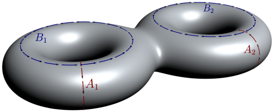
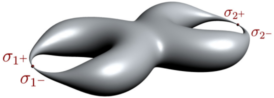
Two-loop integrands from the bi-nodal Riemann sphere

Yvonne Geyer

IAS





Amplitudes 2018

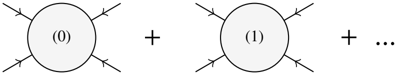
SLAC

arXiv:1805.05344 with R. Monteiro

arXiv:1507.00321, 1511.06315, 1607.08887
with L. Mason, R. Monteiro, P. Tourkine

Worldsheets and Residue Theorems

$\mathscr{M} =$



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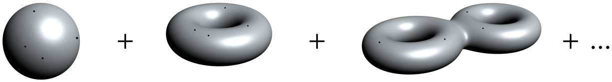
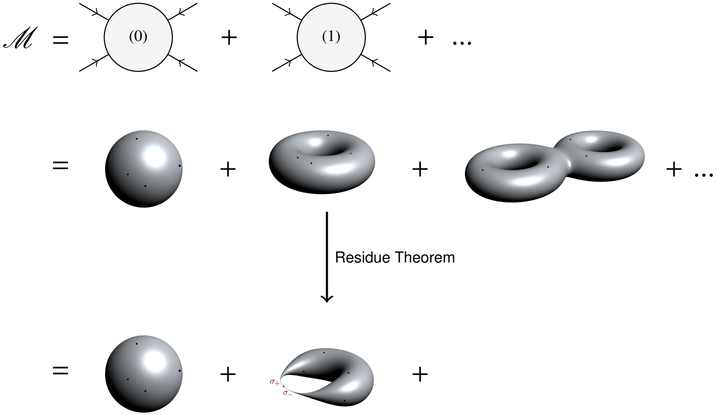


Worldsheets and Residue Theorems



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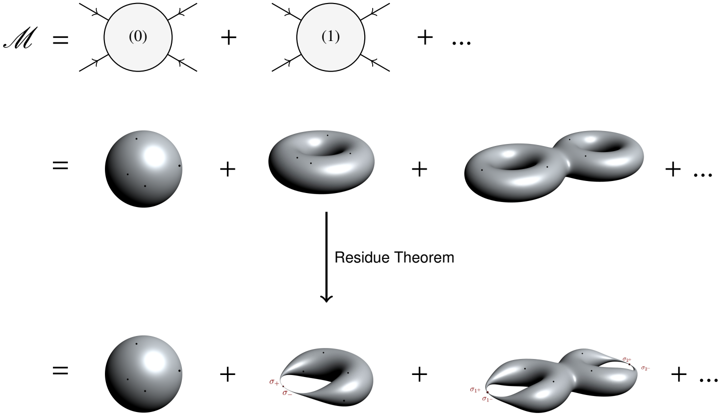


‘RNS’ Ambitwistor String [Mason-Skinner]

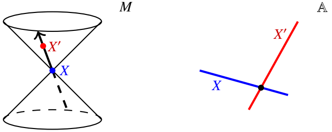
- Chiral worldsheet theory

NO α'

$$S = \frac{1}{2\pi}\int_\Sigma P\cdot\bar\partial X - \frac{\tilde e}{2}P^2 + \frac{1}{2}\Psi_r\cdot\bar\partial\Psi_r - \chi_r P\cdot\Psi_r$$

$$X^\mu \in \Omega^0(\Sigma),\ P_\mu \in \Omega^0(K_\Sigma),\ \Psi^\mu_{r=1,2} \in \Pi\Omega^0(K_\Sigma^{1/2}).$$

- BRST quantisation: free, linear CFTs; $d_{\text{crit}} = 10$.
- target space: $\mathbb{A}$ = phase space of complexified null geodesics



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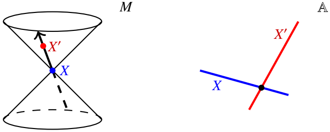


- Spectrum: type II supergravity

NO STRINGY MODES

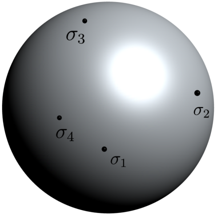
$$V = c\tilde c\delta(\gamma_1)\delta(\gamma_2)\epsilon_{\mu\nu}\Psi_1^\mu\Psi_2^\nu e^{ik\cdot X}$$

with $k^2 = \epsilon_{\mu\nu}k^\nu = \epsilon_{\mu\nu}k^\mu = 0$.

