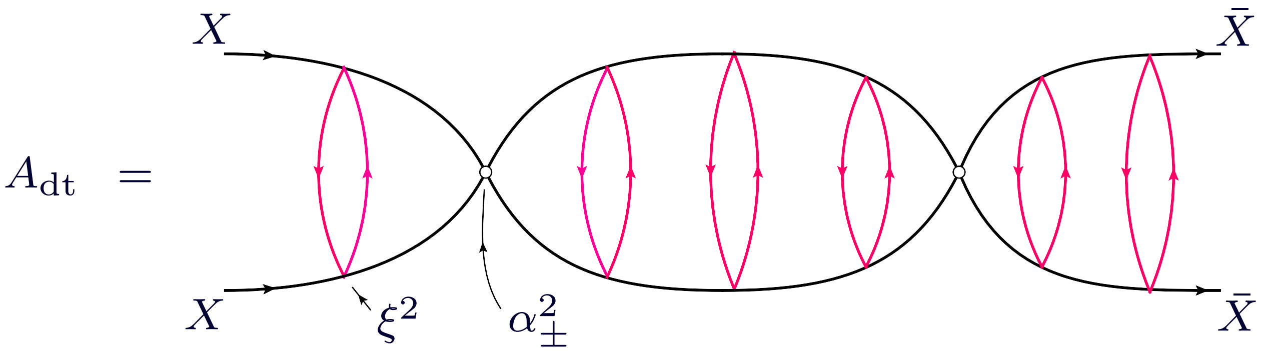
  Coefficient functions $a = -\xi^4 + \xi^8 + \ldots, b = -4\xi^4 + 4\xi^8 + \ldots, c = -4 - 4\xi^4 + \ldots$

- The theory has two lines of fixed points

$$\alpha_1^2 = \xi^2, \qquad \alpha_2^2 = \alpha_\pm^2 = \pm\frac{i\xi^2}{2} - \frac{\xi^4}{2} \mp \frac{3i\xi^6}{4} + \xi^8 \pm \frac{65i\xi^{10}}{48} - \frac{19\xi^{12}}{10} + O\left(\xi^{14}\right)$$

  The bi-scalar theory is a genuine non-unitary CFT on these lines

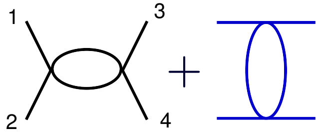
- The only nontrivial four-particle double-trace amplitude

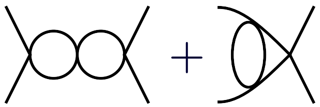


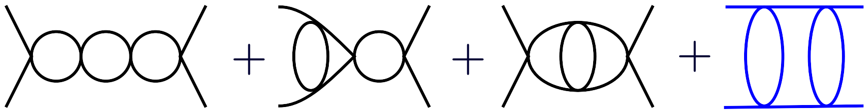
We shall compute this amplitude for an arbitrary coupling ξ^2

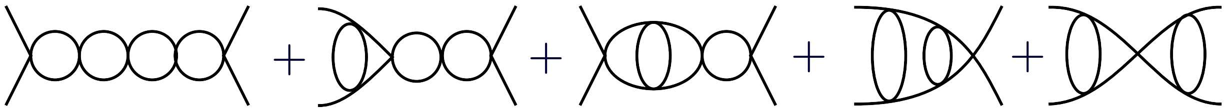
Weak coupling

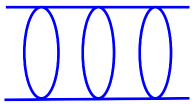
- ✔ Weak coupling expansion $A = 1 + \xi^2 A^{(1)} + \xi^4 A^{(2)} + \xi^6 A^{(3)} + \xi^8 A^{(4)} + \xi^{10} A^{(5)} + \ldots$

$A^{(1)} =$ 

$A^{(2)} =$ 

$A^{(3)} =$ 

$A^{(4)} =$ 

$A^{(5)} =$  $+ \ldots$

Most of diagrams factorize into the product of (UV divergent) form factors

UV divergences cancel in the sum of diagrams at the fixed points

- ✔ Conformal symmetry: $A^{(n)} = A^{(n)}(y)$ are finite functions of $y = s_{13}/s_{12}$

$$A(y,\xi^2) = 1 + i\xi^2\,(1 + \ln y) + \xi^4 \left(\frac{3}{2} + \frac{\pi^2}{3}\right) + i\xi^6 \Big(\underbrace{H_{-1,0,0}(y)}_{\text{harm.polylog}} + \frac{\pi^2}{2} H_{-1}(y) + 4\zeta(3) - 3 \Big) + \ldots$$

