Amplitudes 2018 @ SLAC

Recent developments in string amplitudes

Oliver Schlotterer

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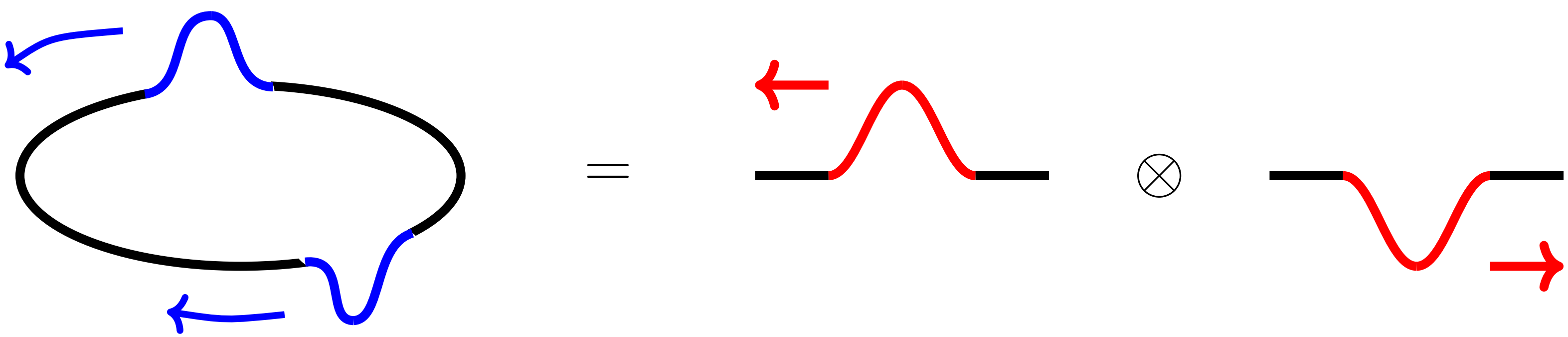
20.06.2018

Intro I: Why string amplitudes?

- key observables in a candidate theory of quantum gravity (testing string dualities, Veneziano amplitude '68 $\Rightarrow$ birth of string theory, etc.)

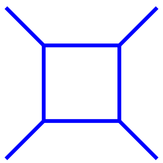
- double-copy structure gravity = (gauge theory)^2 natural from $\alpha' \to 0$ limit of open & closed strings (with chiral splitting for loops)

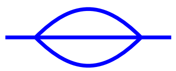


[see e.g. talks of Adamo, Carrasco, Geyer, O'Connell, Zeng]

Intro I: Why string amplitudes?

- key observables in a candidate theory of quantum gravity (testing string dualities, Veneziano amplitude '68 $\Rightarrow$ birth of string theory, etc.)
- double-copy structure gravity = (gauge theory)^2 natural from $\alpha' \to 0$ limit of open & closed strings (with chiral splitting for loops)
- fruitful overlap with number theory: polylogs & multiple zeta values & elliptic / single-valued versions in simple context (cf. Feynman int's)

$\int \mathrm{d}^{D-2\varepsilon}\ell$ 

 etc.

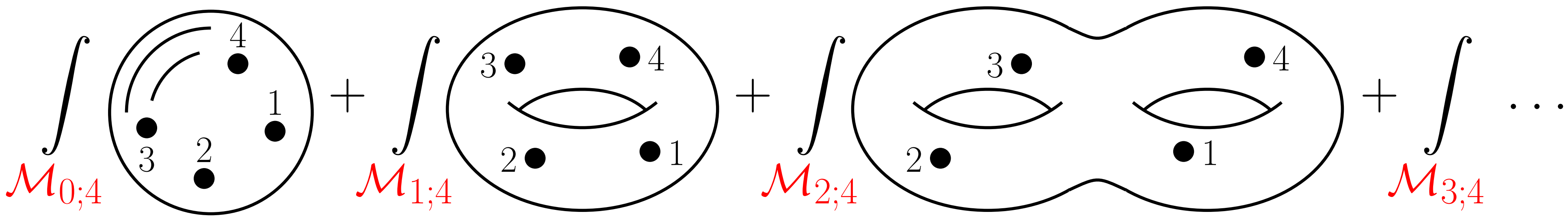
$$\longrightarrow \left\{ \begin{array}{c} \text{multiple polylogarithms } \mathrm{Li}_w(z), \ \ldots \\ \text{elliptic polylogs } \sum_{n=1}^{\infty} \mathrm{Li}_w(q^n z), \ \ldots \\ \text{... and many more ...} \end{array} \right.$$

[see e.g. talks of Bourjaily, Britto, Duhr, McLeod, Mizera, Panzer]

- can often export string correlators to ambitwistors / CHY **[talk of Geyer]**

Intro II: Two major challenges in string amplitudes

Perturbative expansion (drawn for closed strings $\leftrightarrow$ surfaces without bdy)



$$\mathcal{A}^{g\text{-loop}}_{\text{string}}(1,2,\ldots,n) \sim \int_{\mathcal{M}_{g;n}} \Big\langle \Big(\prod_j \mathrm{PCO}(w_j) \Big) V_1(z_1)\, V_2(z_2) \ldots V_n(z_n) \Big\rangle_g$$

moduli space of n-punctured (super-) Riemann surfaces at genus g with or without boundaries

correlation function of n vertex operators V_j for ext. states on Riemann surface of genus g; depending on g & formalism: picture-changing op's (PCOs)

integrals: math

integrands: CFT

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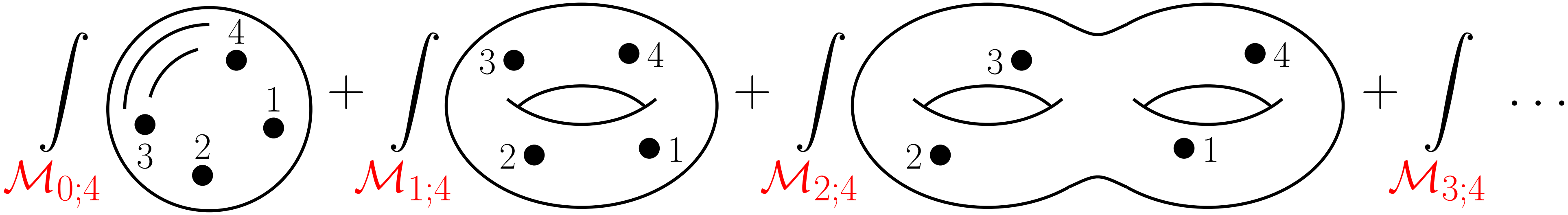

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recent years:

surprisingly rich structures in integrands and integrals !

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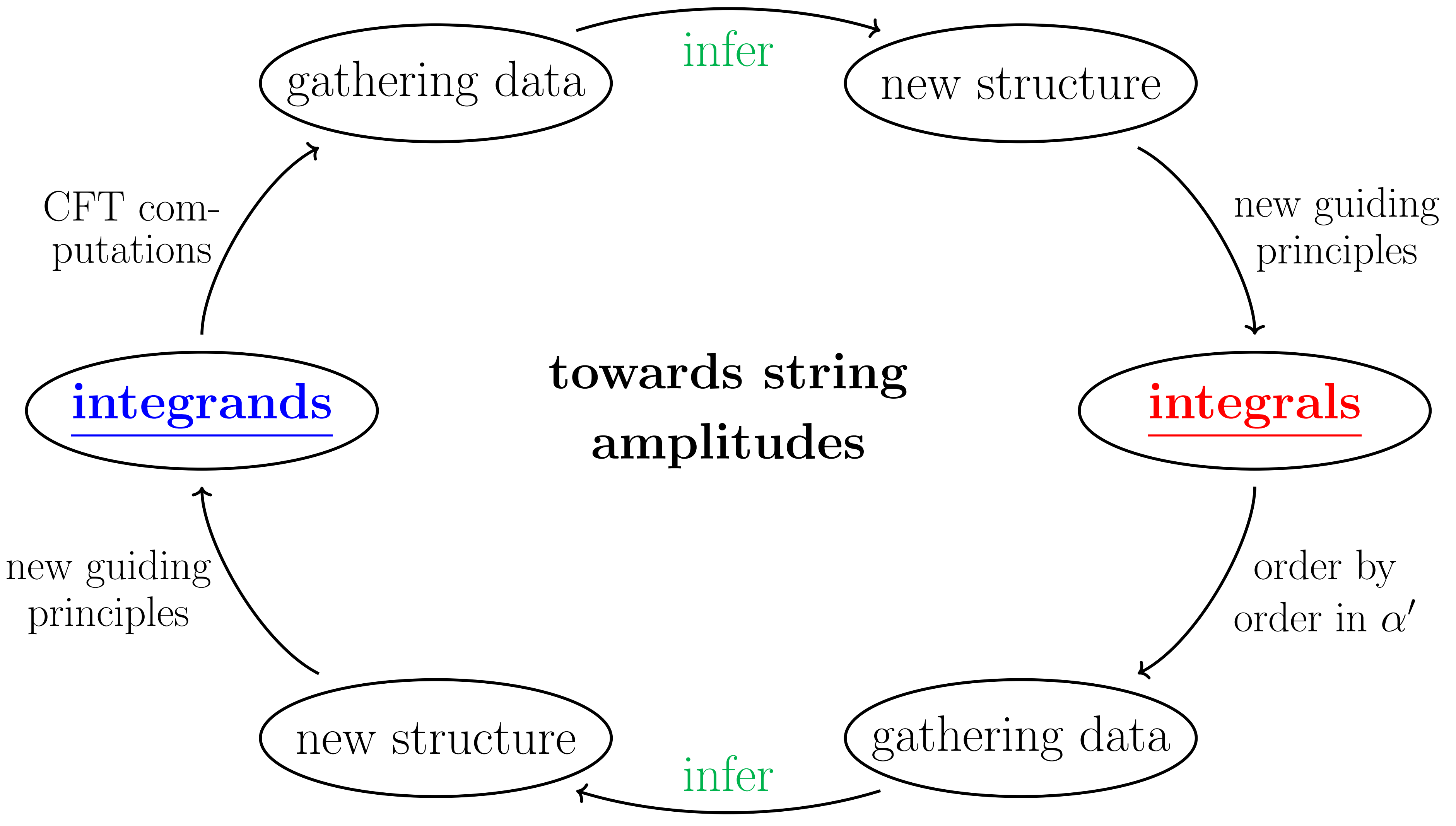
Intro III: This talk's perspective on string amplitudeology

