Continuum limit of fishnet graphs and AdS sigma model

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LPTENS

Amplitudes 2018
SLAC

based on recent work with De-liang Zhong
& older work with Lance Dixon

Motivation

Understand dynamics of planar graphs and its relation to sigma models

['t Hooft]

Best possible starting point: N=4 SYM

[Maldacena'97]

1. String dual is believed to be known
2. Theory is believed to be integrable
   (we have methods for re-summing planar graphs)

Use solution to gain knowledge about other models by deforming / twisting the theory $\longrightarrow$ partial re-summations

Reduce complexity, but maintain as many important properties as possible: conformal symmetry, integrability, etc

Fishnet theory

Baby version of N = 4 SYM

A theory for matrix scalar fields with quartic coupling

[Gurdogan,Kazakov'15]
[Caetano,Gurdogan,Kazakov'16]

$$\mathcal{L}_{\text{fishnet}} = N\text{tr}\left[\partial_\mu\phi_1\partial_\mu\phi_1^* + \partial_\mu\phi_2\partial_\mu\phi_2^* + (4\pi g)^2\phi_1\phi_2\phi_1^*\phi_2^*\right]$$

It can be obtained by twisting N=4 SYM theory, so-called γ deformation, sending the deformation parameter to i-infinity while taking YM coupling to zero

[Frolov'05]
[Lunin,Maldacena'05]
[Grabner,Gromov,Kazakov,Korchemsky'17]
[Sieg,Wilhelm'16]

1. Gluons and gauginos decouple
2. Gauge group becomes a flavour group, no SUSY
3. Conformal symmetry is preserved for any coupling
   (at least in planar limit and for fine-tuned double-trace couplings)
4. Integrability is retained

[Grisha's talk]

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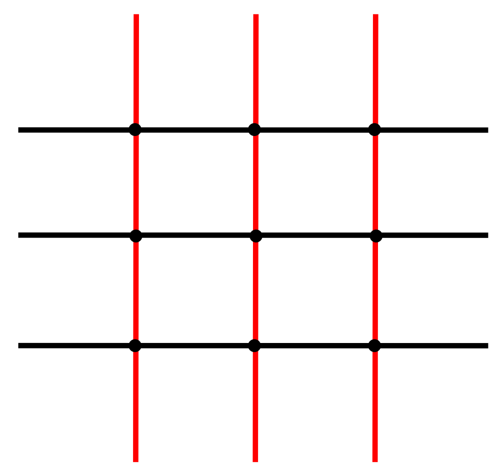
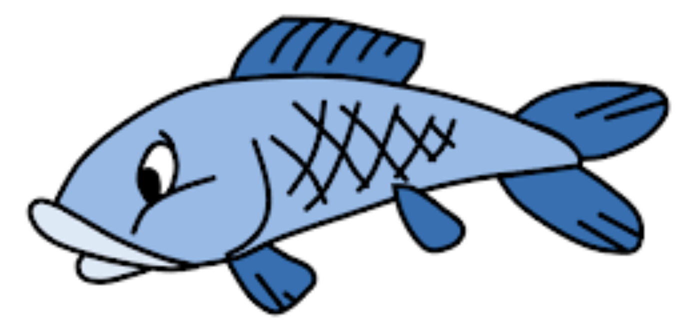
Planar graphs all look the same



bulk graph



Integrability much less mysterious and links directly to properties of the quartic vertex in d=4

[Zamolodchikov'80]
[Isaev'03]
[Gromov,Kazakov,Korchemsky,Negro,Sizov'17]
[Chicherin,Kazakov,Loebbert,Muller,Zhong'16]

Win: simplicity (very few graphs)

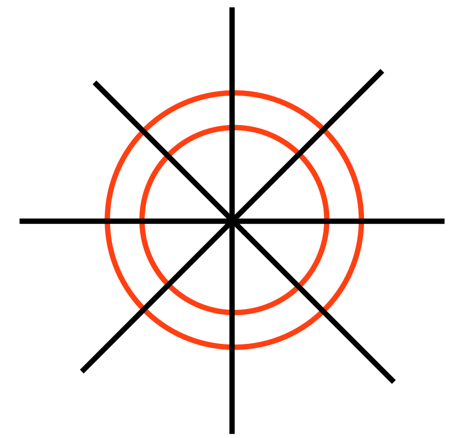
Lose: unitarity

Fishnet graph factory

Fishnet graphs are generalized ladders

Ex 1. Periods of wheel diagrams

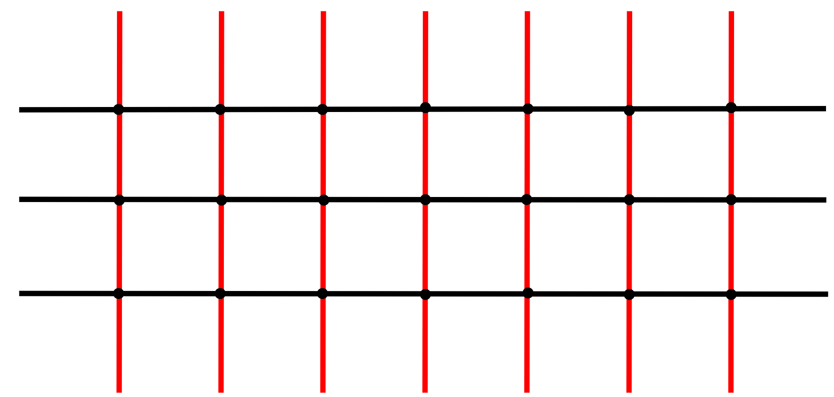
[Erik's talk]



[Gurdogan,Kazakov'15]
[Caetano,Gurdogan,Kazakov'16]
[Gromov,Kazakov,Korchemsky Negro,Sizov'17]

toy model for "Feynman integral transcendental number theory"

[Chicherin,Kazakov, Loebbert,Muller,Zhong'17]



Ex 2. Fishnet amplitudes-correlators

[Grisha's talk]

[Grabner,Gromov, Kazakov,Korchemsky'17]

toy model for polygonal bootstrap program and its re-summed version

[Caron-Huot,Dixon,Von Hippel, McLeod,Papathanasiou'18]

toy model for elliptic integrals (train tracks)

[Bourjaily,He,McLeod, Von Hippel,Wilhelm'18]