$y-$dependence comes from odd number of loops only

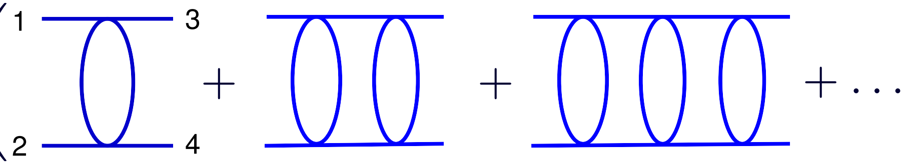
Properties of the amplitude

- Separate contribution from even and odd loops:

$$A(y,\xi^2) = A_{\rm even}(\xi^2) + A_{\rm odd}(y,\xi^2)$$

$$A_{\rm even} = 1 + \xi^4\left(\frac{3}{2}+\frac{\pi^2}{3}\right) + \xi^8\left(-\frac{49}{8}+\frac{\pi^2}{6}+\frac{2\pi^4}{45}\right)+\ldots$$

$A_{\rm odd} = \Big($



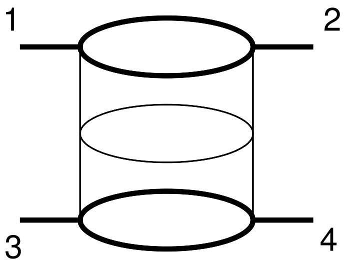
$\Big) -$ [UV divergences]

- General form

$$A_{\rm odd} = \sum_{n\geq 0} (\xi^2)^{2n+1} \times \Big[\text{sum of } {\rm HPL}_{\vec{w}}\,(y) \text{ of weight } |\vec{w}| \leq (2n+1)\Big]$$

- High-energy (Regge) limit $y = s_{13}/s_{12} \gg 1$

$A(y) =$



$$= i\xi^2 \ln y + i\xi^6\left(\tfrac{1}{6}\ln^3 y + \tfrac{\pi^2}{2}\ln y\right) + i\xi^{10}\left(\tfrac{1}{60}\ln^5 y - \tfrac{1}{12}\ln^4 y + \tfrac{\pi^2}{9}\ln^3 y + \ldots\right)$$

Corrections are enhanced by powers of $\ln y$

Properties of the amplitude II

✔ Leading logarithmic approximation $L = \xi^2 \ln y =$ fixed and $y \to \infty$

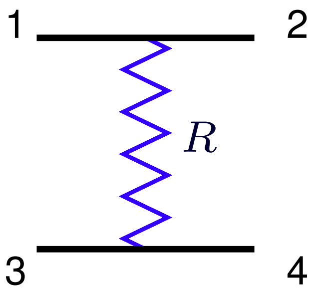
$$A_{\rm LLA}(y,\xi^2) = i \sum_{n\geq 0} \frac{L^{2n+1}}{(2n+1)n!(n+1)!} \sim \frac{i}{4\sqrt{\pi}} L^{-3/2}\, {\rm e}^{2L}$$

Scattering amplitudes grow with the energy

$$A_{\rm LLA}(y,\xi^2) \sim \frac{y^{2\xi^2}}{(\ln y)^{3/2}}\,, \qquad\qquad y = \frac{s_{13}}{s_{12}} \to \infty$$

✔ Regge theory expectations

$$A(y,\xi^2) =$$



$$\sim \left(\frac{s_{13}}{s_{12}}\right)^{\alpha_R}$$

Leading (color singlet) Regge trajectory $\alpha_R = 2\xi^2 + O(\xi^6)$

Questions:

✗ All-loop prediction for the scattering amplitude?

✗ Regge trajectories for arbitrary coupling?

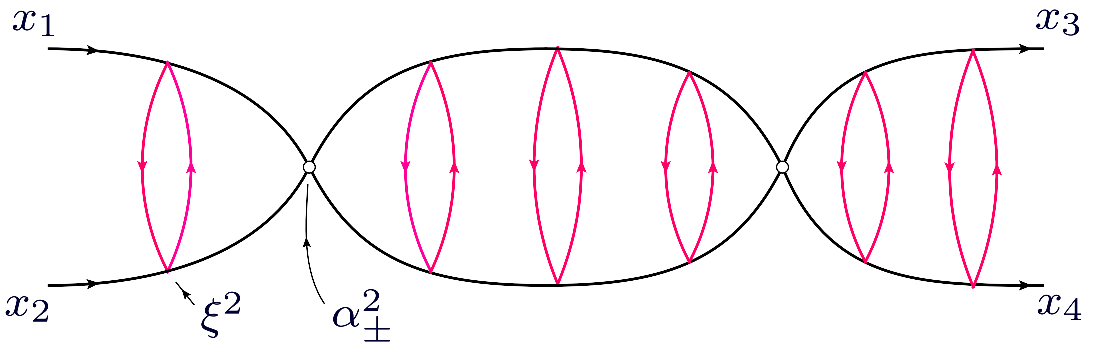
Amplitude from correlation function

Apply Lehmann–Symanzik–Zimmermann (LSZ) reduction formula

Four-point correlation function

$$G(x_1, x_2|x_3, x_4) = \langle \mathrm{tr}(X(x_1)X(x_2))\,\mathrm{tr}(\bar{X}(x_3)\bar{X}(x_4))\rangle$$

$$=$$



Relation to the scattering amplitude

$$\lim_{p_i^2\to 0} p_1^2\, p_2^2\, p_3^2\, p_4^2 \int d^4x_i\, \mathrm{e}^{ip_ix_i}\, G(x_1, x_2|x_3, x_4) = (2\pi)^4\delta^{(4)}(\sum_i p_i)A_{\mathrm{dt}}(y, \xi^2)$$

Why correlation functions?