Path forward: mixture of explicit computation and pattern spotting



Intro IV: Recent progress on "integrand" & "integral" side

n-point tree-level correlators for bosonic & heterotic strings: field-theory double copies involving massive gauge theories

[Azevedo, Chiodaroli, Johansson, OS 1803.05452]

integrands ⟵ string amplitudes ⟶ **integrals**

↓

loop order

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n-point tree-level correlators

for bosonic & heterotic strings:

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ving massive gauge theories

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evidence for double-copy structure

in open superstring at one loop:

gravitational R^4 matrix elements

versus SUSY genus-one correlators

[Mafra, OS 1711.09104 & work in progress]

loop order

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getting closed-string integrals from "single-valued" open-string integrals: modular invariants from single-valued elliptic multiple zeta values @ genus 1

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integrands ← string amplitudes → **integrals**

evidence for double-copy structure in open superstring at one loop: gravitational R^4 matrix elements versus SUSY genus-one correlators

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evidence for double-copy structure in open superstring at one loop: gravitational R^4 matrix elements versus SUSY genus-one correlators

[Mafra, OS 1711.09104 & work in progress]

closed-string low-energy expansion at higher loops: systematics of modular invariants at higher genus and their degenerations to lower genus

[D'Hoker, Green, Pioline 1712.06135 & 1806.02691]

loop order

Outline

I. Tree-level double copy of various string theories

[Azevedo, Chiodaroli, Johansson, OS 1803.05452]

II. One-loop double copy of open superstrings

[Mafra, OS 1711.09104 & in progress]

III. Closed strings as single-valued open strings at one loop

[Brödel, OS, Zerbini 1803.00527]

IV. Closed-string low-energy expansion at higher genus

[D'Hoker, Green, Pioline 1712.06135, 1806.02691]

V. Conclusions & Outlook

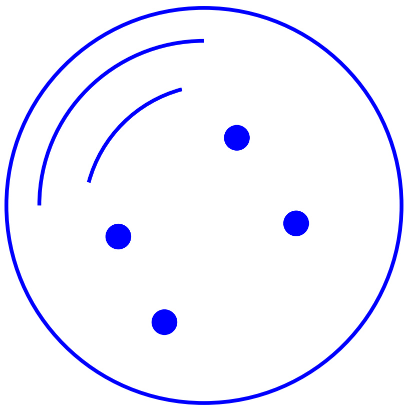
I. Tree-level double copy of various string theories

I. 1 KLT formulae as a diagnosis for double copy

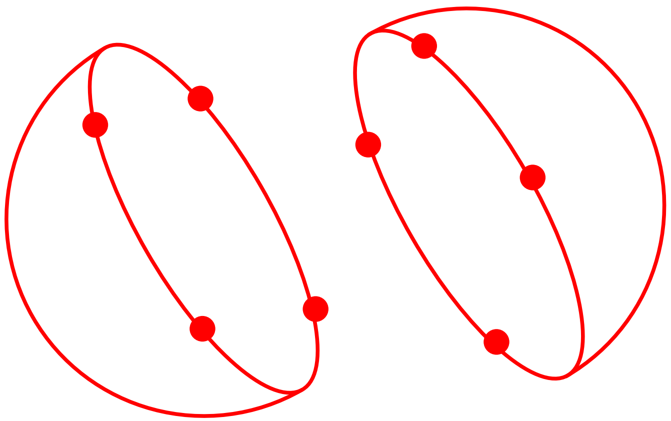
Birth of double copy: KLT relations among string amplitudes at tree-level

$$\mathcal{M}^{\text{tree},4}_{\text{closed}}(\alpha') \;=\; \bar{\mathcal{A}}^{\text{tree}}_{\text{open}}(1,2,4,3;\alpha')\,\sin\left(\frac{\pi\alpha'}{2}k_1\cdot k_2\right)\mathcal{A}^{\text{tree}}_{\text{open}}(1,2,3,4;\alpha')\;.$$

closed strings: sphere



$\longleftrightarrow$



open strings: disk

[Kawai, Lewellen, Tye 1986]

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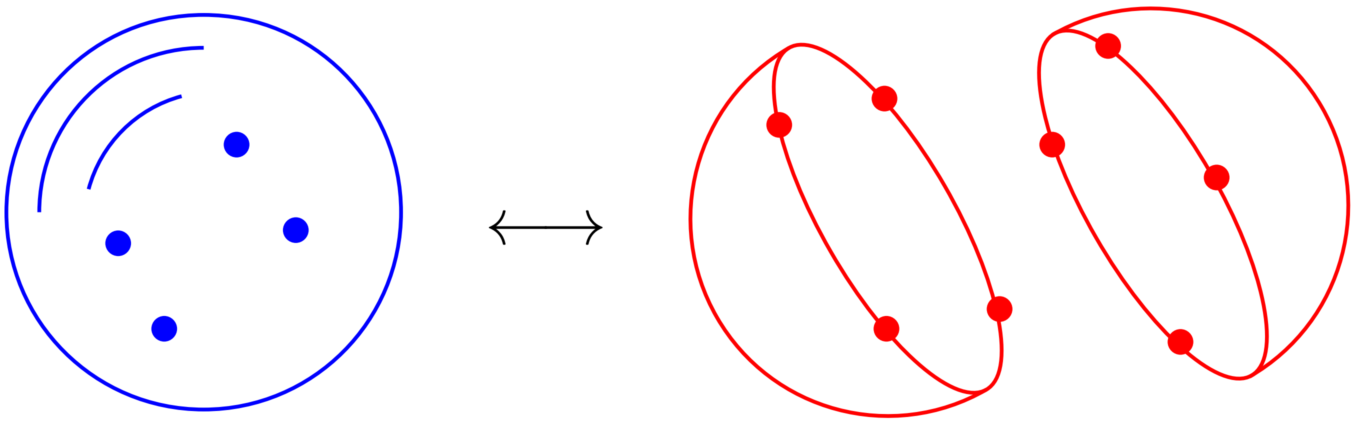

open strings: disk

[**Kawai, Lewellen, Tye 1986**]

Field-theory limit $\alpha' \to 0$: relate gravity to double copy of gauge theories:

$$M^{\text{tree},4}_{\text{SUGRA}} \;=\; \bar{A}^{\text{tree}}_{\text{SYM}}(1,2,4,3)\, k_1\cdot k_2\, A^{\text{tree}}_{\text{SYM}}(1,2,3,4) \;\equiv\; \bar{A}^{\text{tree}}_{\text{SYM}} \otimes_{\text{KLT}} A^{\text{tree}}_{\text{SYM}} \,.$$

Will refer to operation $\otimes_{\text{KLT}}$ at $\alpha' \to 0$ as **field-theory double copy**.