Tree-level amplitudes

correlator: $$\mathscr{M}_n^{(0)} \sim \left\langle \prod_{i=1}^{n} V(\sigma_i) \right\rangle$$

- P_μ localizes onto its classical EoM
$$\bar{\partial} P_\mu = 2\pi i \sum_i k_{i\,\mu} \bar{\delta}(\sigma - \sigma_i)\, \mathrm{d}\sigma\,.$$
- On the Riemann Sphere:
$$P_\mu = \sum_{i=1}^{n} \frac{k_{i\,\mu}}{\sigma - \sigma_i} \mathrm{d}\sigma$$



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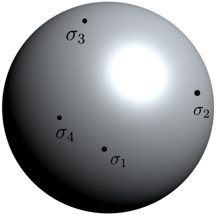

Amplitude localizes on the *scattering equations*: [Cachazo-He-Yuan]

$$E_i = \mathrm{Res}_{\sigma_i} P^2(\sigma) = k_i \cdot P(\sigma_i) = \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0\,.$$

Scattering Equations

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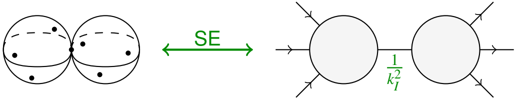
- Encode $P^2 = 0$.
- Factorisation: [Dolan-Goddard, YG-Mason-Monteiro-Tourkine, ...]



boundary of $\mathfrak{M}_{0,n}$ factorisation channel

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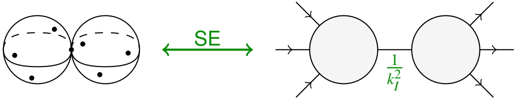
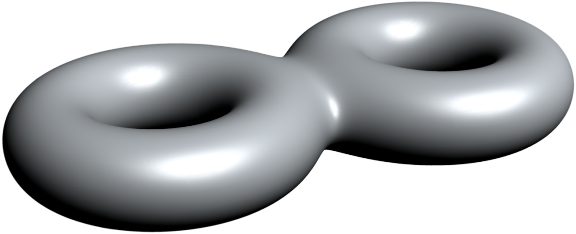


boundary of $\mathfrak{M}_{0,n}$ factorisation channel

- correlator = CHY representation of the amplitude [Cachazo-He-Yuan]

$$\mathcal{M}_n^{(0)} = \int_{\mathfrak{M}_{0,n}} \frac{\mathrm{d}^n\sigma}{\text{vol}\, G} \prod_i \bar{\delta}\left(\text{Res}_i P^2\right) \mathcal{I}(\sigma_i, k_i, \epsilon_i)\,.$$

What about loops?

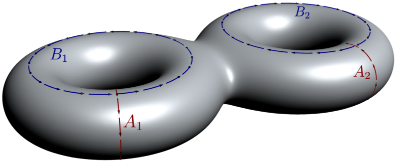


Genus Two [Adamo-Casali, YG-Monteiro]

Moduli space $\mathfrak{M}_{2,n}$ at genus two:

- n marked points z_i
- $\Omega_{IJ} = \oint_{B_I} \omega_J$, where $\delta_{IJ} = \oint_{A_I} \omega_J$,

ω_J holomorphic 1-forms



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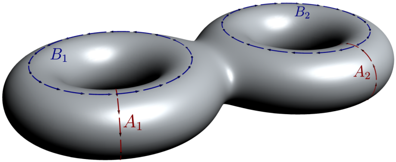

Localization equation for P_μ:

$$\bar{\partial} P_\mu = 2\pi i \sum_i k_{i\,\mu} \bar{\delta}(z - z_i)\, \mathrm{d}z \qquad \Rightarrow \qquad P_\mu = \ell^I_\mu \omega_I + \sum_i k_{i\,\mu} \omega_{i,z_0}$$

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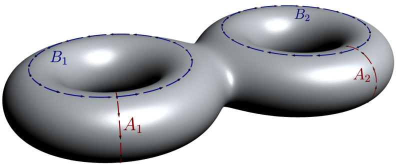