Laboratory for integrability and amplitudes techniques

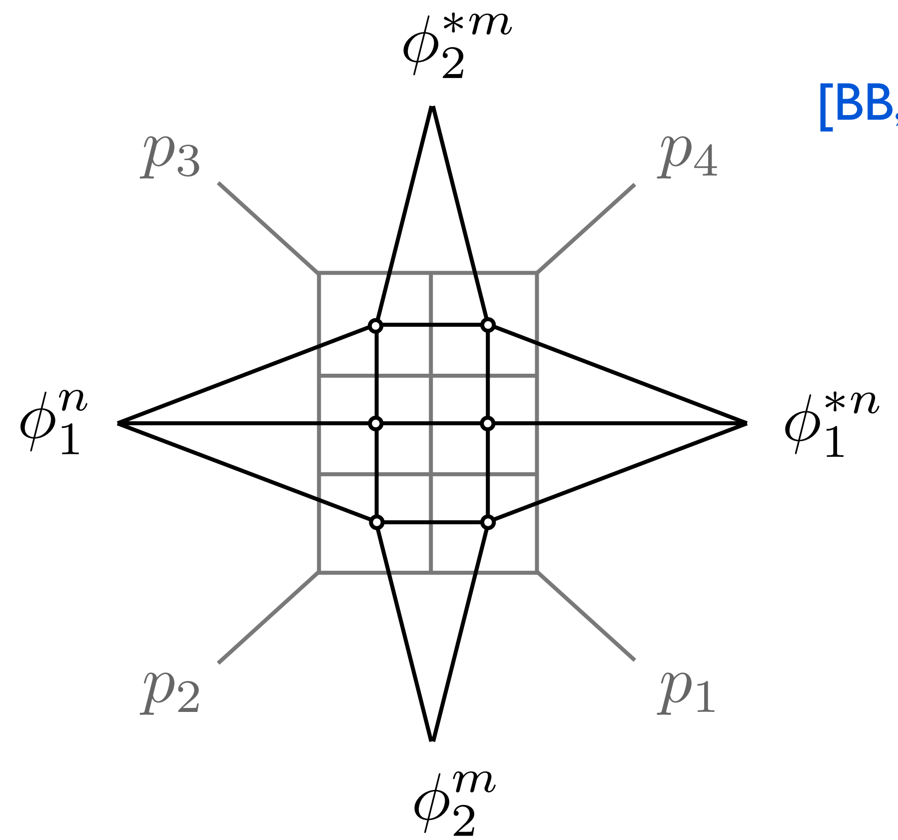
Example : 4pt function

Correlator of local composite operators

[BB,Dixon'17]

$$G_{n,m} = \langle \phi_2^n(x_1)\phi_1^m(x_3)\phi_2^{\dagger n}(x_2)\phi_1^{\dagger m}(x_4)\rangle$$

$$= \frac{g^{2mn}}{(x_{12}^2)^n(x_{34}^2)^m} \times \Phi_{m,n}(u,v)$$



- UV&IR finite, function of 2 cross ratios
- Ladders when m or n = 1 [Usyukina,Davydychev'93]

Integrability provides us with tractable integral representations
Analyticity requirements (Steinmann's relations) allow us to translate them into a conjecture for the end result

[BB,Sever,Vieira'13]
[BB,Komatsu,Vieira'16]
[Fleury,Komatsu'17]
[Eden,Sfondrini'17]
[Caron-Huot,Dixon,McLeod,Von Hippel'16]
[Dixon,Drummond,Harrington,McLeod,Papathanasiou,Spradlin'16]

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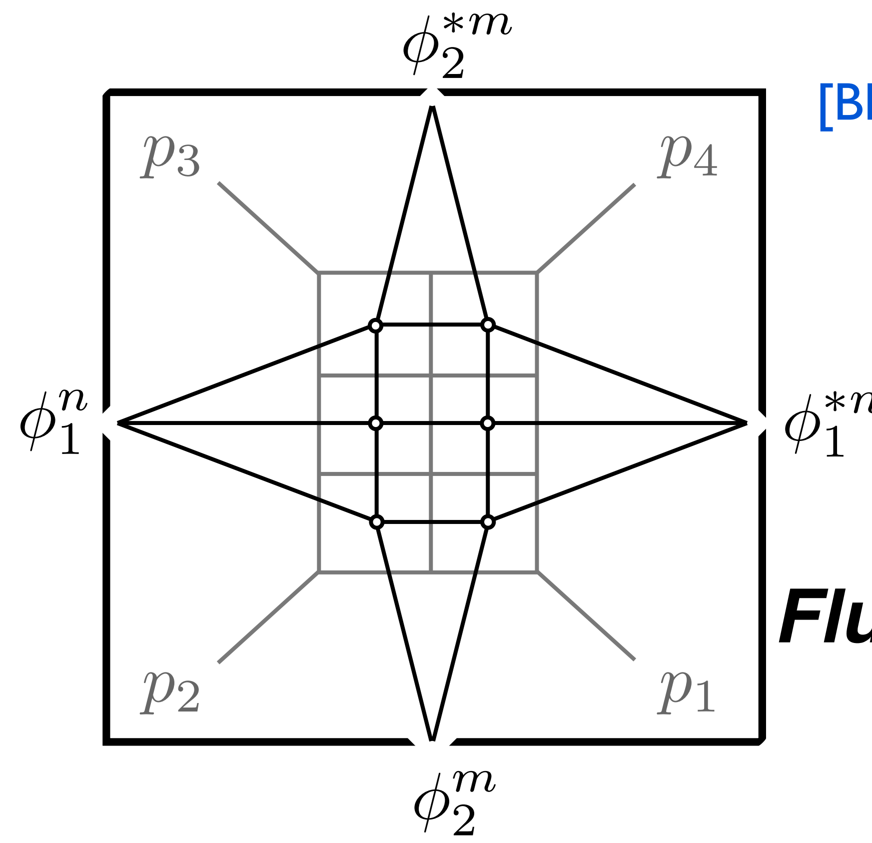

[BB,Dixon'17]

Flux tube

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Example : 4pt function

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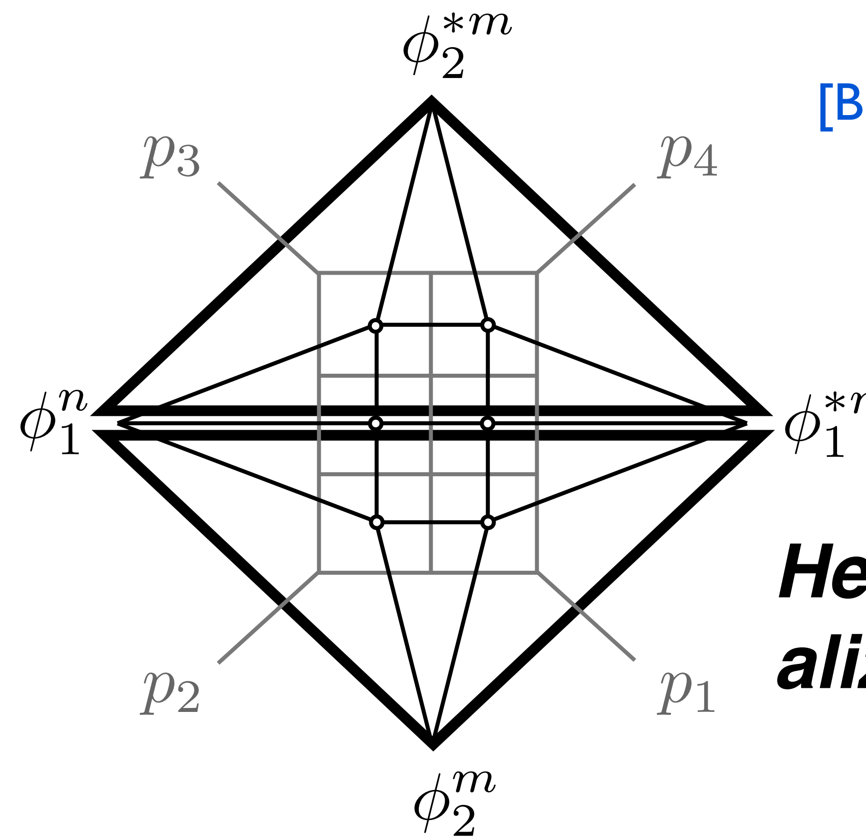

