# Light-ray operators in conformal field theory

David Simmons-Duffin
(w/ Petr Kravchuk)

Caltech

June 21, 2018

# The OPE in Lorentzian signature

Consider a CFT correlation function $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$

- Euclidean signature: all singularities described by the OPE

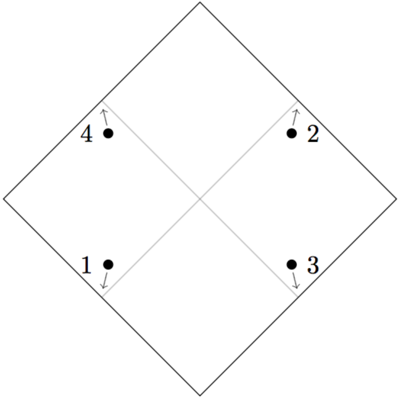
$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_k f_{12k} x_{12}^{\Delta_k - \Delta_1 - \Delta_2} \mathcal{O}_k(x_2)$$

- Lorentzian signature: the OPE is valid when both operators act on the vacuum [Mack]

$$\mathcal{O}_1 \mathcal{O}_2 |\Omega\rangle = \sum_k f_{12k} \mathcal{O}_k |\Omega\rangle$$

But it's easy to find situations where $x_{12}^2 \to 0$ and the $\mathcal{O}_1 \times \mathcal{O}_2$ OPE doesn't work.

# Example: Regge limit




- Position-space version of high-energy scattering.
- $\mathcal{O}_1, \mathcal{O}_3$ create excitations that scatter, measured by $\mathcal{O}_2, \mathcal{O}_4$.
- $x_{12}^2 \to 0$ but the $\mathcal{O}_1 \times \mathcal{O}_2$ OPE is not valid

$$\langle \Omega | T\{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4\} | \Omega \rangle = \langle \Omega | \mathcal{O}_4 \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 | \Omega \rangle$$

# Conformal Regge theory

- The problem of describing the Regge limit in CFT was partially solved in [Brower, Polchinski, Strassler, Tan '06] [Cornalba '07] [Costa, Goncalves, Penedones '12]
- From Grisha's slides:

$$A(y) = \oint \frac{d\Delta}{2\pi i} \oint \frac{dJ}{2\pi i} \frac{\pi}{2\sin(\pi J)} \frac{y^J Q_{\Delta,J}}{h(\Delta,J) - \xi^4}$$

$$h(\Delta,J) = \frac{1}{16}(\Delta+J)(\Delta+J-2)(\Delta-J-2)(\Delta-J-4)$$

  - $y^J Q_{\Delta,J}$ is a conformal partial wave
  - $(h(\Delta,J)-\xi^4)^{-1}$ encodes operator dimensions (poles) and OPE coefficients (residues)
  - Requires analytic continuation in spin of CFT data

$$J = \sqrt{1 + (\Delta-2)^2 + 2\sqrt{(\Delta-2)^2 + 4\xi^4}} - 1$$

# Lorentzian Inversion Formula [Caron-Huot '17]

(Euclidean) harmonic analysis:

$$\langle \phi_1 \cdots \phi_4 \rangle = \sum_{J=0}^{\infty} \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} C(\Delta, J) G_{\Delta,J}(x_i)$$

- Poles of $C(\Delta, J)$ encode operator dimensions, residues encode OPE coefficients.

$$C(\Delta, J) \sim -\sum_i \frac{f_{12i} f_{34i}}{\Delta - \Delta_i}$$

- $G_{\Delta,J}$ is a conformal block.
- Close $\Delta$ contour to the right $\implies$ conformal block expansion.