I. 1 KLT formulae as a diagnosis for double copy

At n points, more combinatorics and $(n-3)!$-element BCJ bases

$$\left\{ A^{\rm tree}_{\rm SYM}(1,\rho(2,3,\ldots,n-2),n-1,n) \ , \quad \text{permutation } \rho \in S_{n-3} \right\}$$

[Bern, Carrasco, Johansson 0805.3993]

$$M^{{\rm tree},n}_{\rm SUGRA} = \sum_{\rho,\tau \in S_{n-3}} \underbrace{\bar A^{\rm tree}_{\rm SYM}(1,\rho,n,n-1)\, S(\rho|\tau)_1 \, A^{\rm tree}_{\rm SYM}(1,\tau,n-1,n)}_{(n-3)! \times (n-3)! \text{ KLT matrix, entries are } \sim (k_i \cdot k_j)^{n-3}}$$

[Bern, Dixon, Perelstein, Rozowsky 1998]

[Bjerrum-Bohr, Damgaard, Feng, Sondergaard, Vanhove 2010]

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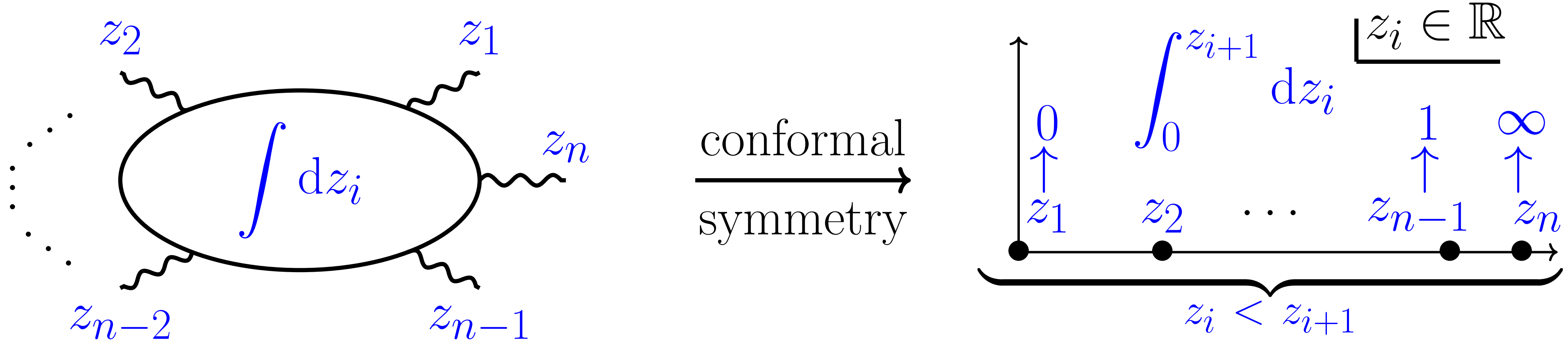
e.g. 2×2 terms at 5 points with
$$S(\rho(2,3)|\tau(2,3))_1 = \begin{pmatrix} (k_1\cdot k_2)(k_{1+2}\cdot k_3) & (k_1\cdot k_2)(k_1\cdot k_3) \\ (k_1\cdot k_2)(k_1\cdot k_3) & (k_1\cdot k_3)(k_{1+3}\cdot k_2) \end{pmatrix}$$

Shorthand for KLT formulae:
$$\boxed{M^{\text{tree}}_{\text{SUGRA}} = \bar{A}^{\text{tree}}_{\text{SYM}} \otimes_{\text{KLT}} A^{\text{tree}}_{\text{SYM}}}$$

I. 2 Open superstrings as a double copy

Map disk boundary to $\mathbb{R}$ $\Longrightarrow$ iterated integrals over $z_j \in (0,1)$.



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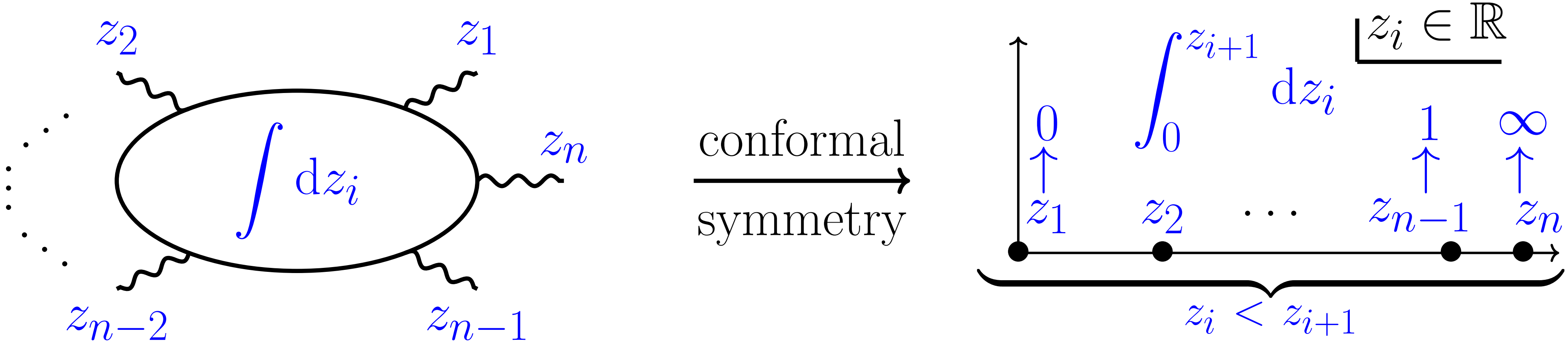

Basis of disk integrals $\subset$ "Parke–Taylor" denominators (with $z_{ij} = z_i - z_j$)

$$Z_\sigma\big(\rho(1,2,\ldots,n)\big) \equiv (2\alpha')^{n-3} \int_{z_{\sigma(i)}<z_{\sigma(i+1)}} \frac{dz_1 \ldots dz_n}{\mathrm{vol}\,\mathrm{SL}_2(\mathbb{R})} \frac{\prod_{i<j}^n |z_{ij}|^{2\alpha' k_i \cdot k_j}}{\rho(z_{12}z_{23}\ldots z_{n-1,n}z_{n,1})}$$

[talks of Arkani-Hamed, He, Mizera]

- BCJ relations in the permutations $\rho \in S_n$ of $(z_{12}z_{23}\ldots z_{n,1})^{-1}$

[Brödel, OS, Stieberger 1304.7267; talk of Mizera]

- $\alpha' \to 0$ limit of $Z_\sigma(\ldots)$ $\Rightarrow$ tree amplitudes of bi-adjoint ϕ^3 & NLSM

[Cachazo, He, Yuan 1309.0885; Carrasco, Mafra, OS 1608.02569]

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behave like amplitudes ... in putative effective theory of bicolored scalars

$\Longrightarrow$ refer to $Z_\sigma(\ldots)$ as amplitudes of "Z-theory"

[Carrasco, Mafra, OS Aug. – Dec. 2016]

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Factorized expression for n-point function of open superstrings

$$\mathcal{A}^{\rm tree}_{\rm open}(\sigma) \;=\; Z_\sigma \otimes_{\rm KLT} A^{\rm tree}_{\rm SYM}$$

[Mafra, OS, Stieberger 1106.2645, Brödel, OS, Stieberger 1304.7267]

- all polarizations in field-theory amplitudes $A^{\rm tree}_{\rm SYM}$ of $D=10$ SYM (!)
- KLT formula signals field-theory double copy

"open superstring = Z-theory $\otimes$ SYM"

I. 3 Towards bosonic & heterotic strings

$\exists$ similar double-copy form for bosonic-string amplitudes

... involving a peculiar gauge theory "$(DF)^2 + \mathrm{YM}$"

[Azevedo, Chiodaroli, Johansson, OS 1803.05452]

$$\mathcal{L}_{(DF)^2+\mathrm{YM}} \equiv \frac{1}{2}(D_\mu F^{a\,\mu\nu})^2 - \frac{1}{4}m^2(F^a_{\mu\nu})^2 - \frac{1}{3}f^{abc}F^{a\nu}_\mu F^{b\lambda}_\nu F^{c\mu}_\lambda$$
$$+\frac{1}{2}(D_\mu\varphi^\alpha)^2 - \frac{1}{2}m^2(\varphi^\alpha)^2 + \frac{1}{2}C^{\alpha ab}\varphi^\alpha F^a_{\mu\nu}F^{b\,\mu\nu} + \frac{1}{3!}d^{\alpha\beta\gamma}\varphi^\alpha\varphi^\beta\varphi^\gamma\,\Big|_{m^2=-\frac{1}{\alpha'}}$$

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- kin. operator $\sim (\partial^4 - m^2\partial^2) \Rightarrow$ 2 gluon modes: $\left(\begin{smallmatrix}\text{massless}\\ \text{physical}\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}\text{massive}\\ \text{ghost}\end{smallmatrix}\right)$
- massive-ghost scalar φ^α: index $\alpha \leftrightarrow$ real representation of gauge group

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- massive-ghost scalar φ^α: index $\alpha \leftrightarrow$ real representation of gauge group
- tachyon poles of bosonic string fix mass parameter to $m^2 = -\frac{1}{\alpha'}$
- constructing $\mathcal{L}_{(DF)^2+\text{YM}}$ guided by BCJ relations **[Johansson, Nohle 1707.02965]**
- $\exists$ canonical extension "$(DF)^2 + \text{YM} + \phi^3$" involving bi-adj. scalars ϕ

I. 4 String tree amplitudes as field-theory double copies

Double-copy pattern (field theory) $\otimes_{\mathrm{KLT}}$ (stringy building block)

$\longrightarrow$ universal to tree amplitudes in various string theories !