[BB,Dixon'17]

Hexagon alization

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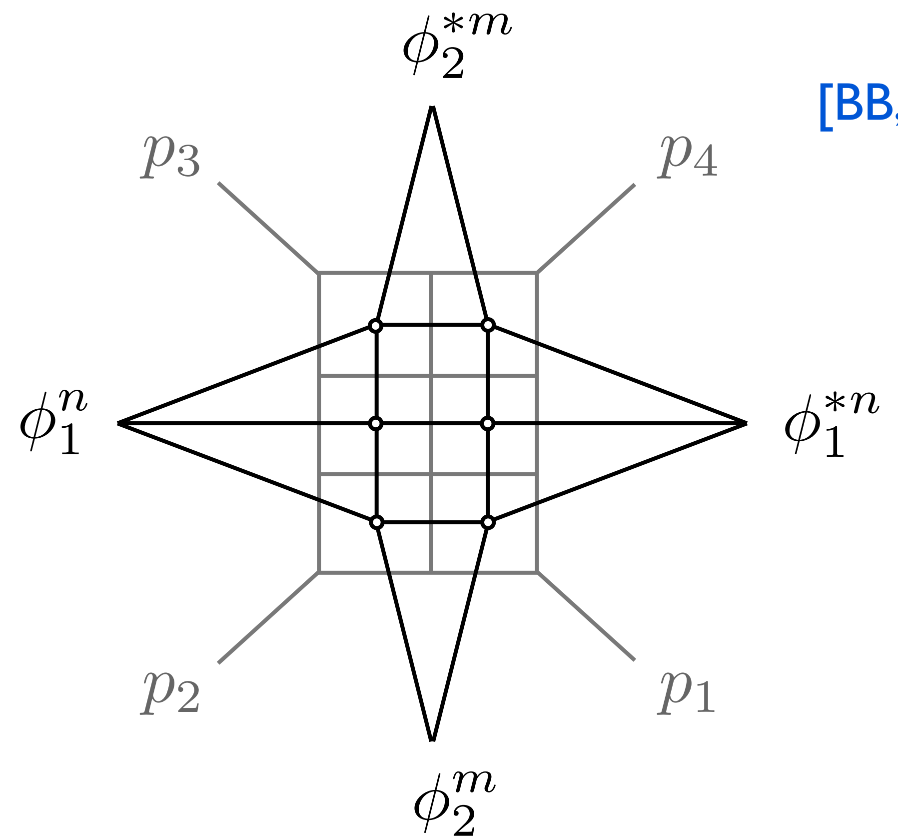
Example : 4pt function

[BB,Dixon'17]

Correlator of local composite operators



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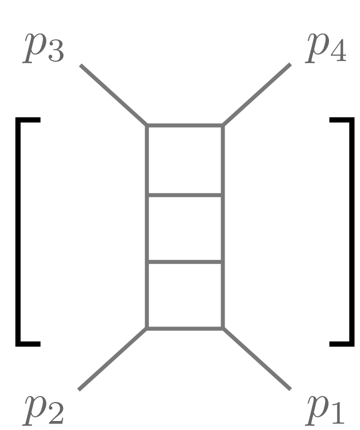
$\longrightarrow \quad \Phi_{m,n} \propto \ \det$



Correlator as a determinant of a m-by-m Hankel matrix M of Ladders L

$$M_{ij} = (n-m+i+j-2)!(n-m+i+j-1)! \times L_{n-m+i+j-1}(z,\bar{z})$$

Continuum limit & string?

What about duality to string in AdS?

Extremal twisting procedure forces the YM coupling to be small
→ *string in highly curved AdS?*

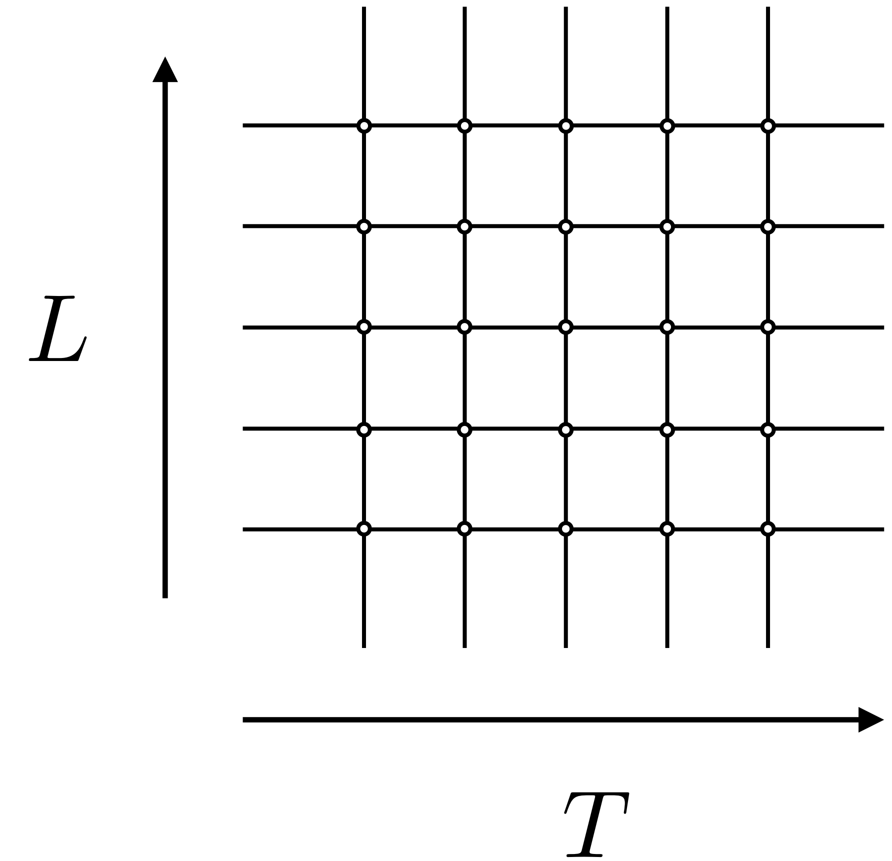
Related question: continuum limit of fishnet graphs?

