ABJM flux-tube and scattering amplitudes

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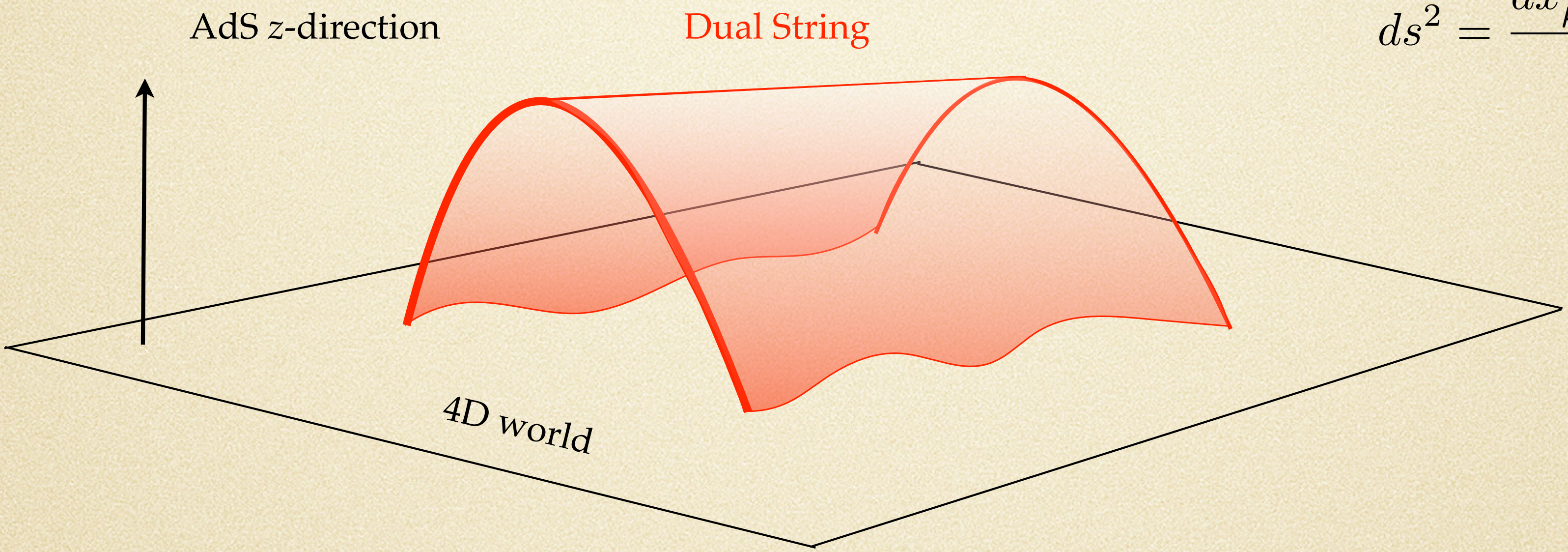
with Benjamin Basso

Gauge/string duality

Gauge/string duality is a gift for non-perturbative studies of gauge theories: D dimensional space-time physics is mapped into dynamics of the 2-dimensional string world-sheet!



$$ds^2 = \frac{dx_\mu^2 + dz^2}{z^2}$$

Planar pairs:

✔ N = 4 super Yang-Mills with 't Hooft coupling $g_{\mathrm{YM}}^2 N$ (4D) — Type-IIB superstring on $AdS_5 \times S^5$

✔ N = 6 Chern-Simons with matter with 't Hooft coupling N/k (3D ABJM) — Type-IIA superstring on $AdS_4 \times CP^3$