Caron-Huot's Lorentzian inversion formula:

$$C(\Delta, J) = \frac{\kappa_{\Delta+J}}{4} \int_0^1 \int_0^1 dz d\bar{z} \mu(z, \bar{z}) G_{J+d-1,\Delta-d+1}(z, \bar{z}) \\ \times \langle \Omega | [\phi_4, \phi_1][\phi_2, \phi_3] | \Omega \rangle$$

# Lorentzian Inversion Formula [Caron-Huot '17]

$$C(\Delta,J) = \frac{\kappa_{\Delta+J}}{4}\int_0^1\int_0^1 dz d\bar{z}\mu(z,\bar{z})G_{J+d-1,\Delta-d+1}(z,\bar{z}) \times \langle\Omega|[\phi_4,\phi_1][\phi_2,\phi_3]|\Omega\rangle$$

Awesome:

- Analytic in spin — justifies conformal Regge theory in a general nonperturbative CFT. (Analog of Froissart-Gribov formula for amplitudes.)
- Many applications to analytic bootstrap, holography, etc.

But mysterious:

- Can operators themselves be analytically continued in spin?
- What is the space of continuous-spin operators in a CFT? What is their physical meaning?
- What's up with the funny block?

$$(\Delta,J)\to(J+d-1,\Delta-d+1)$$

# Continuous spin operators

**Integer spin** (traceless symmetric): $\mathcal{O}^{\mu_1\cdots\mu_J}(x)$. Let

$$\mathcal{O}(x,z) \equiv \mathcal{O}^{\mu_1\cdots\mu_j}(x) z_{\mu_1}\cdots z_{\mu_J}, \quad (z^2=0)$$

- $z^\mu$ is a null polarization vector
- $\mathcal{O}(x,z)$ is a homogeneous polynomial in $z$ of degree $J$
- Can go between $\mathcal{O}^{\mu_1\cdots\mu_J}(x)$ and $\mathcal{O}(x,z)$
- Transforms as $U_g \mathcal{O}(x,z) U_g^{-1} = \Omega(x')^\Delta \mathcal{O}(x', R(x')z)$

**Continuous spin**: $\mathbb{O}(x,z)$

- Homogeneous in $z$, but not polynomial

$$\mathbb{O}(x,\lambda z) = \lambda^J \mathbb{O}(x,z), \qquad \lambda > 0, J \in \mathbb{C}$$

- Transforms in the same way as above.

# Continuous spin operators

**Lemma:** Continuous spin operators kill the vacuum

$$\mathbb{O}(x,z)|\Omega\rangle = 0$$

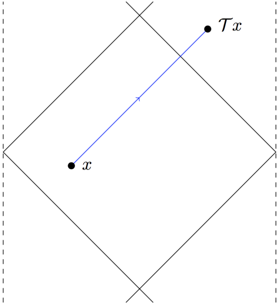
**Proof:** Positive-energy representations of the conformal group $\widetilde{\mathrm{SO}}(d,2)$ have been classified [Mack '77] and they have integer spin.

- By the state-operator correspondence, continuous spin operators must be *nonlocal*.

**Puzzle:** How can we analytically continue local operators $\mathcal{O}_{\Delta,J}$ in $J$, since they don't kill the vacuum?
**Answer:** Build something nonlocal out of $\mathcal{O}_{\Delta,J}$ that does kill the vacuum, and analytically continue that.

# The light transform

$$\mathbf{L}[\mathcal{O}](x,z) \equiv \int_{-\infty}^{\infty} d\alpha(-\alpha)^{-\Delta-J}\mathcal{O}\left(x-\frac{z}{\alpha},z\right)$$




If we put the initial point at past null infinity,

$$\mathbf{L}[\mathcal{O}](-\infty z,z) = \int_{-\infty}^{\infty} dx^{-}\,\mathcal{O}_{-\cdots-}(x^{-})$$

Example: $\mathbf{L}[T]$ is the "average null energy" operator.

# Knapp-Stein intertwiners

- $\mathbf{L}$ is a conformally-invariant integral transform

$$\mathbf{L} : (\Delta, J) \to (1-J, 1-\Delta)$$

(e.g. $\mathbf{L}[T]$ is a primary with dimension $-1$ and spin $1-d$.)