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- zero modes $\ell^I_\mu = \frac{1}{2\pi} \oint_{A_I} P_\mu \;\sim$ loop momenta
- correlator localizes again on $P^2 = 0$:

Genus two scattering equations:

$$E_i = \mathrm{Res}_{z_i} P^2(z) = 0\,, \qquad \Rightarrow P^2 = u^{IJ} \omega_I \omega_J \text{ for some } u^{IJ}$$

$$u^{IJ} = 0 \qquad \text{for } I, J = 1, 2\,.$$

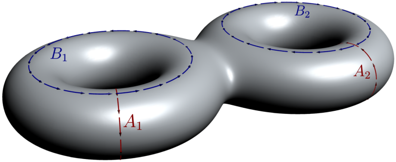
Genus two amplitude

Genus two n-point amplitude:

$$\mathscr{M}^{(2)} = \int \mathrm{d}^{10}\ell_1 \mathrm{d}^{10}\ell_2 \quad \int_{\mathfrak{M}_{2,n}} \mathrm{d}^3\Omega \prod \bar{\delta}\left(u^{IJ}\right) \bar{\delta}\left(\mathrm{Res}_i P^2\right) \mathcal{I} \,.$$

$\mathcal{I} = \sum_{\delta,\tilde{\delta}} \mathcal{Z}_{\delta|\tilde{\delta}} \langle \Upsilon_1 \Upsilon_2 V_1 \ldots V_n \rangle$

- sum over spin structures
- partition function $\mathcal{Z}_{\delta|\tilde{\delta}}$
- correlator over PCOs Υ, VOs V



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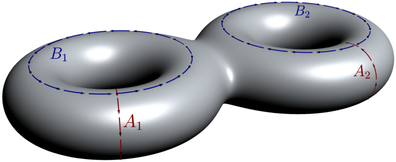

The 4pt amplitude

Simplification on SE:

$$I = \mathcal{K}\tilde{\mathcal{K}}\, \mathcal{Y}^2 \,, \qquad \text{where } \mathcal{Y} = s\Delta_{14}\Delta_{23} - t\Delta_{12}\Delta_{34} \,,$$

with $\Delta_{ij} = \epsilon^{IJ}\omega_I(z_i)\omega_J(z_j)$ and $\mathcal{K} = stA^{\text{SYM}}_{g=0,n=4}$.

Two Comments

Genus two integrand:

$$\mathscr{M}^{(2)} = \int \mathrm{d}^{10}\ell_1 \mathrm{d}^{10}\ell_2 \; \mathfrak{I} \qquad \mathfrak{I} = \int_{\mathfrak{M}_{2,n}} \mathrm{d}^3\Omega \prod_{i,I,J} \bar{\delta}\left(u^{IJ}\right) \bar{\delta}\left(\mathrm{Res}_i P^2\right) \mathcal{I}$$

- Localization
  Loop integrand $\mathfrak{I}$ is *fully localized* on SE.

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- Localization

  Loop integrand $\mathfrak{I}$ is *fully localized* on SE.

- Modular invariance

  $g = 2$ modular transformations: $\mathrm{Sp}(4, \mathbb{Z})$

  This does NOT imply finiteness of the amplitude!

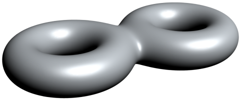
non-compact moduli space	$\Leftrightarrow$	integration over loop momenta ℓ^I

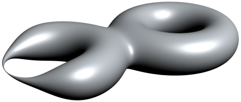
The residue theorem

Key features:

- localization
- modular invariance

$\Rightarrow$ Residue Theorem





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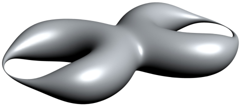

$\rightsquigarrow$

A simpler case: the torus [YG-Mason-Monteiro-Tourkine]

$$\mathscr{M}^{(1)} = \int d^{10}\ell\, \mathfrak{I}^{(1)}\,, \qquad \mathfrak{I}^{(1)} = \int_{\mathfrak{M}_{1,n}} d\tau\ \underbrace{\bar{\delta}\left(P^2(z_0)\right)\prod_{i=2}^{n}\bar{\delta}\left(\mathrm{Res}_{z_i}P^2\right)}_{\text{1-loop SE}}\,\mathcal{I}^{(1)}$$