Aharony, Bergman, Jefferis, Maldacena '08

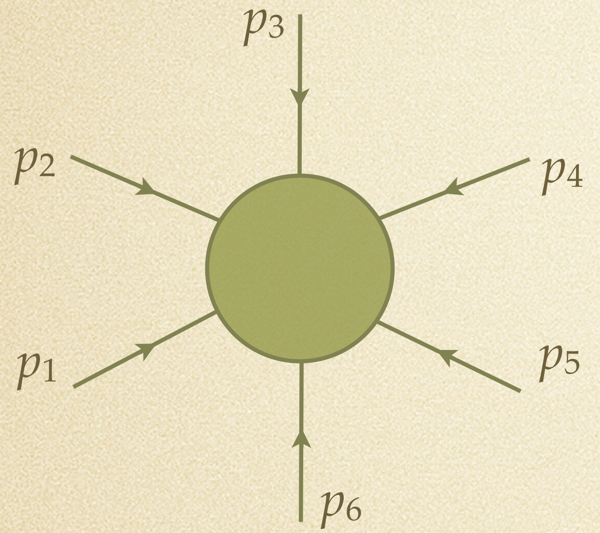
While D-dimensional physics on the QFT side cannot be integrable per se, the 2D world-sheet theory can (and better be integrable).

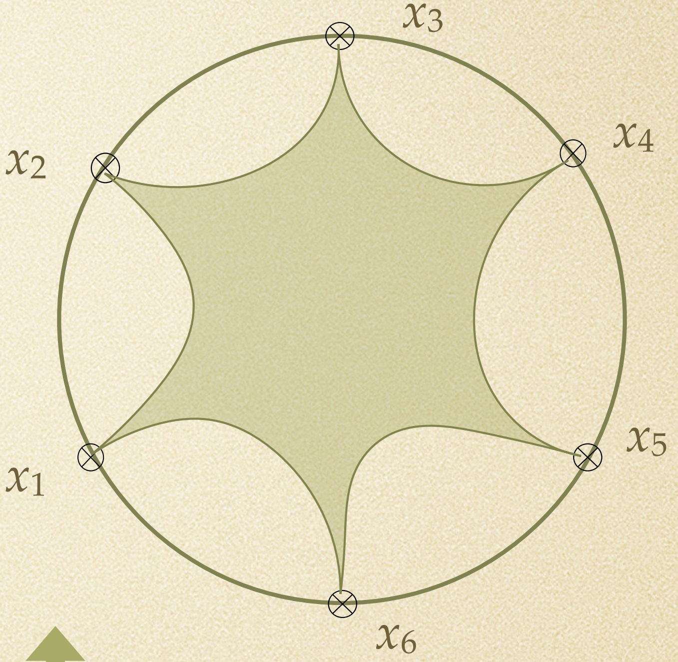
But we need a dictionary … let us review a well-studied example first

2D map for amplitudes

Alday, Maldacena '07

Planar gluon amplitudes (in *N* = 4 SYM) correspond to scattering of open strings with disk world-sheet topology (ending on D branes in the bulk $z \to \infty$).





Under T-duality (from coordinate *x* to momentum *y* space)