- ✔ All lines are off-shell
- ✔ Feynman diagrams have a nice iterative structure
- ✔ The correlation function can be computed exactly for any ξ^2 [Grabner,Gromov,Kazakov,GK]

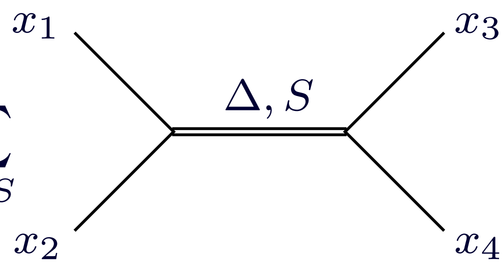
Four-point correlation function

Operator product expansion

$$\mathrm{tr}(X(x_1)X(x_2)) = \sum_{\Delta,S} C_{\Delta,S} O_{\Delta,S}(x_2) + \text{descendants}$$

Conformal partial wave expansion

$$G(x_1,x_2|x_3,x_4) = \sum_{\Delta,S} C^2_{\Delta,S} \times [\text{Conformal block}_{\Delta,S}] \; = \; \sum_{\Delta,S}$$



Exact correlation function

$$G(x_1,x_2|x_3,x_4) = \sum_{S\geq 0} \int_{-\infty}^{\infty} \frac{d\nu}{h(\nu,S)-\xi^4} \underbrace{\int d^4x_0 \, \Phi_{\nu}^{\mu_1\ldots\mu_S}(x_{10},x_{20}) \Phi_{-\nu}^{\mu_1\ldots\mu_S}(x_{30},x_{40})}_{\text{Conformal block}_{\Delta=2+2i\nu,S}}$$

Integral over ν reduces to the sum over residues at $\nu=\nu_\star$

$$h(\nu_\star,S) = (\nu_\star^2 + S^2/4)(\nu_\star^2 + (S+2)^2/4) = \xi^4 \,, \qquad\qquad \mathrm{Im}\,\nu_\star < 0$$

$G(x_1,x_2|x_3,x_4)$ = the sum over states with $\Delta = 2+2i\nu_\star$ and spin S

LSZ reduction

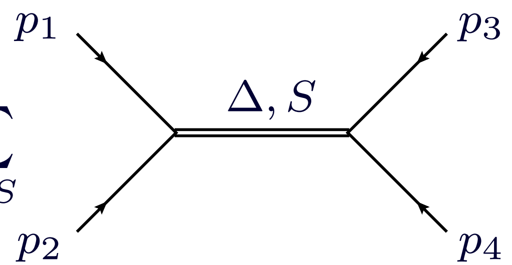
$$G(x_1,x_2|x_3,x_4) = \sum_{S\geq 0}\int_{-\infty}^{\infty}\frac{d\nu}{h(\nu,S)-\xi^4}\int d^4x_0\,\underbrace{\Phi_{\nu}^{\mu_1\cdots\mu_S}(x_{10},x_{20})}_{\text{conf. 3-point function}}\,\Phi_{-\nu}^{\mu_1\cdots\mu_S}(x_{30},x_{40})$$

Residue at two particle pole

$$\widetilde{\Phi}_{\nu}^{\mu_1\cdots\mu_S}(p_1,p_2) = \lim_{p_i^2\to 0} p_1^2\,p_2^2\int d^4x_1 d^4x_2\,\mathrm{e}^{ip_1x_1+ip_2x_2}\,\Phi_{\nu}^{\mu_1\cdots\mu_S}(x_1,x_2)$$

Exact scattering amplitude

$$A_{\mathrm{dt}}(p_i) = \sum_{\Delta,S}$$



$$= \sum_{S\geq 0}\int_{-\infty}^{\infty}\frac{d\nu}{h(\nu,S)-\xi^4}\underbrace{P_{\nu,S}(s_{13}/s_{12})}_{\text{Conformal block}}$$

Conformal block for the amplitude

$$P_{\nu,S}(s_{13}/s_{12}) = \widetilde{\Phi}_{\nu}^{\mu_1\cdots\mu_S}(p_1,p_2)\widetilde{\Phi}_{-\nu}^{\mu_1\cdots\mu_S}(p_3,p_4) = y^S Q_{\nu,S} + O(y^{S-1})$$