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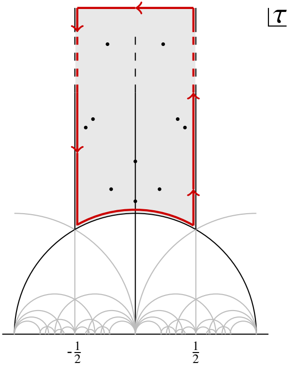
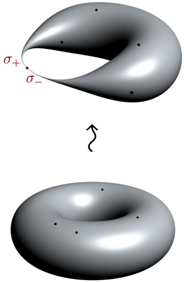
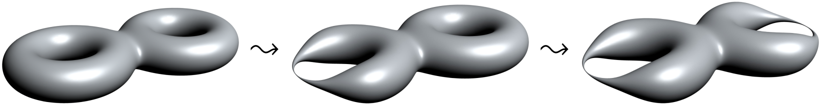
- modular invariance: $\tau \sim \tau + 1 \sim -1/\tau$
- Residue theorem: localizes $\mathfrak{I}^{(1)}$ to $\tau = i\infty \Leftrightarrow q \equiv e^{2i\pi\tau} = 0$.





Requirements



Requirements:

- ☑ simple pole at non-separating degeneration
- ☐ no other poles
- ☐

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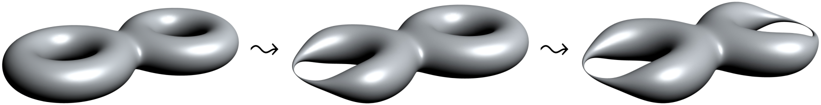

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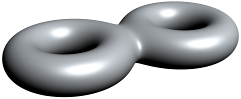
Freedom:

- representation of integrand (up to SE)
- choice of scattering equations: $\{u_1, u_2, u_3\}$

$$\mathrm{Span}(u_1, u_2, u_3) = \mathrm{Span}(u^{11}, u^{12}, u^{22})$$

- modular invariance

Parametrisation of $\mathfrak{M}_{2,n}$: $\Omega = \begin{pmatrix} \tau_1 + \tau_3 & \tau_3 \\ \tau_3 & \tau_2 + \tau_3 \end{pmatrix}, \quad q_r = e^{i\pi\tau_r}$



Number of modular parameters:

$$n + 3$$

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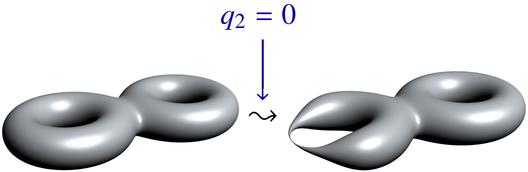

Number of modular parameters:

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$n + 2$ $\{z_i, q_1, q_3\}$

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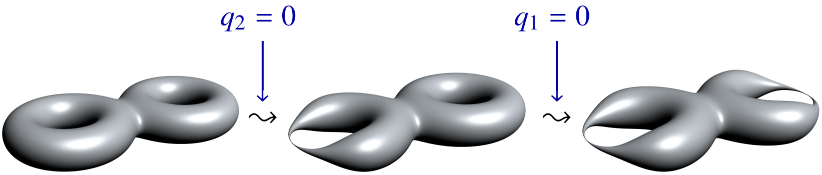
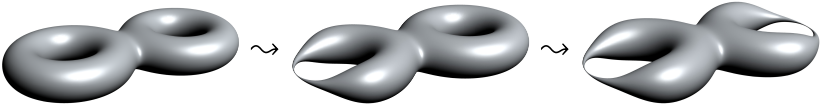


Number of modular parameters:

$n + 3$	$n + 2$	$n + 1$
$\{z_i, q_1, q_2, q_3\}$	$\{z_i, q_1, q_3\}$	$\{z_i, q_3\}$

Requirements



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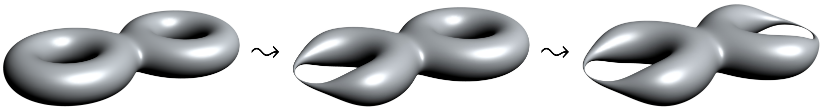