$$dy_\mu = \frac{*dx_\mu}{z^2} \qquad r = \frac{1}{z}$$

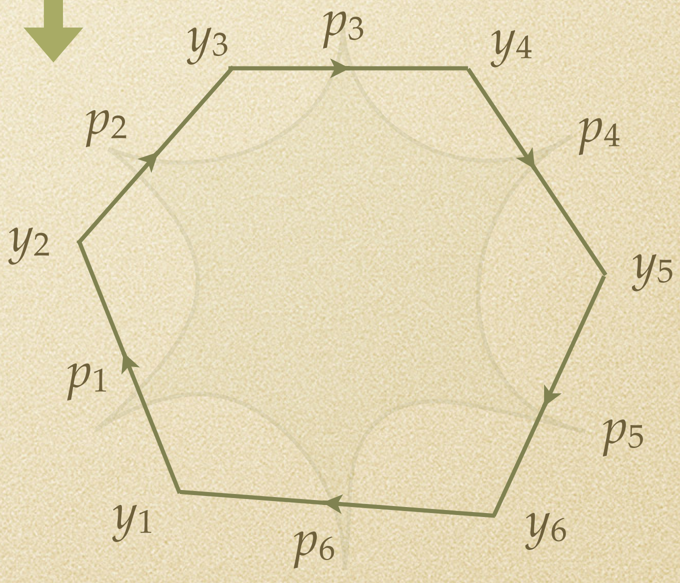
the AdS_5 goes into itself

$$ds^2 = \frac{dy_\mu^2 + dr^2}{r^2}$$

and the boundary conditions change to

$$\Delta y_\mu = 2\pi p_\mu$$

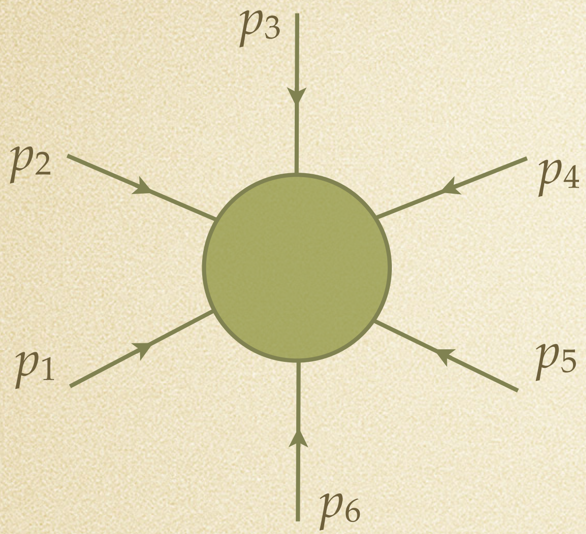
On the boundary (*r*=0) it becomes a holonomy of the gauge field on a piece-wise light-like contour.



Drummond, Henn, Korchemsky, Sokatchev '07
Brandhuber, Heslop, Travaglini '08

Amplitude/Wilson loop duality

In fact, the $AdS_5 \times S^5$ super sigma-model is self T-dual under generalization involving Grassmann coordinates as well!



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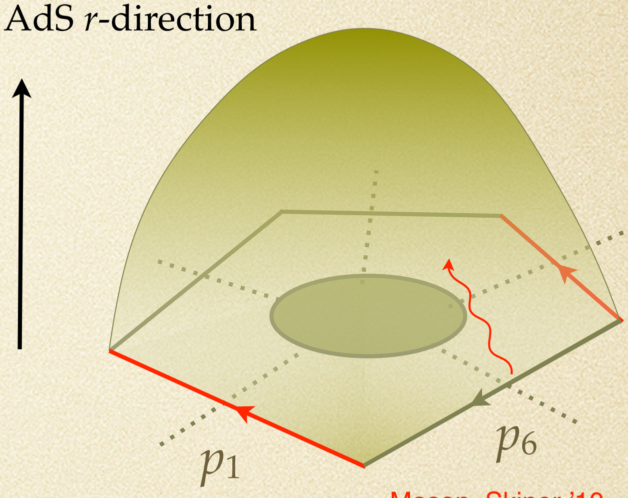


Mason, Skiner ’10
Caron-Huot ’10
AB, Korchemsky, Sokatchev ’11

✔ The asymptotic states of (regularized) $N = 4$ SYM

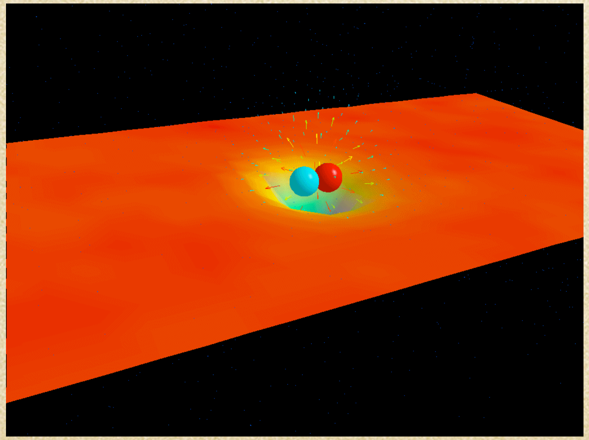
$$A_\mu,\ \psi^A,\ \bar{\psi}_A,\ \phi^{AB}$$