$\otimes_{\mathrm{KLT}}$	SYM		
Z-theory	open superstring		

- "Z-theory" $\leftrightarrow$ α'-dependent disk integrals

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sv(open superstring)	closed superstring		

- "Z-theory" $\leftrightarrow$ α'-dependent disk integrals
- "**sv**" $\leftrightarrow$ formal "single-valued" projection on zeta values in α'-expansion

[Schnetz 1302.6445 & Brown 1309.5309, see later]

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$\otimes_{\text{KLT}}$	SYM	$(DF)^2$+YM	
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- gauge theory $(DF)^2 + \text{YM}$

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$\otimes_{\text{KLT}}$	SYM	$(DF)^2$+YM	$(DF)^2$+YM+ϕ^3
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sv(open superstring)	closed superstring	heterotic string (grav)	het. string (gauge/grav)
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- gauge theory $(DF)^2$ + YM and scalar extension $(DF)^2$ + YM + ϕ^3

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sv(open superstring)	closed superstring	heterotic string (grav)	het. string (gauge/grav)
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II. One-loop double copy of open superstrings

II. Double-copy structure of 1-loop open-string amplitudes

Recall the double-copy structure of the open superstring @ tree level

$$\boxed{\begin{matrix}\text{open superstring}\\ \text{at tree level}\end{matrix}} \ = \ \boxed{\begin{matrix}\text{gauge invariant}\\ \text{amplitudes } A^{\text{tree}}_{\text{SYM}}\end{matrix}} \ \otimes_{\text{KLT}} \ \boxed{\begin{matrix}\text{disk integrals over}\\ (z_{12}z_{23}\ldots z_{n1})^{-1}\end{matrix}}$$

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∃ conjectural loop-level extension for maximal supersymmetry

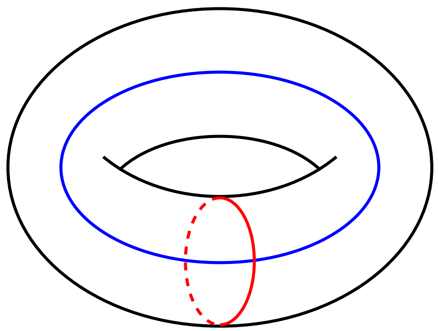
$$\boxed{\begin{array}{c}\text{open superstring}\\ \text{at one loop}\end{array}} = \boxed{\begin{array}{c}\text{gauge inv. SYM}\\ \text{tensors } C^{\mu\nu\cdots}\end{array}} \otimes_{R^4} \boxed{\begin{array}{c}\text{cylinder int's over}\\ E^{\mu\nu\cdots}(\ell, z_j, \tau)\end{array}}$$

Functions on the cylinder are **“generalized elliptic integrands”** with

$$E^{\mu\nu\cdots}(\ell, z_j, \tau) = E^{\mu\nu\cdots}(\ell - 2\pi i k_j, z_j + \tau, \tau)$$

periodic modulo shift of loop momentum ℓ

e.g. $E^{\mu}_{1|2,3,4,5} = \ell^{\mu} + \left[k_2^{\mu}\, \partial_z \log \theta_1(z_{12}) + (2 \leftrightarrow 3,4,5)\right]$



[D’Hoker, Phong ’88 & ’89] **[Mafra, OS 1711.09104 & in progress]**

II. Double-copy structure of 1-loop open-string amplitudes

open superstring at one loop	$=$	gauge inv. SYM tensors $C^{\mu\nu\ldots}$	$\otimes_{R^4}$	cylinder int's over $E^{\mu\nu\ldots}(\ell, z_j, \tau)$

What is the "double-copy metric" $\otimes_{R^4}$?

$\longrightarrow$ defined by tree-level matrix elements of SUSY R^4 operator

matrix element of SUSY R^4	$=$	gauge inv. SYM tensors $C^{\mu\nu\ldots}$	$\otimes_{R^4}$	gauge inv. SYM tensors $\tilde{C}^{\mu\nu\ldots}$

E.g. at 5pt, ($\longrightarrow$ so far, double-copy structure verified up to 7pt)

$$\mathcal{A}^{1-\text{loop}}_{\text{open},5} = \int_{\mathcal{M}_{1;5}} \int d^{10}\ell \, \Big\{ C^{\mu}_{1|2,3,4,5} \, E^{\mu}_{1|2,3,4,5} + \big[s_{23} C_{1|23,4,5} E_{1|23,4,5} + \ldots \big] \Big\}$$

$$E \leftrightarrow \tilde{C} \qquad M^{\text{tree}}_{R^4} = C^{\mu}_{1|2,3,4,5} \, \tilde{C}^{\mu}_{1|2,3,4,5} + \big[s_{23} C_{1|23,4,5} \tilde{C}_{1|23,4,5} + \ldots \big]$$

[Mafra, OS 1711.09104 & in progress]

II. Double-copy structure of 1-loop open-string amplitudes

open superstring at one loop	=	gauge inv. SYM tensors $C^{\mu\nu\ldots}$	$\otimes_{R^4}$	cylinder int's over $E^{\mu\nu\ldots}(\ell, z_j, \tau)$

What is the "double-copy metric" $\otimes_{R^4}$?

$\longrightarrow$ defined by tree-level matrix elements of SUSY R^4 operator

matrix element of SUSY R^4	=	gauge inv. SYM tensors $C^{\mu\nu\ldots}$	$\otimes_{R^4}$	gauge inv. SYM tensors $\tilde{C}^{\mu\nu\ldots}$

Is double copy a generic feature of open-string perturbation theory?

- reduced supersymmetry @ 1 loop: R^2 matrix elements instead of R^4
- higher loops $\leftrightarrow$ different gravitational operators? E.g. D^4R^4 at 2 loops?
- reformulation of amplitude prescription where double copy is manifest?

III. Closed strings as single-valued open strings @ 1 loop

III. 1 Tree level: closed strings vs. single-valued open strings

α'-expansion of $F_\sigma{}^\tau$ & $\mathcal{A}^{\text{tree}}_{\text{open}}$ involves multiple zeta values (MZVs)

$$\zeta_{n_1,n_2,\ldots,n_r} \equiv \sum_{0<k_1<k_2<\ldots<k_r}^{\infty} k_1^{-n_1} k_2^{-n_2} \ldots k_r^{-n_r} \, , \qquad n_r \geq 2$$

[Terasoma 2002 & Brown 2006]

Schematically,

$$\mathcal{A}^{\text{tree}}_{\text{open}}(\sigma) \; = \; \overbrace{\Big(\mathbb{1} \, + \, \zeta_2 \, (\alpha' k_i \cdot k_j)^2 \, + \, \zeta_3 \, (\alpha' k_i \cdot k_j)^3 \, + \, \mathcal{O}(\alpha'^4)\Big)_\sigma{}^\tau}^{\text{from } Z_\sigma(\rho(\ldots)) \text{ and } \otimes_{\text{KLT}}} \, A^{\text{tree}}_{\text{SYM}}(\tau)$$

Polynomial structure in $\alpha' k_i \cdot k_j$ at n points can be determined to any order

[e.g. Brödel, OS, Stieberger, Terasoma 1304.7304 & Mafra, OS 1609.07078]

Explicit results at $n \leq 7$ points available for download

`http://wwwth.mpp.mpg.de/members/stieberg/mzv/index.html`

III. 1 Tree level: closed strings vs. single-valued open strings

α'-expansion of $F_\sigma{}^\tau$ & $\mathcal{A}^{\rm tree}_{\rm open}$ involves multiple zeta values (MZVs)

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Define single-valued projection **sv** of MZVs via their polylogarithm origin

[Schnetz 1302.6445 & Brown 1309.5309]

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{MZVs} \arrow[r, dashed, "\mathbf{sv}"'] & \text{single-valued MZVs} \\
\begin{array}{c}\text{polylogarithms}\\ G(\{\text{word in } 0,1\};z)\end{array} \arrow[u, "z\to 1"] \arrow[r, "\begin{array}{c}\text{Francis Brown 2004:}\\ \exists \text{ unique procedure to}\end{array}", "\text{cancel monodromies}"'] & \begin{array}{c}\text{single-valued polylog's}\\ G_{\rm sv}(\{\text{word in } 0,1\};z)\end{array} \arrow[u, "z\to 1"']
\end{tikzcd}$$

e.g. $G(1;z) \ = \ \log(1-z) \quad \longrightarrow \quad G_{\rm sv}(1;z) = \log|1-z|^2$

III. 1 Tree level: closed strings vs. single-valued open strings

α'-expansion of $F_\sigma{}^\tau$ & $\mathcal{A}^{\rm tree}_{\rm open}$ involves multiple zeta values (MZVs)

$$\zeta_{n_1,n_2,\ldots,n_r} \equiv \sum_{0<k_1<k_2<\ldots<k_r}^{\infty} k_1^{-n_1} k_2^{-n_2} \ldots k_r^{-n_r} \, , \qquad n_r \geq 2$$

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\end{tikzcd}$$

$$\textbf{sv}(\zeta_{2k}) = 0 \, , \quad \textbf{sv}(\zeta_{2k+1}) = 2\,\zeta_{2k+1} \, , \quad \textbf{sv}(\zeta_{3,5}) = -10\,\zeta_3\,\zeta_5 \, , \quad \text{etc.}$$

III. 1 Tree level: closed strings vs. single-valued open strings

Closed-string n-point function simplifies to field-theory double copy

$$\boxed{\mathcal{M}^{\rm tree}_{\rm closed}(\alpha') \ = \ \bar{A}^{\rm tree}_{\rm SYM} \otimes_{\rm KLT} \mathbf{sv}\, \mathcal{A}^{\rm tree}_{\rm open}(\alpha')}$$