Requirements:

- ☑ simple pole at non-separating degeneration
- ☐ no other poles
- ☐ map remaining modular parameter to bi-nodal sphere

Freedom:

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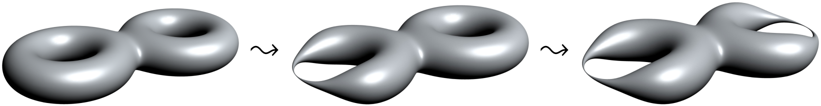
- modular invariance

From $g = 2$ to the bi-nodal Riemann Sphere

Convenient notation: $\mathfrak{I} = \mathfrak{I}(u_1, u_2, u_3)$



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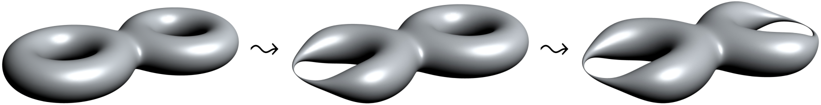

Residue theorem:

$$\mathfrak{I}(u_1, u_2, u_3) = -\underbrace{\mathfrak{I}(u_1, q_1, u_3)}_{\overset{!}{=}0} - \mathfrak{I}(u_1, q_2, u_3) - \underbrace{\mathfrak{I}(u_1, q_3, u_3)}_{\overset{!}{=}0}$$

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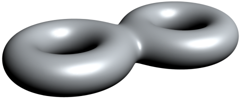
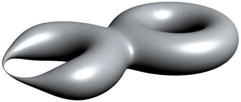
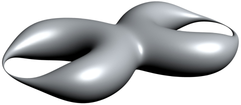
 $\rightsquigarrow$  $\rightsquigarrow$ 

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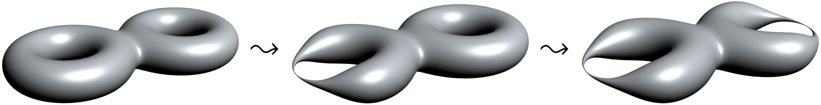
$$= \mathfrak{I}(q_1, q_2, u_3) + \underbrace{\mathfrak{I}(q_3, q_1, u_3)}_{=0} + \underbrace{\mathfrak{I}(u_2, q_1, u_3)}_{\stackrel{!}{=}0}$$

From $g = 2$ to the bi-nodal Riemann Sphere

Convenient notation: $\mathfrak{I} = \mathfrak{I}(u_1, u_2, u_3)$



Residue theorem:

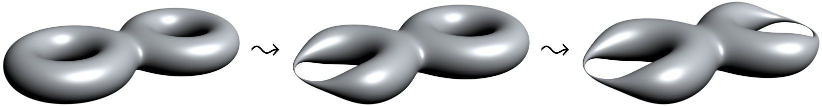
$$\mathfrak{I}(u_1, u_2, u_3) = -\underbrace{\mathfrak{I}(u_1, q_1, u_3)}_{\stackrel{!}{=}0} - \mathfrak{I}(u_1, q_2, u_3) - \underbrace{\mathfrak{I}(u_1, q_3, u_3)}_{\stackrel{!}{=}0}$$

$$= \mathfrak{I}(q_1, q_2, u_3) + \underbrace{\mathfrak{I}(q_3, q_1, u_3)}_{=0} + \underbrace{\mathfrak{I}(u_2, q_1, u_3)}_{\stackrel{!}{=}0}$$

Constraints: $\mathfrak{I}(u_1, q_1, \bullet) = \mathfrak{I}(u_2, q_2, \bullet) = \mathfrak{I}(u_3, q_3, \bullet) = 0$

Solution: $u_1 = u^{11}\,, \qquad u_2 = u^{22}\,, \qquad u_3 = u^{11} + u^{22} + u^{12}$

New features at two loops



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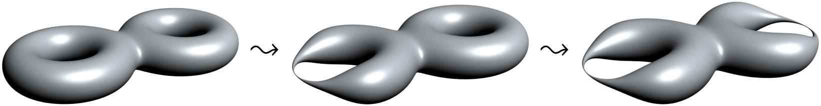