Important observation concerning large order behaviour [Zamolodchikov'80]

Zamolodchikov's thermodynamical scaling $L, T \to \infty$

$$\log Z_{L,T} = -L \times T \log g_{cr}^2$$

$$g_{cr} = \frac{\Gamma(3/4)}{\sqrt{\pi}\Gamma(5/4)} = 0.7...$$



Critical coupling :
graphs become dense, continuum limit

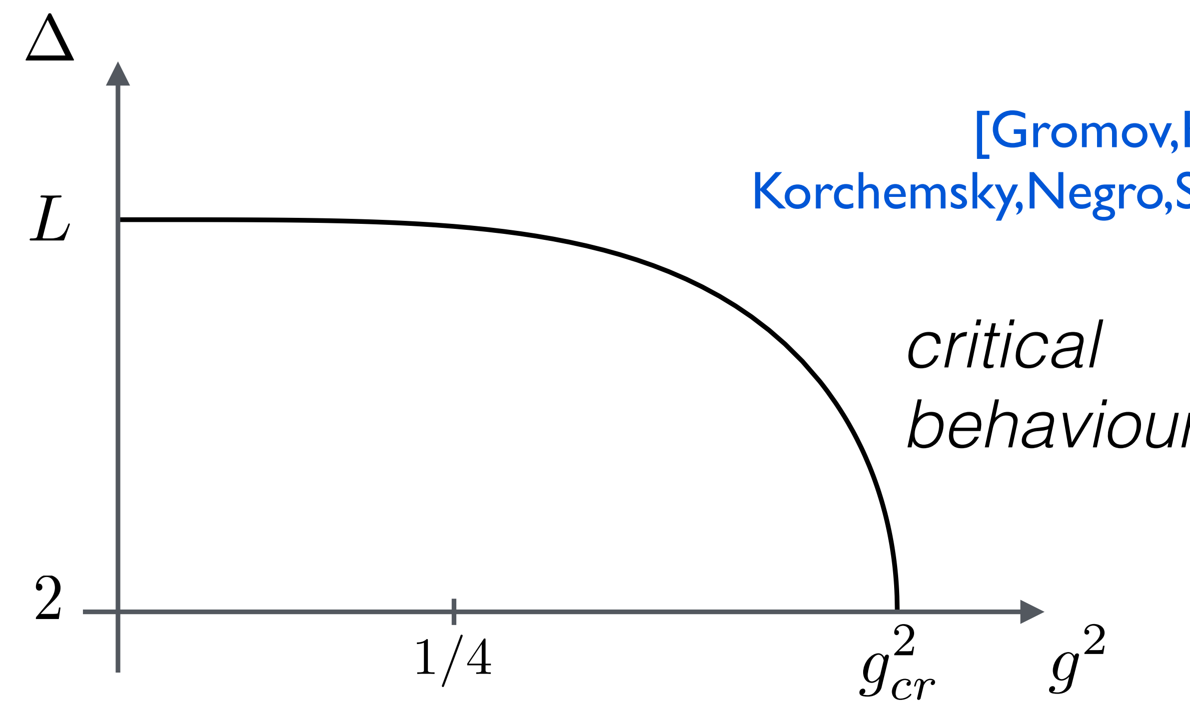
Plan

Study continuum limit using N=4 SYM integrable techniques

Probe: scaling dimension Δ of BMN vacuum operator $\mathcal{O} = \mathrm{tr}\,\phi_1^L$

Qualitative picture of Δ as function of coupling g^2 when $L \to \infty$

Thermodynamical scaling $\Delta \sim L f(g)$



see also [Gromov,Kazakov, Korchemsky,Negro,Sizov'17]

Analysis close to critical point: fishnet graph = 2d AdS5 sigma model

Dictionary:

BMN operator = tachyon $\quad \mathrm{tr}\,\phi_1^L \leftrightarrow V_\Delta \sim e^{-i\Delta t}$

4d coupling = worldsheet energy $\quad \log g^{2L} = \log g_{cr}^{2L} + E_{2d}(\Delta, L)$

Graphs versus integrability

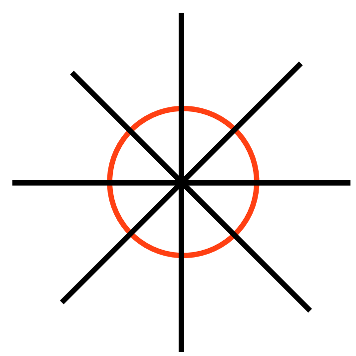
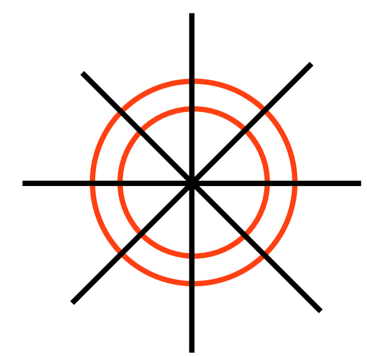
Computation of anomalous dimension

[Gurdogan,Kazakov'15]
[Caetano,Gurdogan,Kazakov'16]
[Gromov,Kazakov,
Korchemsky,Negro,Sizov'17]

BMN vacuum $\mathrm{tr}\,\phi_1^L$
(not protected)

$$\Delta = \Delta_L(g) = L + \gamma$$

Graphs: loop corrections come from the wheel diagrams

$Z \;=\; 1 \;+$  $+$  $+ \;\cdots$

wave-function
renormalization

1 wheel $\sim g^{2L}$

2 wheels $\sim g^{4L}$

Depends on cut off

$$R \sim \log \Lambda_{UV}$$

Anomalous dimension controls the logarithmic dependence on cut off

$$\log Z \sim -\gamma \times R$$

Graphs versus integrability

Computation of anomalous dimension

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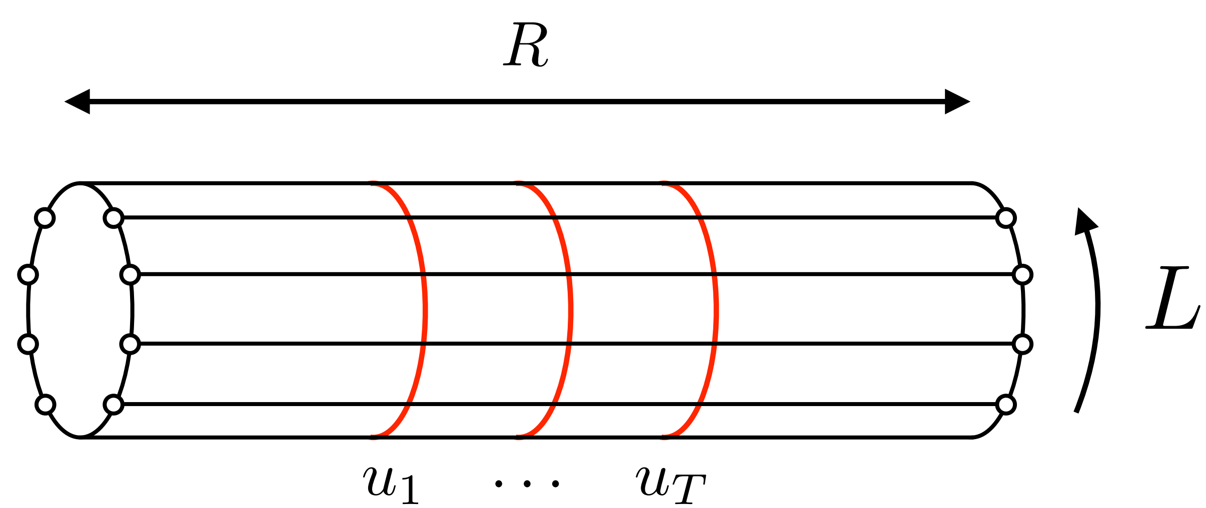
BMN vacuum (not protected) $\mathrm{tr}\,\phi_1^L$