- Part of a dihedral group $D_8$ of Lorentzian transforms.
- Another element of $D_8$ is the "shadow" transform

$$\mathbf{S}[\mathcal{O}](x) = \int d^d y \frac{1}{(x-y)^{2(d-\Delta)}} \mathcal{O}(y)$$

$$\mathbf{S} : (\Delta, J) \to (d-\Delta, J)$$

- $D_8$ preserves the conformal Casimirs, e.g.

$$C_2(\Delta, J) = \Delta(\Delta - d) + J(J + d - 2)$$

- By our lemma, $\mathbf{L}[\mathcal{O}_{\Delta,J}]|\Omega\rangle = 0$ for any local operator $\mathcal{O}_{\Delta,J}$.

# Analyticity in spin at the operator level

**Claim:** $\mathbf{L}[\mathcal{O}_{\Delta,J}]$ has a natural analytic continuation in $J$.

**Example:** double-trace operators in 2d Generalized Free Theory

$$[\phi\phi]_J(u,v) = :\phi(u,v)\partial_v^J\phi(u,v):$$

Analytic continuation of $\mathbf{L}[[\phi\phi]_J]$:

$$\mathbb{O}_J = \frac{i\Gamma(J+1)}{2^J}\int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty}\frac{ds}{2\pi}\left(\frac{1}{(s+i\epsilon)^{J+1}}+\frac{1}{(-s+i\epsilon)^{J+1}}\right) \times :\phi(0,v+s)\phi(0,v-s):$$

When $J \in 2\mathbb{Z}_{\geq 0}$,

$$\frac{i\Gamma(J+1)}{2\pi}\left(\frac{1}{(s+i\epsilon)^{J+1}}+\frac{1}{(-s+i\epsilon)^{J+1}}\right) = \frac{\partial^J\delta(s)}{\partial s^J}$$

$$\implies \mathbb{O}_J = \int_{-\infty}^{\infty} dv :\phi(0,v)\partial_v^J\phi(0,v): = \mathbf{L}[[\phi\phi]_J]$$

# Gauge theory & BFKL physics

- The importance of light-ray operators in gauge theories is well-known [Balitsky, Fadin, Kuraev, Lipatov]
- Single-trace twist-2 trajectory:

$$\int_{-\infty}^{\infty} dy^- \int_0^\infty \frac{du}{u^{j+1}} \mathrm{Tr}[\phi(y^- + u) W(y^- + u, y^- - u)\phi(y^- - u)]$$

  $W(u_1, u_2)$ is a null Wilson line from $u_1$ to $u_2$.
- Reggeized gluons:

$$\int d^2\vec{y}_1 d^2\vec{y}_2 f_\nu(\vec{y}_1, \vec{y}_2, \vec{y}_3) \mathrm{Tr}[U^\dagger(\vec{y}_1) U(\vec{y}_2)]$$

  $U(\vec{y})$ is an infinite null Wilson line at transverse position $\vec{y}$.

What is a null Wilson line in a general nonperturbative CFT? In the 3d Ising model?

# General construction

Using harmonic analysis, we find $K_{\Delta,J}$ such that

$$\mathbb{O}_{\Delta,J}(x,z) = \int d^d x_1 d^d x_2 K_{\Delta,J}(x_1,x_2,x,z)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)$$

- Transforms with dimension $1-J$ and spin $1-\Delta$.
- Is meromorphic in $\Delta$ and $J$ and has poles of the form

$$\mathbb{O}_{\Delta,J}(x,z) \sim \frac{1}{\Delta - \Delta_i(J)}\mathbb{O}_{i,J}(x,z)$$

  Poles come from neighborhood of a lightcone/lightray
- When $J$ is an integer, $\mathbb{O}_{i,J} = \mathbf{L}[\mathcal{O}_{\Delta_i,J}]$ where $\mathcal{O}_{\Delta_i,J} \in \mathcal{O}_1 \times \mathcal{O}_2$ (by construction)
- When $J$ is non-integer, $\mathbb{O}_{i,J}$ are genuinely nonlocal light-ray operators