[OS, Stieberger 1205.1516 & Stieberger 1310.3259]

Schematically, by $\mathbf{sv}(\zeta_{2k}) \ = \ 0$ and $\mathbf{sv}(\zeta_{2k+1}) \ = \ 2\,\zeta_{2k+1}$,

$$\mathcal{A}^{\rm tree}_{\rm open}(\sigma) \ = \ \Big(\mathbb{1} \ + \ \zeta_2(\alpha' k_i \cdot k_j)^2 \ + \ \ \zeta_3(\alpha' k_i \cdot k_j)^3 \ + \ \mathcal{O}(\alpha'^4)\Big)_\sigma{}^\tau\, A_{\rm SYM}(\tau)$$

$$\mathbf{sv}\, \mathcal{A}^{\rm tree}_{\rm open}(\sigma) \ = \ \Big(\mathbb{1} \ + \ 2\zeta_3(\alpha' k_i \cdot k_j)^3 \ + \ \mathcal{O}(\alpha'^4)\Big)_\sigma{}^\tau\, A_{\rm SYM}(\tau)$$

III. 1 Tree level: closed strings vs. single-valued open strings

Closed-string n-point function simplifies to field-theory double copy

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$$\mathbf{sv}\,\mathcal{A}^{\text{tree}}_{\text{open}}(\sigma) = \big(\mathbb{1} + 2\zeta_3(\alpha' k_i \cdot k_j)^3 + \mathcal{O}(\alpha'^4)\big)_\sigma{}^\tau A_{\text{SYM}}(\tau)$$

- emergence of $\mathbf{sv}\,\mathcal{A}^{\text{tree}}_{\text{open}}(\alpha')$ still conjectural (tested to high α'-orders),

  but inductive all-order proof is within reach **[OS, Schnetz: in progress]**

- also: **sv**-relation in worldsheet sigma models of type-I & heterotic strings

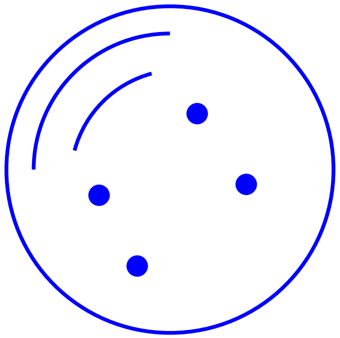
[Fan, Fotopoulos, Stieberger, Taylor 1711.05821]

III. 1 Tree level: closed strings vs. single-valued open strings

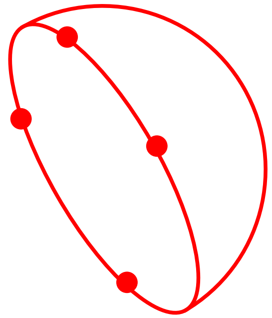
Next, generalize genus-zero correspondence

between open- and closed-string integrals ...

closed strings: sphere



= **sv** {

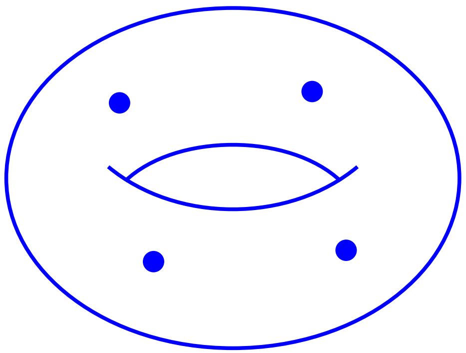


open strings: disk }

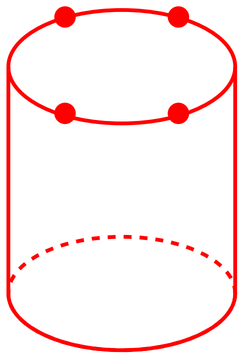
[OS, Stieberger 1205.1516, Stieberger 1310.3259]

... to an empirical elliptic single-valued projection "**esv**" at genus one

closed strings: torus



= **esv** {



open strings: cylinder }

[Brödel, OS, Zerbini 1803.00527]

III. 2 One loop four-point integrals – the toy model

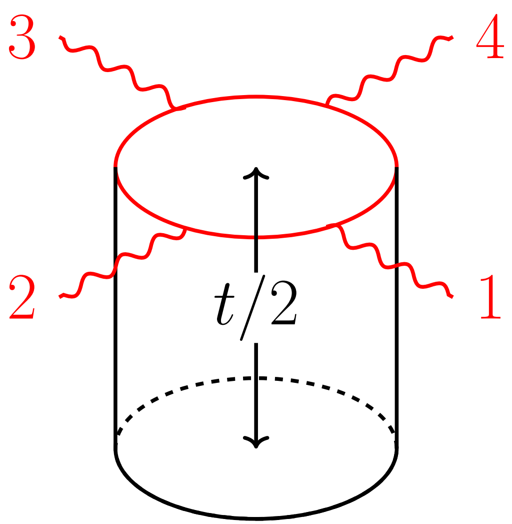
Toy model: *abelian* open-string states $t^a \to \mathbb{1}$ on cylinder worldsheet:

$$\mathcal{A}^{\text{1-loop}}_{\text{open}}(1,2,3,4) \;=\; s_{12}s_{23}A^{\text{tree}}_{\text{YM}}(1,2,3,4)\int_0^\infty \mathrm{d}t\; I_{\text{open}}(s_{ij},\tau=it)$$

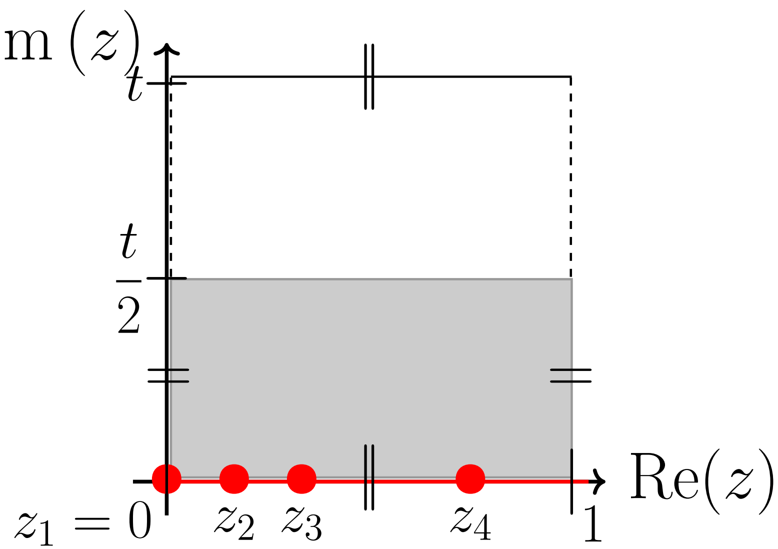
$$I_{\text{open}}(s_{ij},\tau) \;=\; \Big(\prod_{j=2}^4\int_0^1 \mathrm{d}z_j\Big)\,\exp\Big(\frac{1}{2}\sum_{i<j}^4 s_{ij}\,g(z_i-z_j,\tau)\Big)\Big|_{z_1=0}$$

with $s_{ij}=2\alpha' k_i\cdot k_j$ & Green function $\partial_z g(z,\tau)=\partial_z\log\theta(z,\tau)+2\pi i\frac{\mathrm{Im}\,z}{\mathrm{Im}\,\tau}$.

[Brink, Green, Schwarz 1982]



$$\xrightarrow[\text{“half a torus”}]{\text{parametrized as}}$$



III. 2 One loop four-point integrals – the toy model

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with $s_{ij}=2\alpha' k_i\cdot k_j$ & Green function $\partial_z g(z,\tau)=\partial_z \log\theta(z,\tau)+2\pi i\frac{\mathrm{Im}\,z}{\mathrm{Im}\,\tau}$.

Similar integral for 4 closed strings at one loop:

fund. domain $\mathcal{F}$ of $\mathrm{SL}_2(\mathbb{Z})$

$$\mathcal{M}^{\text{1-loop}}_{\text{closed}}(1,2,3,4) \;=\; \big|s_{12}s_{23}A^{\text{tree}}_{\text{YM}}(1,2,3,4)\big|^2\int_{\mathcal{F}}\frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^2}\;J_{\text{closed}}(s_{ij},\tau)$$

$$\underbrace{J_{\text{closed}}(s_{ij},\tau)}_{\text{mod. invariant}} \;=\; \Big(\prod_{j=2}^{4}\int_{\text{torus}(\tau)}\frac{\mathrm{d}^2 z_j}{\mathrm{Im}\,\tau}\Big)\,\exp\Big(\sum_{i<j}^{4} s_{ij}\,g(z_i-z_j,\tau)\Big)\Big|_{z_1=0}$$

[Brink, Green, Schwarz 1982]