- Scattering equations and integrand
  (only poles at non-separating degenerations)

$$u^{11} = u^{22} = u^{12} + u^{11} + u^{22} = 0$$

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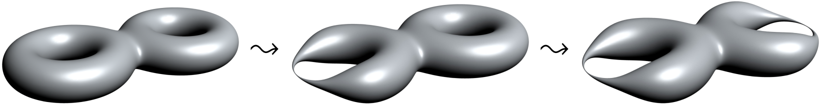

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  Fundamental domain: $|q_3| = |e^{2i\pi\tau_3}| \leq 1$
  On bi-nodal RS: $q_3 \sim \sigma_{1^+} \in \mathbb{C}$

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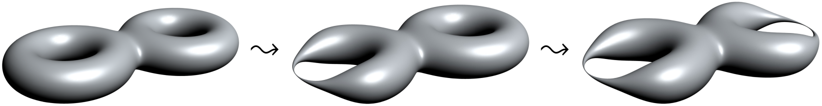

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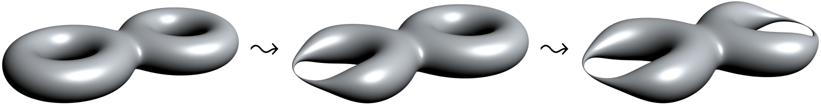
  On bi-nodal RS: $q_3 \sim \sigma_{1^+} \in \mathbb{C}$
  $\Rightarrow$ extend DoI using modular invariance
  $\Rightarrow$ $1/(1 - q_3)$ in integrand

Requirements



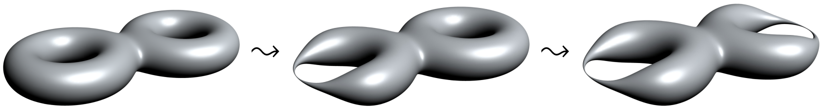
Requirements:

- ☑ simple pole at non-separating degeneration
- ☐ no other poles
- ☐ map remaining modular parameter to bi-nodal sphere

Freedom:

- representation of integrand (up to SE)
- choice of SE: $\{u_1, u_2, u_3\}$
- modular invariance

Requirements



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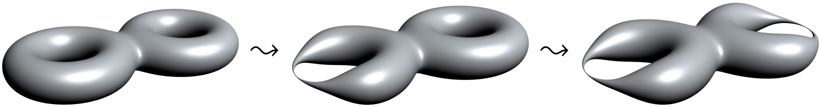

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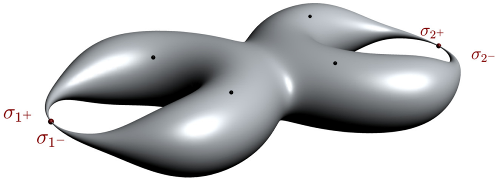
On the bi-nodal Riemann sphere

Basic objects:

- modular parameter $q_3 = \frac{(\sigma_{1^+} - \sigma_{2^+})(\sigma_{1^-} - \sigma_{2^-})}{(\sigma_{1^+} - \sigma_{2^-})(\sigma_{1^-} - \sigma_{2^+})}$
- holomorphic differentials $\omega_I = \frac{1}{\sigma - \sigma_{I^+}} - \frac{1}{\sigma - \sigma_{I^-}}$
- P resembles tree-level, but with 4 more particles:

$$P = \sum_{J=1,2} \frac{\ell_J}{\sigma - \sigma_{J^+}} - \frac{\ell_J}{\sigma - \sigma_{J^-}} + \sum_i \frac{k_i}{\sigma - \sigma_i}$$



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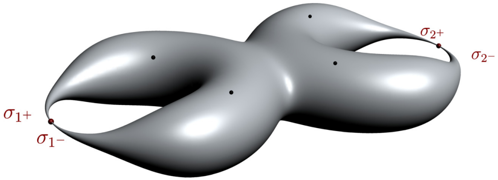

Amplitudes:

$$\mathcal{M}_n = \int \frac{\mathrm{d}^{10}\ell_1\, \mathrm{d}^{10}\ell_2}{\ell_1^2\, \ell_2^2} \int_{\mathfrak{M}_{0,n+4}} \frac{\mathrm{d}^{n+4}\sigma_A}{\text{vol SL}(2,\mathbb{C})^2} \prod_A \bar{\delta}(\mathcal{E}_A)\, \mathcal{I}_n^{(2)}$$