# Lorentzian inversion revisited

Computing matrix elements of light-ray operators gives a proof/generalization of the Lorentzian inversion formula

$$C(\Delta, J) = -\frac{1}{2\pi i}\int \frac{d^dx_1 \cdots d^dx_4}{\mathrm{vol}\,\widetilde{\mathrm{SO}}(d,2)}\langle\Omega|[\mathcal{O}_4,\mathcal{O}_1][\mathcal{O}_2,\mathcal{O}_3]|\Omega\rangle \times \frac{\langle\mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}]\rangle^{-1}\langle\mathcal{O}_4\mathcal{O}_3\mathbf{L}[\mathcal{O}]\rangle^{-1}}{\langle\mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}]\rangle^{-1}}$$

- $\mathcal{O}$ has dimension $\Delta$ and spin $J$
- $\langle\cdots\rangle^{-1}$ uses natural pairing between two/three-pt structures
- "Ratio" of structures means a conformal block
- Formula works for arbitrary $\mathcal{O}_i$ (not just scalars)
- "Funny" quantum numbers are dual to light-transform

$$d^dx\, d^dz\delta(z^2)\mathcal{O}_{1-J,1-\Delta}(x,z)\mathcal{O}_{J+d-1,\Delta-d+1}(x,z)$$

# Conformal Regge theory revisited

A natural formula for Regge correlators

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_4 \rangle \overset{\text{Regge}}{\sim} \oint dJ \oint \frac{d\Delta}{2\pi i} \frac{C(\Delta, J)}{1 - e^{-2\pi i J}} \frac{\langle \mathcal{O}_1 \mathcal{O}_2 \mathbf{L}[\mathcal{O}] \rangle \langle \mathcal{O}_3 \mathcal{O}_4 \mathbf{L}[\mathcal{O}] \rangle}{\langle \mathbf{L}[\mathcal{O}] \mathbf{L}[\mathcal{O}] \rangle}$$

- Interpretation: the Regge limit is an expansion in light-ray operators
- For example, the Reggeon/Pomeron are (families of) light-ray operators

# The average null energy condition (ANEC)

- The ANEC says that $\mathbf{L}[T]$ is positive semidefinite

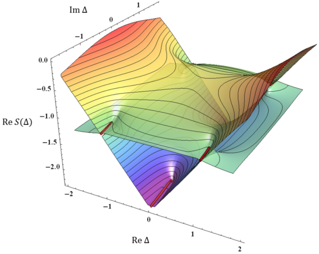
$$\langle\Psi|\mathbf{L}[T]|\Psi\rangle \geq 0$$

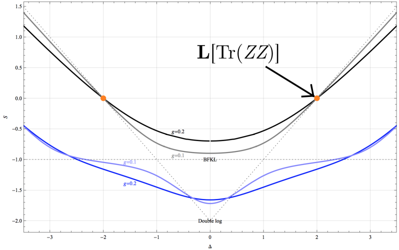
- Recently proved using information theory [Faulkner, Leigh, Parrikar, Wang '16] and causality [Hartman, Kundu, Tajdini '16]. Applications: [Hofman, Maldacena '08], [Cordova, Maldacena, Turiaci '17], [Cordova, Diab '17], [Delacrétaz, Hartman, Hartnoll, Lewkowycz '18]
- The inversion formula for matrix elements of light-ray operators gives a new proof of the ANEC.
- Can be generalized to continuous spin: every light-ray operator on the leading Regge trajectory is positive

$$\langle\Psi|\mathbb{O}_{\Delta_{\min}(J),J}|\Psi\rangle \geq 0, \qquad (J > J_0)$$

# Discussion

- Light-ray operators fit into a more rigid structure than local operators, due to analyticity in spin.
- They control the Regge limit of CFT correlators.
- What does the Riemann surface of light-ray operators look like in the $\Delta$-$J$ plane? Nonplanar $\mathcal{N}=4$? 3d Ising? Beautiful pictures in planar $\mathcal{N}=4$ [Gromov, Levkovich-Maslyuk, Sizov '15].




- Is there a "complete basis" of Lorentzian operators that gives a convergent expansion around Lorentzian singularities?
- Applications to Lorentzian observables like amplitudes/PDFs/energy correlators in general CFTs?