III. 2 One loop four-point integrals – the toy model

This talk's main interest: integrals $I_{\rm open}$, $J_{\rm open}$ over punctures z_j @ fixed τ

$$I_{\rm open}(s_{ij},\tau) \ = \ \Big(\prod_{j=2}^4 \int_0^1 \mathrm{d}z_j \Big) \ \exp\Big(\frac{1}{2} \sum_{i<j}^4 s_{ij} \, g(z_i{-}z_j,\tau) \Big) \Big|_{z_1=0}$$

$$J_{\rm closed}(s_{ij},\tau) \ = \ \Big(\prod_{j=2}^4 \int_{\mathrm{torus}(\tau)} \frac{\mathrm{d}^2 z_j}{\mathrm{Im}\, \tau} \Big) \ \exp\Big(\sum_{i<j}^4 s_{ij} \, g(z_i{-}z_j,\tau) \Big) \Big|_{z_1=0}$$

- low-energy expansion in $s_{ij} = 2\alpha' k_i \cdot k_j$: Taylor expand the $\exp(s_{ij} \ldots)$
- $\int \prod_{i<j} (g(z_i{-}z_j,\tau))^{n_{ij}}$ over cyl. boundary $\Rightarrow$ elliptic MZVs (eMZVs)

[Brödel, Mafra, Matthes, OS 1412.5535]

- eMZVs & closed-string integrals $\Rightarrow$ iterated Eisenstein integrals

iterated τ-integral $\mathcal{E}_0(k_1,k_2,\ldots,k_r;\tau)$ over holo' Eisenstein series $\mathrm{G}_k(\tau)$

[Enriquez 1301.3042 & Brödel, Matthes, OS 1507.02254]

[see Claude Duhr's talk]

III. 3 Iterated Eisenstein integrals in one-loop amplitudes

Symmetrized open-string integral (note modular transformation $\tau \rightarrow -\frac{1}{\tau}$)

$$I_{\rm open}(s_{ij},-\tfrac{1}{\tau}) = 1 - (s_{12}^2+s_{12}s_{23}+s_{23}^2)\Big[\frac{T^2}{90}-\frac{\pi^2}{9}-\frac{\pi^4}{30T^2}-\frac{2i\zeta_3}{T}+12\,\mathcal{E}_0(4,0;\tau)+\frac{12i}{T}\mathcal{E}_0(4,0,0;\tau)\Big]$$
$$+\, s_{12}s_{13}s_{23}\Big[-\frac{iT^3}{756}+\frac{i\pi^2T}{45}-\frac{\zeta_3}{2}-\frac{7i\pi^4}{72T}-\frac{2\pi^2\zeta_3}{T^2}+\frac{15\zeta_5}{2T^2}+\frac{17i\pi^6}{1890T^3}+\frac{12\pi^2}{T^2}\mathcal{E}_0(4,0,0;\tau)$$
$$+\,300\,\mathcal{E}_0(6,0,0;\tau)+\frac{900i}{T}\mathcal{E}_0(6,0,0,0;\tau)-\frac{900}{T^2}\mathcal{E}_0(6,0,0,0,0;\tau)\Big]+\mathcal{O}(\alpha'^4)$$

Open string: Laurent polynomials in $T\equiv\pi\tau$ along with series in $q{=}e^{2\pi i\tau}$

"double expansion in $(\log q)^{\pm 1}$ and q"

III. 3 Iterated Eisenstein integrals in one-loop amplitudes

Symmetrized open-string integral (note modular transformation $\tau \rightarrow -\frac{1}{\tau}$)

$$I_{\rm open}(s_{ij},-\tfrac{1}{\tau}) = 1 - (s_{12}^2+s_{12}s_{23}+s_{23}^2)\Big[\frac{T^2}{90}-\frac{\pi^2}{9}-\frac{\pi^4}{30T^2}-\frac{2i\zeta_3}{T}+12\,\mathcal{E}_0(4,0;\tau)+\frac{12i}{T}\mathcal{E}_0(4,0,0;\tau)\Big]$$
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Open string: Laurent polynomials in $T \equiv \pi\tau$ along with series in $q{=}e^{2\pi i\tau}$

... fits to closed strings: Laurent poly's in $y \equiv \pi\,{\rm Im}\,\tau$ & series in q & $\bar{q}$

$$J_{\rm closed}(s_{ij},\tau) = 1 + (s_{12}^2+s_{12}s_{23}+s_{23}^2)\Big[\frac{2y^2}{45}+\frac{2\zeta_3}{y}-24\,{\rm Re}\,\mathcal{E}_0(4,0;\tau)-\frac{12}{y}\,{\rm Re}\,\mathcal{E}_0(4,0,0;\tau)\Big]$$
$$+\, s_{12}s_{13}s_{23}\Big[-\frac{2y^3}{189}-\zeta_3-\frac{15\zeta_5}{4y^2}+600\,{\rm Re}\,\mathcal{E}_0(6,0,0;\tau)$$
$$+\frac{900}{y}\,{\rm Re}\,\mathcal{E}_0(6,0,0,0;\tau)+\frac{450}{y^2}\,{\rm Re}\,\mathcal{E}_0(6,0,0,0,0;\tau)\Big] + \mathcal{O}(\alpha'^4)$$

α'-expanding $J_{\rm closed}$ & generalizations $\Rightarrow$ "*modular graph functions*"

[D'Hoker, Green, Gürdogan, Vanhove 1512.06779; various authors 1999 - 2017]

III. 4 Towards an elliptic single-valued projection

How to map open-string data to closed-string data?

$$I_{\rm open}(s_{ij},-\tfrac{1}{\tau})\,\Big|_{(\alpha')^2} \sim -\frac{T^2}{90}+\frac{\pi^2}{9}+\frac{\pi^4}{30T^2}+\frac{2i\zeta_3}{T}-12\,\mathcal{E}_0(4,0;\tau)-\frac{12i}{T}\,\mathcal{E}_0(4,0,0;\tau)$$

$$J_{\rm closed}(s_{ij},\tau)\,\Big|_{(\alpha')^2} \sim \frac{2y^2}{45}+\frac{2\zeta_3}{y}-24\,{\rm Re}\,\mathcal{E}_0(4,0;\tau)-\frac{12}{y}\,{\rm Re}\,\mathcal{E}_0(4,0,0;\tau)$$

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Engineer candidate **esv** for *elliptic single-valued projection*

$$\textbf{esv}\ :\ \begin{cases} \text{(i)}: & \zeta_{n_1,n_2,\ldots} \rightarrow \textbf{sv}(\zeta_{n_1,n_2,\ldots}) \\ \text{(ii)}: & T \rightarrow 2iy \ \ \text{i.e.}\ \ \tau \rightarrow 2i\,{\rm Im}\,\tau \\ \text{(iii)}: & \mathcal{E}_0(k_1,\ldots;\tau) \rightarrow 2\,{\rm Re}\,\mathcal{E}_0(k_1,\ldots;\tau) \end{cases}$$

to match above expressions @ $(\alpha')^2$ and in fact complete $(\alpha')^{\leq 6}$ orders!

III. 4 Towards an elliptic single-valued projection

How to map open-string data to closed-string data?

$$I_{\rm open}(s_{ij},-\tfrac{1}{\tau})\,\big|_{(\alpha')^2} \sim -\frac{T^2}{90}+\frac{\pi^2}{9}+\frac{\pi^4}{30T^2}+\frac{2i\zeta_3}{T}-12\,\mathcal{E}_0(4,0;\tau)-\frac{12i}{T}\,\mathcal{E}_0(4,0,0;\tau)$$

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$$\textbf{esv}\ :\ \begin{cases} \text{(i)}: & \zeta_{n_1,n_2,\ldots} \rightarrow \textbf{sv}(\zeta_{n_1,n_2,\ldots}) \\ \text{(ii)}: & T \rightarrow 2iy \ \ \text{i.e.}\ \ \tau \rightarrow 2i\,{\rm Im}\,\tau \\ \text{(iii)}: & \mathcal{E}_0(k_1,\ldots;\tau) \rightarrow 2\,{\rm Re}\,\mathcal{E}_0(k_1,\ldots;\tau) \end{cases}$$

to match above expressions @ $(\alpha')^2$ and in fact complete $(\alpha')^{\leq 6}$ orders!

conjecture:

$$\boxed{\textbf{esv}\ I_{\rm open}(s_{ij},-\tfrac{1}{\tau}) \ = \ J_{\rm closed}(s_{ij},\tau)}$$

[Brödel, OS, Zerbini 1803.00527]

III. 4 Towards an elliptic single-valued projection

Conjectural *elliptic single-valued projection* **esv** (works to order α'^6)

$$\mathbf{esv}\ :\ \begin{cases} \text{(i)}: & \zeta_{n_1,n_2,\ldots} \ \to\ \mathbf{sv}(\zeta_{n_1,n_2,\ldots}) \\ \text{(ii)}: & T \ \to\ 2iy \quad \text{i.e.} \quad \tau \ \to\ 2i\,\mathrm{Im}\,\tau \\ \text{(iii)}: & \mathcal{E}_0(k_1,\ldots;\tau) \ \to\ 2\,\mathrm{Re}\,\mathcal{E}_0(k_1,\ldots;\tau) \end{cases}$$

conjecture:

$$\boxed{\mathbf{esv}\, I_{\text{open}}(s_{ij}, -\tfrac{1}{\tau}) \ = \ J_{\text{closed}}(s_{ij}, \tau)}$$

[Brödel, OS, Zerbini 1803.00527]