$$\Delta = \Delta_L(g) = L + \gamma$$

Integrability: quantum mechanical interpretation of the graphs

1+1d picture: partition function on $\mathbb{R} \times S_L$

$$\mathcal{Z}_{L,R} = \sum_{T \geqslant 0} g^{2LT} \times$$



dof = magnon = wheel

magnon carries a rapidity "u", momentum along euclidean time direction, and a discrete label "a", for harmonics on 3-sphere

scaling dimension = free energy of a gas of magnon

$$\log \mathcal{Z}_{L,R} = -\Delta_L(g) R + O(R^0)$$

at temperature $1/L$ and chemical potential $h = \log g^2$

Thermodynamical Bethe Ansatz

Factorized scattering allows us to obtain free energy from TBA

$$\Delta = L - 2\sum_{a\geqslant 1}\int \frac{du}{2\pi}\mathbf{Y}_a(u)$$
$$-\sum_{a\geqslant 1}\int \frac{du}{2\pi}\mathbf{Y}_a^2(u) - 2\sum_{a,b\geqslant 1}\int \frac{dudv}{(2\pi)^2}\mathbf{Y}_a(u)\mathcal{K}_{a,b}(u,v)\mathbf{Y}_b(v) + O(\mathbf{Y}^3)$$

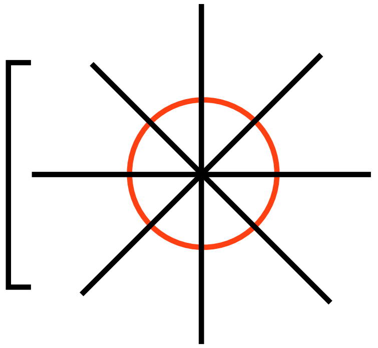
Boltzmann weight: $\mathbf{Y}_a(u) = a^2 e^{Lh - L\epsilon_a(u)} \ll 1$

Magnon energy: $\epsilon_a(u) = \log\left(u^2 + a^2/4\right)$

Scattering kernel: $\mathcal{K}_{a,b}(u,v) = \dfrac{\partial}{i\partial u}\log S_{a,b}(u,v) + \text{matrix part}$

Coupling constant: fugacity for the magnons (wheels) $g^2 = e^h$

[Gurdogan,Kazakov'15]

free wheel example: div $\left[\right.$  $\left.\right] = -\sum_{a\geqslant 1} a^2 \int \frac{du}{\pi}\frac{g^{2L}}{(u^2+a^2/4)^L} \propto g^{2L}\zeta(2L-3)$

[Erik's talk]

Thermodynamic limit

Thermodynamic limit $L \to \infty$

Interesting when chemical potential gets bigger than mass of lightest magnon

$$h > \log \epsilon(u=0) = \log 1/4$$

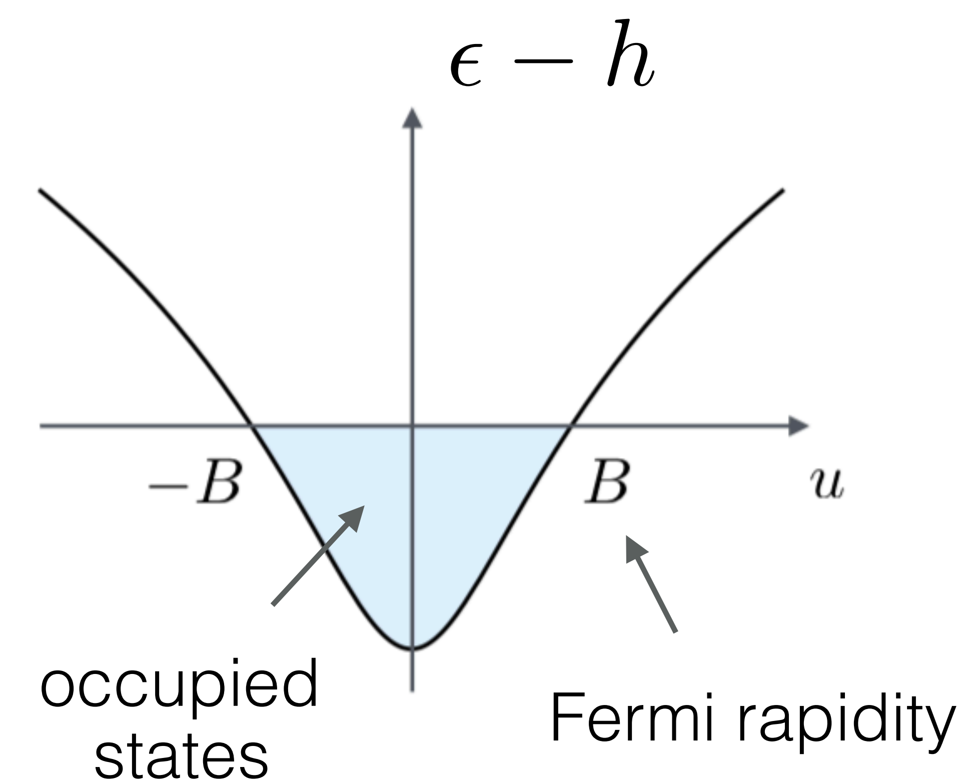
that is for

$$g > 1/2$$

A Fermi sea forms

All states below the Fermi rapidity are filled

Increasing coupling amounts to increasing B



Comment: only the s-wave (lightest) magnons condense
(higher Lorentz harmonics decouple)

Linear integral equation

In the thermodynamic limit the TBA linearizes

Integral equation for the rapidity distribution of energy levels

$$\chi(u) = C - \epsilon(u) + \int_{-B}^{B} \frac{du}{2\pi} \mathcal{K}(u-v)\chi(v)$$

BC: $\chi(u = \pm B) = 0$

Effective chemical potential: $C = \log g^2 - \int_{-B}^{B} \frac{du}{2\pi} k(u)\chi(u)$

Kernel: $\mathcal{K}(u) = 2\psi(1+iu) + 2\psi(1-iu) + \frac{2}{1+u^2}$

Scaling dimension: $\Delta/L = 1 - \int_{-B}^{B} \frac{du}{\pi} \chi(u)$

Critical regime

Small B : dilute gas, density of magnons is small, free regime

$$j = -df/dh \sim 0 \qquad \varepsilon = f + hj \sim 1 \qquad f \sim 1$$

Critical regime : dense gas, large magnon density $B \to \infty$

$$\varepsilon \sim j \log g_{cr}^2 \quad \longleftrightarrow \quad \log Z_{L,T} \sim -LT \log g_{cr}^2$$

All energy levels are filled, distribution stretches all over the real axis

Equation immediately solved in Fourier space

$$\chi_{cr}(u) = C_{cr} - \epsilon(u) + \int_{-\infty}^{\infty} \frac{dv}{2\pi} \mathcal{K}(u-v) \chi_{cr}(v) \quad \Rightarrow \quad \chi_{cr} = \log \frac{\sqrt{2}\cosh\theta + 1}{\sqrt{2}\cosh\theta - 1}$$

with $\theta = \pi u/2$

One verifies:

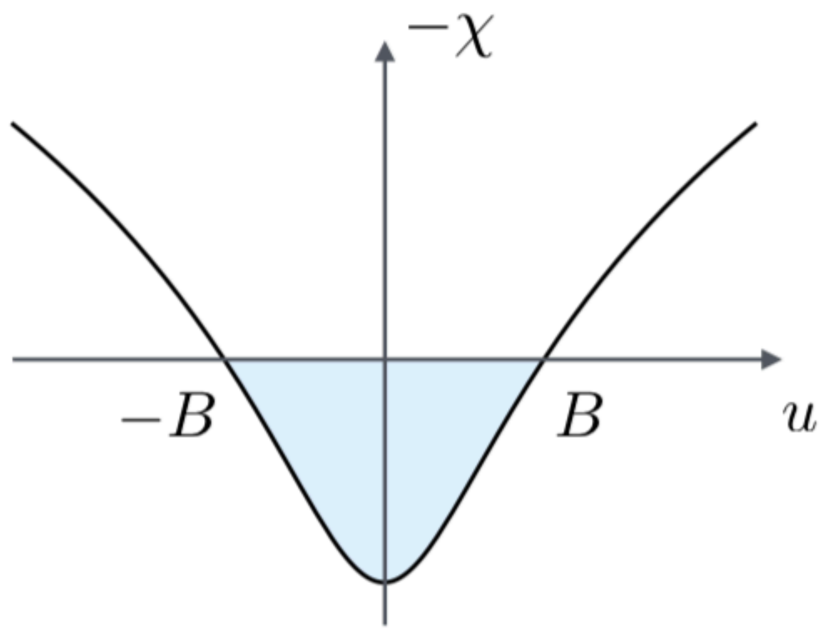
(i) Scaling function vanishes $f = \Delta/L \to 0$

(ii) Chemical potential (coupling) approaches predicted value

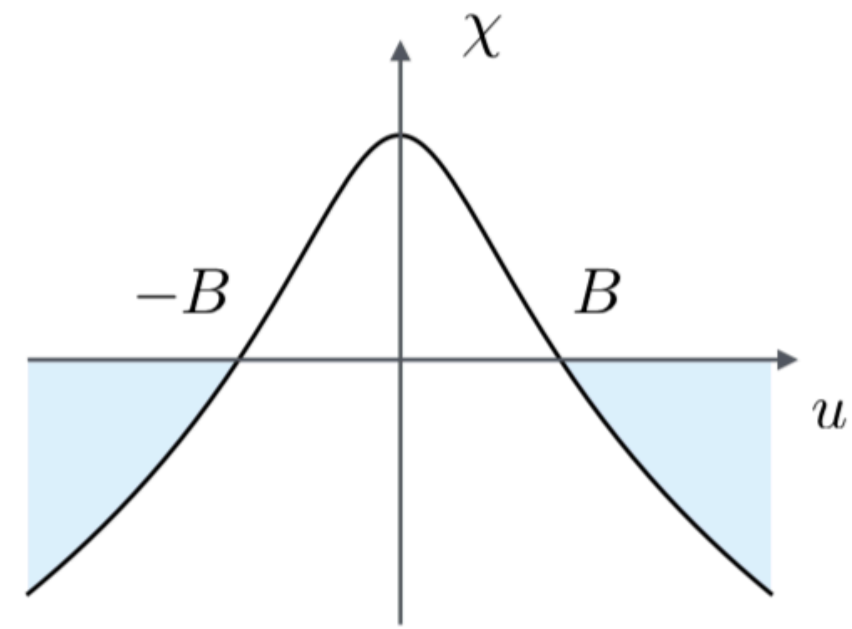
$$\chi_{cr} \sim e^{-|\theta|} \quad \Rightarrow \quad C_{cr} = 0 \quad \Rightarrow \quad g_{cr} = \Gamma(3/4)/\sqrt{\pi}\Gamma(5/4)$$

Near-critical regime

Particle-hole transformation:



Fermi sea of magnons



dual Fermi sea

Equation for dual excitations:

Dualize kernel by means of

$$K = -\frac{\mathcal{K}}{1-\mathcal{K}*} = -\mathcal{K} - \mathcal{K} * \mathcal{K} - \ldots$$

Act on both sides of the equation with $1 - K*$

Dual equations

Dual equation: $$\chi(\theta) = E(\theta) + \int\limits_{\theta^2 > B^2} \frac{d\theta'}{2\pi} K(\theta - \theta')\chi(\theta')$$

Dual energy formula: $$\log g^2 = \log g_{cr}^2 + \int\limits_{\theta^2 > B^2} \frac{d\theta}{2\pi} P'(\theta)\chi(\theta)$$

No chemical potential but extra BC: $\chi(\theta) \sim -2\rho \log\theta$ $\qquad \rho = \Delta/L =$ charge density

1) Dual kernel:

$$K(\theta) = \frac{\partial}{i\partial\theta} \log \frac{\Gamma(1 + \frac{i\theta}{2\pi})\Gamma(\frac{1}{2} - \frac{i\theta}{2\pi})\Gamma(\frac{3}{4} + \frac{i\theta}{2\pi})\Gamma(\frac{1}{4} - \frac{i\theta}{2\pi})}{\Gamma(1 - \frac{i\theta}{2\pi})\Gamma(\frac{1}{2} + \frac{i\theta}{2\pi})\Gamma(\frac{3}{4} - \frac{i\theta}{2\pi})\Gamma(\frac{1}{4} + \frac{i\theta}{2\pi})}$$

2) Dual dispersion relation:

$$E(\theta) = \chi_{cr}(\theta) = \log\left[\frac{\sqrt{2}\cosh\theta + 1}{\sqrt{2}\cosh\theta - 1}\right] \sim \frac{m}{2} e^{-|\theta|}$$

$$P(\theta) = -iE(\theta + i\frac{\pi}{2}) = i\log\left[\frac{\sqrt{2}\sinh\theta + i}{\sqrt{2}\sinh\theta - i}\right] \sim \mp\frac{m}{2} e^{-|\theta|}$$

where $m = 4\sqrt{2}$

Interpretation

What is the dual system describing?