Polynomial in $y = s_{13}/s_{12}$ of degree S

Exact scattering amplitude

$$A_{\rm dt}(p_i) = \sum_{S\ge 0}\int_{-\infty}^{\infty}\frac{d\nu}{h(\nu,S)-\xi^4}P_{\nu,S}(s_{13}/s_{12}) = A_{\rm even}(\xi^2)+A_{\rm odd}(y,\xi^2)$$

The $\nu-$integral reduces to the sum over residues at $\nu=\nu_\star$ and $\mathrm{Im}\,\nu_\star<0$

$$A_{\rm dt}(p_i) = \sum_{S\ge 0}\frac{P_{\nu_\star,S}(s_{13}/s_{12})}{\nu_\star(4\nu_\star^2+(S+1)^2+1)}\Bigg|_{\left(\nu_\star^2+\frac{S^2}{4}\right)\left(\nu_\star^2+\frac{(S+2)^2}{4}\right)=\xi^4}$$

ξ^2-even contribution comes from the states with $S=0$

$$A_{\rm even}(\xi^2) = \frac{\left(4\nu_\star^2+1\right)\sinh^2(\pi\nu_\star)}{\pi^2\nu_\star\left(2\nu_\star^2+1\right)}\Bigg|_{\nu_\star^2(\nu_\star^2+1)=\xi^4}$$

$$= \frac{2\left(1-2\sqrt{4\xi^4+1}\right)}{\xi^2\sqrt{2(4\xi^4+1)(\sqrt{4\xi^4+1}-1)}}\sinh^2\left(\pi\sqrt{(\sqrt{4\xi^4+1}-1)/2}\right)$$

Weak coupling expansion

$$A_{\rm even}(\xi^2) = 1+\left(\frac{3}{2}+\frac{\pi^2}{3}\right)\xi^4+\left(-\frac{49}{8}+\frac{\pi^2}{6}+\frac{2\pi^4}{45}\right)\xi^8+O\left(\xi^{12}\right)$$

Perfect agreement with perturbative calculation!

Regge limit $y = s_{13}/s_{12} \to \infty$

$$A(y) = \sum_{S\geq 0} \int_{-\infty}^{\infty} \frac{d\nu}{h(\nu,S) - \xi^4} P_{\nu,S}(y)\,, \qquad P_{\nu,S}(y) \sim y^S Q_{\nu,S}$$

High-energy limit is controlled by high-spin $S \gg 1$ contribution

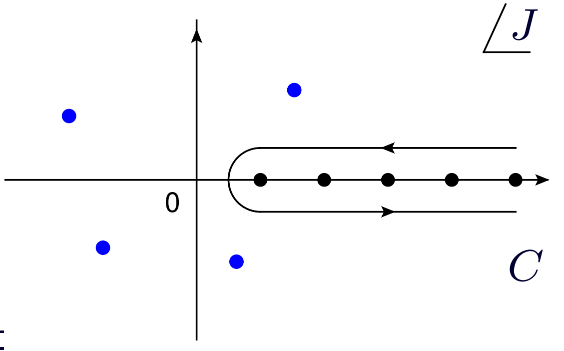
Watson-Sommerfeld representation

$$A(y) = \int_{-\infty}^{\infty} d\nu \int_C \frac{dJ}{2\pi i} \frac{\pi}{2\sin(\pi J)} \frac{y^J Q_{\nu,J}}{h(\nu,J) - \xi^4} + (y \to -y)$$

Integration contour encircles nonnegative integer J

Deform the integration contour and pick up residue at Regge poles $j = j_i(\nu)$

$$h(\nu,j) = (\nu^2 + j^2/4)(\nu^2 + (j+2)^2/4) = \xi^4$$



Exact leading Regge trajectory $\quad j = \sqrt{1 - 4\nu^2 + 4\sqrt{\xi^4 - \nu^2}} - 1$

$$A(y) \sim \int d\nu\, y^{j(\nu)} \sim y^{j(0)} = y^{\sqrt{1+4\xi^2}-1}$$

Agreement with the leading log approximation $A_{\rm LLA} \sim y^{2\xi^2}$

Conclusions and open questions

- $\gamma-$deformed $\mathcal{N} = 4$ SYM / fishnet theory :
  - Two lines of the fixed points
  - Non-unitary 4d conformal integrable field theory
- Closed expression for planar four-particle amplitudes
  - Exact conformal symmetry + integrability
  - Trivial single-trace amplitudes
  - Double-trace amplitudes have an interesting structure
  - Exact (color singlet) Regge trajectories
- Exact expressions for higher point amplitudes?
- Does conformal symmetry/integrability survive for arbitrary values of deformation parameters?
- Dual AdS description of the scattering amplitudes?