Two-loop SE

Compact form:

$$\mathcal{E}_A = \text{Res}_A\Big(P^2 - \ell_I^2\omega_I^2 + (\ell_1^2 + \ell_2^2)\omega_1\omega_2\Big) = 0$$

with $A \in \{\sigma_{1^\pm}\sigma_{2^\pm}, \sigma_1, \ldots, \sigma_n\}$.

$$\pm\,\mathcal{E}_{1^\pm} = \frac{1}{2}(\ell_1 + \ell_2)^2\left(\frac{1}{\sigma_{1^\pm} - \sigma_{2^+}} - \frac{1}{\sigma_{1^\pm} - \sigma_{2^-}}\right) + \sum_j \frac{\ell_1 \cdot k_j}{\sigma_{1^\pm} - \sigma_j}\,,$$

$$\pm\,\mathcal{E}_{2^\pm} = \frac{1}{2}(\ell_1 + \ell_2)^2\left(\frac{1}{\sigma_{2^\pm} - \sigma_{1^+}} - \frac{1}{\sigma_{2^\pm} - \sigma_{1^-}}\right) + \sum_j \frac{\ell_2 \cdot k_j}{\sigma_{2^\pm} - \sigma_j}\,,$$

$$\mathcal{E}_i = k_i \cdot \ell_1\left(\frac{1}{\sigma_i - \sigma_{1^+}} - \frac{1}{\sigma_i - \sigma_{1^-}}\right) + k_i \cdot \ell_2\left(\frac{1}{\sigma_i - \sigma_{2^+}} - \frac{1}{\sigma_i - \sigma_{2^-}}\right) + \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j}$$

Two-loop SE

Compact form: from $u_3 = u^{11} + u^{22} + u^{12}$

$$\mathcal{E}_A = \mathrm{Res}_A\Big(P^2 - \ell_I^2\omega_I^2 + (\ell_1^2 + \ell_2^2)\omega_1\omega_2\Big) = 0$$

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Integrands on the bi-nodal Riemann sphere

$$\mathscr{M}_n = \int \frac{\mathrm{d}^{10}\ell_1\,\mathrm{d}^{10}\ell_2}{\ell_1^2\,\ell_2^2} \int_{\mathfrak{M}_{0,n+4}} \frac{\mathrm{d}^{n+4}\sigma_A}{\text{vol}\,\mathrm{SL}(2,\mathbb{C})^2} \prod_A \bar{\delta}(\mathcal{E}_A)\;\mathscr{I}_n^{(2)}$$

- complexity of integrands $\sim$ tree-level

$$\mathscr{I}^{\text{sugra}}_{(2),n} = \mathcal{I}_n^{(2)}(\epsilon)\;\mathcal{I}_n^{(2)}(\tilde{\epsilon})\;\frac{(\sigma_{1^+}-\sigma_{2^-})(\sigma_{1^-}-\sigma_{2^+})}{(\sigma_{1^+}-\sigma_{1^-})(\sigma_{2^+}-\sigma_{2^-})}$$

Integrands on the bi-nodal Riemann sphere

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- Super Yang-Mills via *'inverse double copy'*:

$$\mathscr{I}^{\text{sYM}}_{(2),n} = \mathcal{I}_n^{(2)}(\epsilon)\ \mathcal{I}_n^{\text{PT}(2)}$$

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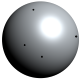
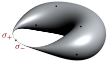
- formulae valid in any dimension d

Outlook

Summary:

$\mathscr{M}$ =  + 

+ 

+ ...

Outlook

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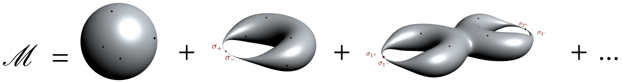

Future directions:

- Non-supersymmetric theories c.f. [YG-Mason-Monteiro-Tourkine]
- ‘gluing operator’ to implement the nodes c.f. [Roehrig-Skinner]
- factorisation proof c.f. [Dolan-Goddard, YG-Mason-Monteiro-Tourkine]
- BCJ numerators? c.f. [He-Schlotterer-Zhang, Du-Feng-Fu-Huang, YG-Monteiro, ...]

Thank you!