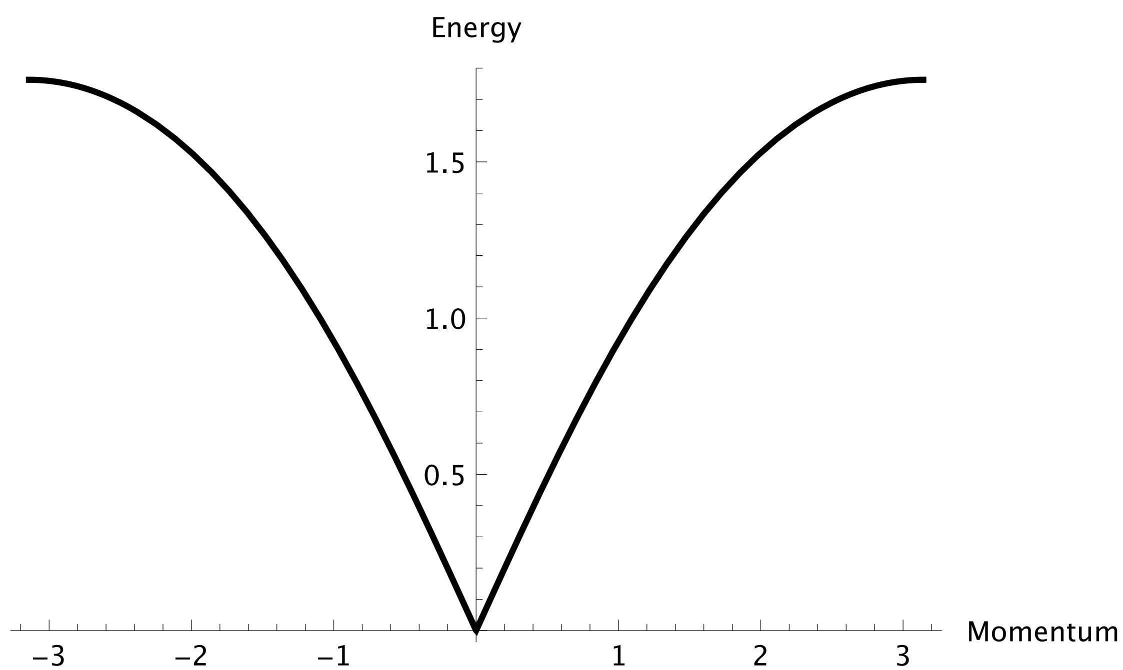
1) Kernel: $$K = -i\partial_\theta \log S_{O(6)}$$

Excitations scatter as particles in 2d O(6) non-linear sigma model

[Zamolodchikov&Zamolodchikov]

2) Dispersion relation:

$$\sinh^2(\frac{1}{2}E) = \sin^2(\frac{1}{2}P)$$



Gapless excitations (unlike O(6) model)

Besides, E decreases when θ increases
Support is non-compact and density is not normalizable (cannot count excitations)

No mass gap + continuous spectrum
Suggest: sigma model with ***non-compact*** target space
Proposal: integrable lattice completion of AdS_5 sigma model

Dual model: hyperbolic model

Sigma model with curved target space

$$\mathcal{L} = -\frac{1}{2e^2}\, G^{AB} \partial_a X_A \partial^a X_B$$

Weak coupling (large AdS radius) $\quad e^2 \ll 1$

Beta function related to Ricci scalar

For AdS_{d+1} : $\quad -Y_0^2 + Y_\perp^2 - Y_{d+1}^2 = -1$

One loop running: $\quad \mu \frac{\partial}{\partial \mu} e^2(\mu) = \frac{d}{2\pi} e^4(\mu) + \ldots$

Alternatively: $\quad \frac{1}{e^2(\mu)} = \frac{d}{2\pi} \log\left(\Lambda/\mu\right)$

1. Theory is weakly coupled in IR
2. There is no mass gap
3. There is no isolated vacuum

Dual state: tachyon

Consider sigma model on cylinder of radius L

Interested in 2d “ground state” energy: tachyon

(best candidate for extremum of energy at given charge = global time energy)

$$V_\Delta \sim e^{-i\Delta t}$$

Classically, it corresponds to solution
$$Y^0 \pm iY^{d+1} = e^{\pm iH\tau}$$
$$Y_\perp = 0$$

Charge density $\quad \Delta/L = \frac{1}{e^2}(Y^0\dot{Y}_{d+1} - Y^{d+1}\dot{Y}_0) = -H/e^2$

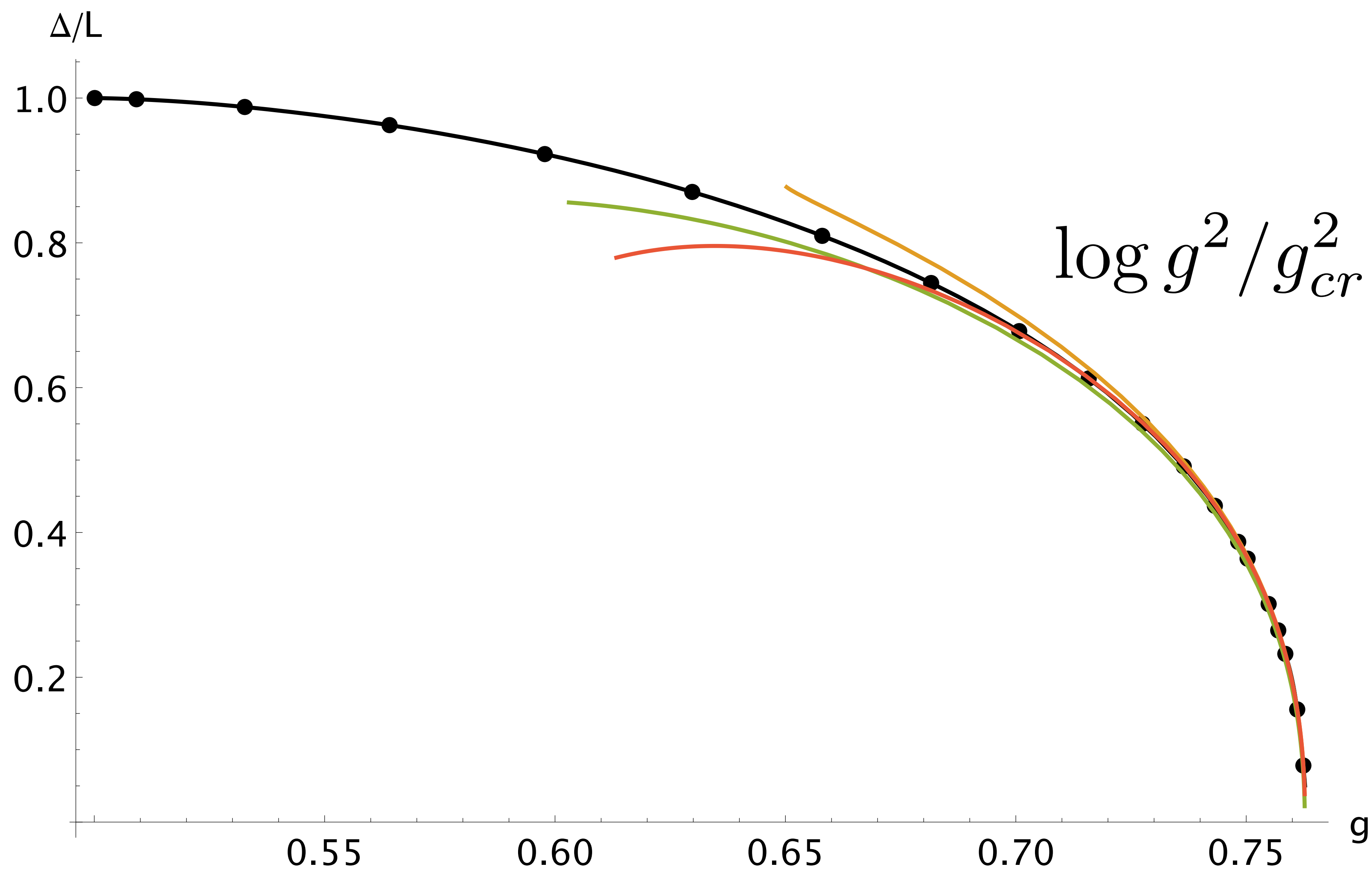
Energy density $\quad E/L = \frac{1}{2e^2}(\dot{Y}^0\dot{Y}_0 + \dot{Y}^{d+1}\dot{Y}_{d+1}) = -H^2/(2e^2)$