- so far requires ad-hoc convention how to use shuffle multiplication of $\mathcal{E}_0$
- should connect **esv** rules with single-valued elliptic polylogarithms

[D'Hoker, Green, Gürdogan, Vanhove 1512.06779]

- **esv** should relate to equivariant iterated Eisenstein integrals of Brown

[Brown 1407.5167, 1707.01230, 1708.03354]

IV. Closed-string low-energy expansion @ higher genus

IV. 1 Modular graph functions at higher genus

Four-point closed-string integral @ 1 loop $\Rightarrow$ modular graph functions

$$J_{\rm closed}(s_{ij},\tau) \, = \, \bigg(\prod_{j=2}^4 \int_{\Sigma(\tau)} \frac{ \mathrm{d}^2 z_j }{ \mathrm{Im}\, \tau } \bigg) \, \exp \Big(\sum_{i<j}^4 s_{ij} \, g(z_i-z_j,\tau) \Big) \Big|_{z_1=0}$$

Crucial information from degeneration near the cusp $\tau \rightarrow i\infty$

$$J_{\rm closed}(s_{ij},\tau) \, = \, 1 + (s_{12}^2{+}s_{12}s_{23}{+}s_{23}^2) \Big(\frac{2y^2}{45} + \frac{2\zeta_3}{y} + \mathcal{O}(e^{-2\pi \mathrm{Im}\, \tau}) \Big) + \mathcal{O}(\alpha'^3)$$

$\longrightarrow$ Laurent polynomials in $y \equiv \pi \, \mathrm{Im} \, \tau$ of **bounded** degree

IV. 1 Modular graph functions at higher genus

Four-point closed-string integral @ 1 loop $\Rightarrow$ modular graph functions

$$J_{\text{closed}}(s_{ij},\tau) \;=\; \left(\prod_{j=2}^{4} \int_{\Sigma(\tau)} \frac{\mathrm{d}^2 z_j}{\operatorname{Im}\tau} \right) \exp\left(\sum_{i<j}^{4} s_{ij}\, g(z_i - z_j,\tau) \right)\Big|_{z_1=0}$$

Crucial information from degeneration near the cusp $\tau \to i\infty$

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$\longrightarrow$ Laurent polynomials in $y \equiv \pi \operatorname{Im}\tau$ of **bounded** degree

Impressive novel achievements [D'Hoker, Green, Pioline 1712.06135, 1806.02691]

- generalize gen. series J_{closed} of modular graph functions to any genus g
- identify "good" parameter like y for non-separating degeneration @ $g>1$
- degeneration of Green functions and modular graph functions @ $g>1$

IV. 1 Modular graph functions at higher genus

At higher loops, need the following ingredients

- abelian differentials $\omega_{I=1,2,\ldots,g}$ on genus-g surface (@ g=1 only ω_1=dz)
- $g \times g$ period matrix Ω with positive definite imaginary part $\mathrm{Im}\,\Omega$
- Arakelov Green function $\mathcal{G}(z_i, z_j|\Omega)$ inverse to $\partial_z \overline{\partial}_z$ subject to

$$\int \sum_{I,J=1}^{g} (\mathrm{Im}\,\Omega^{-1})^{IJ}\, \omega_I(z)\, \overline{\omega}_J(z)\, \mathcal{G}(z,y|\Omega) \;=\; 0$$

At two loops, integrand of $\mathcal{M}^{\text{2-loop}}_{\text{closed}}$ w.r.t. d$^6\Omega$ reads **[D'Hoker, Phong 2005]**

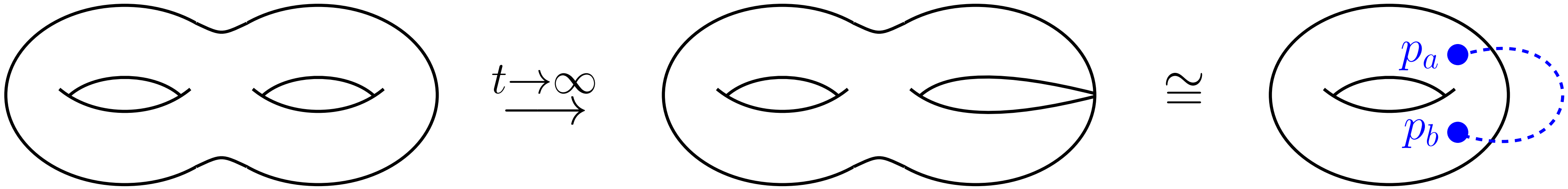
$$J^{\text{2-loop}}_{\text{closed}}(s_{ij}|\Omega) \;=\; \frac{1}{16} \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \overline{\mathcal{Y}}}{(\det \mathrm{Im}\,\Omega)^2}\; \exp\left\{ \sum_{1<i\leq j}^{4} s_{ij}\, \mathcal{G}(z_i, z_j|\Omega) \right\}$$

with $\Delta(z_i, z_j) \equiv \omega_1(z_i) \wedge \omega_2(z_j) - \omega_1(z_j) \wedge \omega_2(z_i)$ and

$$3\,\mathcal{Y} \;\equiv\; (s_{13} - s_{14})\, \Delta(z_1, z_2) \wedge \Delta(z_3, z_4) \;+\; \mathrm{cyc}(2,3,4)$$

IV. 2 Degeneration of higher-genus modular graph functions

Non-separating degeneration from genus g to genus $g-1$



- two extra punctures $\Sigma_g \to \Sigma_{g-1} \backslash \{p_b, p_a\}$

- block decompose $g \times g$ period matrix and send $\Omega_{gg} \to i\infty$

$$\Omega_{g\times g} = \begin{pmatrix} \tau_{(g-1)\times(g-1)} & v \\ v^t & \Omega_{gg} \end{pmatrix}, \qquad v_{I=1,2,\ldots,g-1} = \int_{p_a}^{p_b} \omega_I$$

Optimal degeneration parameter for bounded-degree Laurent polynomials

$$t \equiv \frac{\det \operatorname{Im} \Omega}{\det \operatorname{Im} \tau} = \operatorname{Im} \Omega_{gg} - \sum_{I,J=1}^{g-1} \operatorname{Im} v_I \, (\operatorname{Im} \tau^{-1})^{IJ} \operatorname{Im} v_J$$

[D'Hoker, Green, Pioline 1712.06135]

IV. 2 Degeneration of higher-genus modular graph functions

2-loop amplitude as generating series for genus-2 modular graph functions

$$J^{\text{2-loop}}_{\text{closed}}(s_{ij}|\Omega) = \frac{1}{16}\int_{\Sigma^4} \frac{\mathcal{Y}\wedge\overline{\mathcal{Y}}}{(\det \operatorname{Im}\Omega)^2} \exp\Big\{ \sum_{1<i\leq j}^{4} s_{ij}\, \mathcal{G}(z_i,z_j|\Omega)\Big\}$$

Using degeneration parameter $t = \operatorname{Im}\Omega_{22} - \frac{(\operatorname{Im} v)^2}{\operatorname{Im}\tau}$ at $\Omega_{22} \to i\infty$,

- Arakelov Green function has degree $(1,1)$ in t (modulo exp. suppressed)

$$\mathcal{G}\Big(z_i,z_j \,\Big|\, \begin{pmatrix} \tau & v \\ v & \Omega_{22}\end{pmatrix}\Big) = \frac{\pi t}{3g^2} + f_0(z_i,z_j,p_a,p_b|\tau) + \frac{f_1(z_i,z_j,p_a,p_b|\tau)}{t} + \mathcal{O}(e^{-2\pi t})$$

- modular graph functions from $J^{\text{2-loop}}_{\text{closed}}(s_{ij}|\Omega)$ with w powers of $\mathcal{G}(z_i,z_j)$

  $\to$ Laurent polynomials of degree (w,w) in t (modulo exp. suppressed)

  $\to$ Laurent coeff's $=$ v-dependent modular graph functions @ genus 1

- real power in the use of t: **bounded**-degree Laurent polynomials

IV. 2 Degeneration of higher-genus modular graph functions

α'-expansion of the two-loop integral

$$J^{\text{2-loop}}_{\text{closed}}(s_{ij}|\Omega) \ = \ 2\, \overbrace{(s_{12}^2+s_{13}^2+s_{14}^2)}^{\to\, D^4R^4} + 64\, \varphi(\Omega)\ \overbrace{(s_{12}^3+s_{13}^3+s_{14}^3)}^{\to\, D^6R^4} + \frac{1}{2}\, \Phi(\Omega)\ \overbrace{(s_{12}^2+s_{13}^2+s_{14}^2)^2}^{\to\, D^8R^4} + \mathcal{O}(\alpha'^5)$$

Subleading order D^6R^4: Kawazumi–Zhang inv. **[D'Hoker, Green 1308.4597]**

$$\varphi(\Omega) \ = \ -\frac{(\operatorname{Im}\Omega^{-1})^{IL}\,(\operatorname{Im}\Omega^{-1})^{JK}}{4} \int_{\Sigma^2} \omega_I(z_1)\, \overline{\omega}_J(z_1)\, \mathcal{G}(z_1,z_2)\, \omega_K(z_2)\, \overline{\omega}_L(z_2)$$

with degeneration **[D'Hoker, Green, Pioline 1712.06135, 1806.02691]**

$$\varphi\begin{pmatrix} \tau & v \\ v & \Omega_{22} \end{pmatrix} \ = \ \frac{\pi t}{6} + \frac{g_1(v|\tau)}{2} + \frac{5}{4\pi t}\Big[g_2(0|\tau) - g_2(v|\tau)\Big] + \mathcal{O}(e^{-2\pi t})$$

Generalized genus-1 modular graph functions with $v = r + \tau s$ & $r, s \in \mathbb{R}$

$$g_k(v|\tau) \ = \ \Big(\frac{\operatorname{Im}\tau}{\pi}\Big)^k \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{e^{2\pi i(nr-ms)}}{|m+\tau n|^{2k}}$$

IV. 2 Degeneration of higher-genus modular graph functions

α'-expansion of the two-loop integral

$$J^{\text{2-loop}}_{\text{closed}}(s_{ij}|\Omega) = 2\,\overbrace{(s_{12}^2+s_{13}^2+s_{14}^2)}^{\to D^4R^4} + 64\,\varphi(\Omega)\,\overbrace{(s_{12}^3+s_{13}^3+s_{14}^3)}^{\to D^6R^4} + \frac{1}{2}\,\Phi(\Omega)\,\overbrace{(s_{12}^2+s_{13}^2+s_{14}^2)^2}^{\to D^8R^4} + \mathcal{O}(\alpha'^5)$$

Higher-order invariant $\to D^8R^4$

$$\Phi(\Omega) = \frac{1}{64}\int_{\Sigma^4} \frac{|\Delta(z_1,z_3)\Delta(z_2,z_4)|^2}{(\det \operatorname{Im}\Omega)^2} \left\{\mathcal{G}(z_1,z_2)+\mathcal{G}(z_3,z_4)-\mathcal{G}(z_2,z_3)-\mathcal{G}(z_1,z_4)\right\}^2$$

with degeneration (numerators of t^{-1} and t^{-2} known)

$$\Phi\begin{pmatrix}\tau & v\\ v & \Omega_{22}\end{pmatrix} = \frac{8\pi^2t^2}{45} + \frac{2\pi t}{3}\,g_1(v|\tau) + \frac{7}{3}\,g_2(0|\tau) - \frac{2}{3}\,g_2(v|\tau) + \big(g_1(v|\tau)\big)^2$$
$$+ \frac{\text{more involved}}{\pi t} + \frac{\text{even more involved}}{\pi^2t^2} + \mathcal{O}(e^{-2\pi t})$$

Similar results for separating degenerations and worldline “tropical” limit.

[D’Hoker, Green, Pioline 1806.02691]

V. Conclusions & Outlook

V. 1 Exciting news I did not have a chance to talk about

- hyperbolic geometry & string perturbation theory / string field theory

  [Moosavian, Pius 1703.10563, 1706.07366, 1708.04977]

- low-energy expansion of 1-loop 4-graviton amplitude of heterotic strings, get modular graph forms ($\rightarrow$ nonzero modular weights)

  [Basu 1708.08409, 1710.01993]

- all-loop exact & non-perturbative coefficient of D^6R^4 interaction in dim $d < 10$ type-IIB theory (using new Poincaré-series techniques)

  [Ahlén, Kleinschmidt 1803.10250]

- exact non-perturbative coefficient of D^2F^4 in certain heterotic orbifolds

  [Bossard, Cosnier-Horeau, Pioline 1806.03330]

V. 2 Visions & long term goals for string amplitudes

- sidestep CFT: obtain open-string correlators from first principles

  $\longrightarrow$ double copy is likely to be part of the answer

- closed-string integrals from suitable projection of open-string data:

  $\longrightarrow$ expect higher-genus version of the single-valued projection

- mathematical framework for open-string integrals:

  $\longrightarrow$ combine the input from coaction principle and KZB associators

Thank you for your attention !

Backup I: Iterated Eisenstein integrals

Holomorphic Eisenstein series ($k \geq 4$ even, $q = e^{2\pi i\tau}$) & $\mathrm{G}_0 \equiv -1$

$$\mathrm{G}_k(\tau) \;=\; \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k} \;=\; 2\,\zeta_k \;+\; \frac{2\,(2\pi i)^k}{(k-1)!} \sum_{m,n=1}^{\infty} m^{k-1} q^{mn}$$

Backup I: Iterated Eisenstein integrals

Holomorphic Eisenstein series ($k \geq 4$ even, $q = e^{2\pi i\tau}$) & $\mathrm{G}_0 \equiv -1 \equiv \mathrm{G}_0^0$

$$\mathrm{G}_k(\tau) = \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k} = 2\,\zeta_k + \underbrace{\frac{2\,(2\pi i)^k}{(k-1)!} \sum_{m,n=1}^{\infty} m^{k-1} q^{mn}}_{\mathrm{G}_k^0(\tau)}$$

Define iterated Eisenstein integrals recursively by $\mathcal{E}_0(\,;\tau) = 1$ and

$$\mathcal{E}_0(k_1,k_2,\ldots,k_r;\tau) = (2\pi i)^{1-k_r} \int_\tau^{i\infty} \mathrm{d}\tau'\, \mathrm{G}^0_{k_r}(\tau')\, \mathcal{E}_0(k_1,k_2,\ldots,k_{r-1};\tau')$$

$k_1 \geq 4 \;\Rightarrow$ convergent integrals by zero-mode subtraction $\mathrm{G}^0_{k_1} = \mathrm{G}_{k_1} - 2\zeta_{k_1}$

$$\text{e.g.} \quad \mathcal{E}_0(k,\underbrace{0,0,\ldots,0}_{p-1};\tau) = \frac{-2}{(k-1)!} \sum_{m,n=1}^{\infty} \frac{m^{k-1}}{(mn)^p} q^{mn}$$

q-expansion straightforwardly inherited from above $\mathrm{G}^0_k(\tau)$.

Backup II: Modular transformation for open strings @ 1 loop

Back to symmetrized open-string integrals (recall abelianization $t^a \to \mathbb{1}$)

$$
\begin{aligned}
I_{\rm open}(s_{ij},\tau) \;=\; 1 \;+\; (s_{12}^2+s_{12}s_{23}+s_{23}^2)\left[\zeta_2 - 12\,\mathcal{E}_0(4,0;\tau)\right] \\
+\; s_{12}s_{13}s_{23}\Big[12\,\mathcal{E}_0(4,0,0;\tau)+300\,\mathcal{E}_0(6,0,0;\tau)-\frac{5}{2}\,\zeta_3\Big] \;+\; \mathcal{O}(\alpha'^4)
\end{aligned}
$$

Backup II: Modular transformation for open strings @ 1 loop

Back to symmetrized open-string integrals (recall abelianization $t^a \to \mathbb{1}$)

$$
\begin{aligned}
I_{\text{open}}(s_{ij},\tau) \; &= \; 1 \; + \; (s_{12}^2 + s_{12}s_{23} + s_{23}^2)\left[\zeta_2 - 12\,\mathcal{E}_0(4,0;\tau)\right] \\
&+ \; s_{12}s_{13}s_{23}\left[12\,\mathcal{E}_0(4,0,0;\tau) + 300\,\mathcal{E}_0(6,0,0;\tau) - \frac{5}{2}\zeta_3\right] \; + \; \mathcal{O}(\alpha'^4)
\end{aligned}
$$

Modular S-transformation $\tau \to -\frac{1}{\tau}$ follows from $\mathrm{G}_k(-\frac{1}{\tau}) = \tau^k \mathrm{G}_k(\tau)$

$$
\begin{aligned}
I_{\text{open}}(s_{ij},-\tfrac{1}{\tau}) = 1 - (s_{12}^2{+}s_{12}s_{23}{+}s_{23}^2)&\left[\frac{T^2}{90} - \frac{\pi^2}{9} - \frac{\pi^4}{30T^2} - \frac{2i\zeta_3}{T} + 12\,\mathcal{E}_0(4,0;\tau) + \frac{12i}{T}\mathcal{E}_0(4,0,0;\tau)\right] \\
+ \; s_{12}s_{13}s_{23}&\left[-\frac{iT^3}{756} + \frac{i\pi^2 T}{45} - \frac{\zeta_3}{2} - \frac{7i\pi^4}{72T} - \frac{2\pi^2\zeta_3}{T^2} + \frac{15\zeta_5}{2T^2} + \frac{17i\pi^6}{1890T^3} + \frac{12\pi^2}{T^2}\mathcal{E}_0(4,0,0;\tau) \right. \\
&\left. + 300\,\mathcal{E}_0(6,0,0;\tau) + \frac{900i}{T}\mathcal{E}_0(6,0,0,0;\tau) - \frac{900}{T^2}\mathcal{E}_0(6,0,0,0,0;\tau)\right] \; + \; \mathcal{O}(\alpha'^4)
\end{aligned}
$$

$\Longrightarrow$ coeff's of $q^{0,1,2,\dots}$ are Laurent polynomials in $T \equiv \pi\,\tau$ with MZVs.

$\Longrightarrow$ right structure to compare with closed-string quantities ...