Classical result :
(c-o-m energy)
$$E = -\frac{e^2\Delta^2}{2L}$$

Same as in O(d+2) model if not for the sign of the coupling $\quad e^2 \leftrightarrow -e^2$

Numerics

"Quadratic Casimir" scaling near critical point fits numerics



$$\log g^2/g_{cr}^2 = -\frac{e^2\Delta^2}{2L^2}$$

WS PT at 1st, 2nd and 3rd order

$g_{\rm thr} = 0.5$ $g_{\rm cr} = 0.76276$

thermodynamical window

All order perturbation theory

[Volin'09]

Solutions to sphere and hyperboloid model are the same, to any order in $1/B$
if we formally flip the sign of Fermi rapidity $B \to -B$

Fermi parameter $B \sim 1/e^2 \sim \log(L/\Delta) \gg 1$

plays role of inverse running coupling at energy scale $\sim \rho = \Delta/L$

Hence, flipping is sign: $e^2 \leftrightarrow -e^2$

= changing the sign of curvature

non-compact		compact
$\Delta \ll L$	$\leftrightarrow$	$\Delta \gg L$

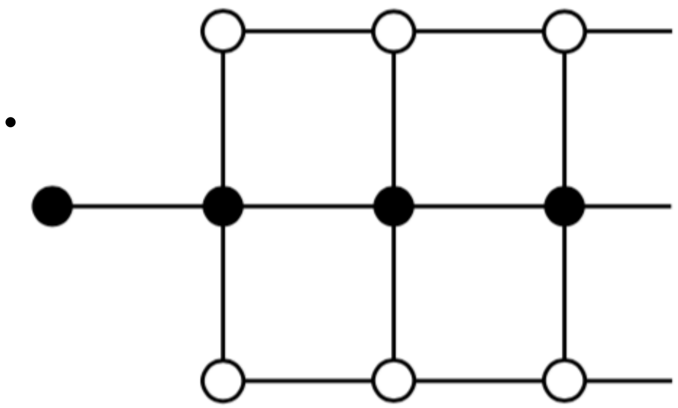
= changing "direction of RG flow"

Conclusion : dual integral equation describes the tachyon of the AdS model to *all* orders in perturbation theory

Two descriptions co-exist

Original magnon TBA (massive + chemical potential)

$$\log Y_1 = L \log g^2 - L\epsilon + \mathcal{K} * \log(1+Y_1) + \dots$$



massive TBA

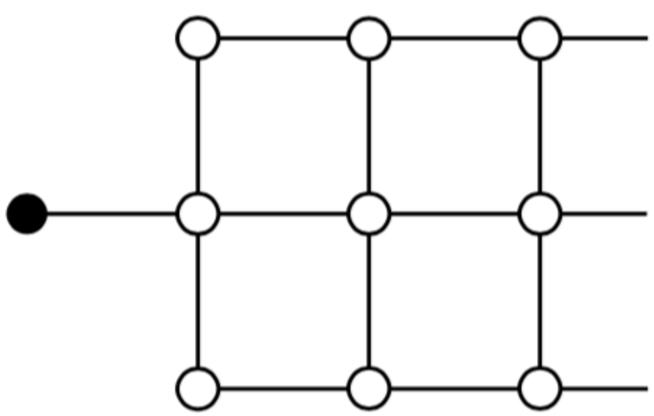
coupling = input

(all blacks nodes contribute to energy)

output :

$$\Delta = L - \sum_{a=1}^{\infty} \int \frac{du}{\pi} \log(1+Y_a)$$

Dual TBA (massless + no chemical potential)



massless TBA

(only 1 momentum carrier)

$$\log Y_1 = LE - K_{O(6)} * \log(1+1/Y_1) + \dots$$

Δ = input (tachyon rep)

coupling = output (sigma model energy)

$$\log g^{2L}/g_{cr}^{2L} = -\int \frac{d\theta}{2\pi} P'(\theta) \log(1+1/Y_1)$$

Finite size effect : central charge

TBA analysis in CFT limit (1/L effect = Casimir energy)

[Zamolodchikov'90s]
[Klassen-Melzer'90s]

Standard analysis gives TBA central charge $c = c_0 - c_\infty$

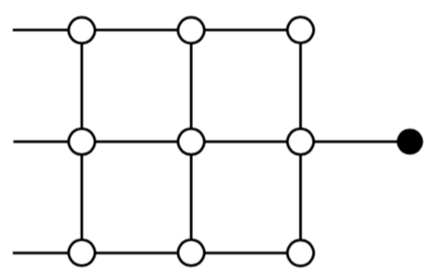
where $c_\star = \sum_i \mathcal{L}(\frac{Y_i^\star}{1+Y_i^\star})$

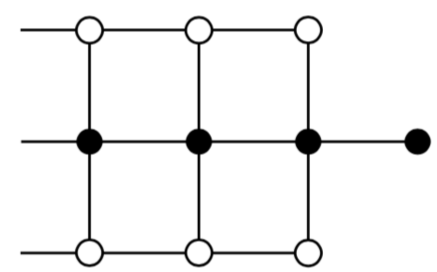
with Rogers dilogarithm $\mathcal{L}(x) = \frac{6}{\pi^2}(\mathrm{Li}_2(x) + \frac{1}{2}\log x \log(1-x))$

Stationary solutions to Y-system are known

$c_0 = 7$

symmetric phase
O(6)





$c_\infty = 2$

broken phase
O(4)

TBA central charge $c = 5$

CFT : Operator-state corr. maps E to 2d anomalous dimension

$$V_\Delta \sim e^{-i\Delta t} \qquad E_{2d} = -\frac{\pi c_{eff}(L)}{6L} - \frac{e^2 \Delta(\Delta - d)}{2L} + O(e^4)$$

Effective central charge at distance L: $c_{eff}(L) = 5 + O(e^2)$ (count of # WS dofs)

Summary

Conformal fishnet theory: laboratory for amplitudes, correlators, integrability, … techniques

Dual proposal: fishnet graphs define an integrable lattice regularization of the 2d AdS5 sigma model

Dual sigma model description is weakly coupled when fishnet length scales are large

i.e. large L and “small” quantum numbers = low worldsheet energy

Outlook

String worldsheet or not?

Marginality condition of sort $0 = L\mu + E_{2d}(L)$

with cosmological constant $\mu = \log g_{cr}^2 / g^2$

On-shell condition comes from the geometric sum over the wheels [Grisha's talk]

$$\sum_{T \geqslant 0} (g/g_{cr})^{2LT} e^{-TE_{2d}(L,\Delta)} = \frac{1}{1-(g/g_{cr})^{2L} e^{-E_{2d}(L,\Delta)}}$$

with T acting as a discrete proper time (Schwinger parameter)

Non-critical string with a tunable intercept exists in flat space, at least classically

Could it be an AdS version of it?

THANK YOU!