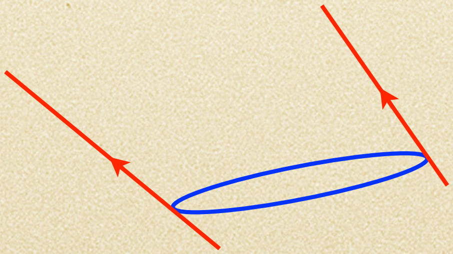
Fields of superconnection(s) living on the contour

$$A_\mu,\ \psi^A,\ \bar{\psi}_A,\ \phi^{AB}$$

These source excitations propagating on 2D world-sheet

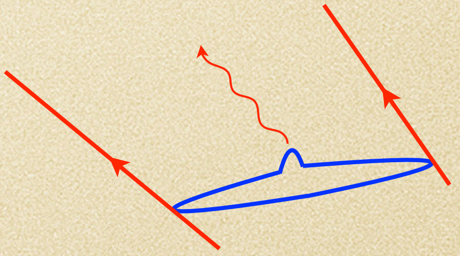
✔ The physics of excitations propagating on the color flux sourced by two fast SU(N) charged particles





vacuum

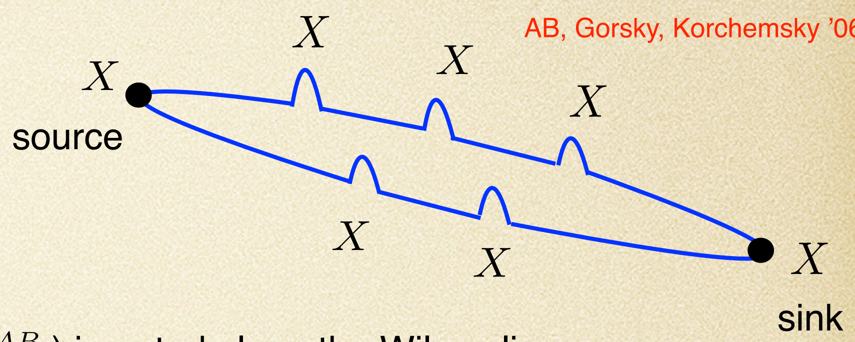
+



one excitation

+…

N=4 flux tube



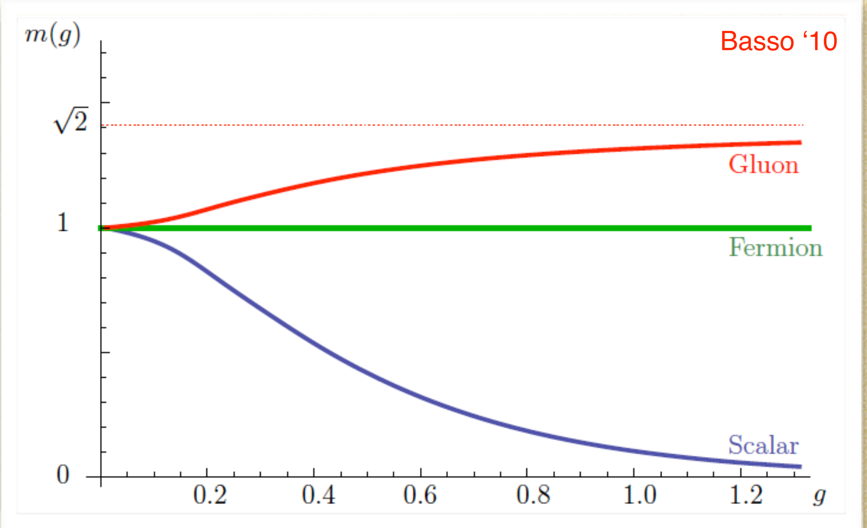
- ✔ Extracted from the large spin limit of the $N = 4$ spectral problem for single-trace operators

$$D_+^s X(0) \overset{s\to\infty}{\to} P\exp\left(ig\int_{[0,\infty)} dx_\mu A_\mu(x)\right) X(0)$$

- ✔ Multi-particle state $|\psi\rangle$ are adjoint fields ($X = A_\mu,\ \psi^A,\ \bar\psi_A,\ \phi^{AB}$) inserted along the Wilson line

- ✔ Spectral data:

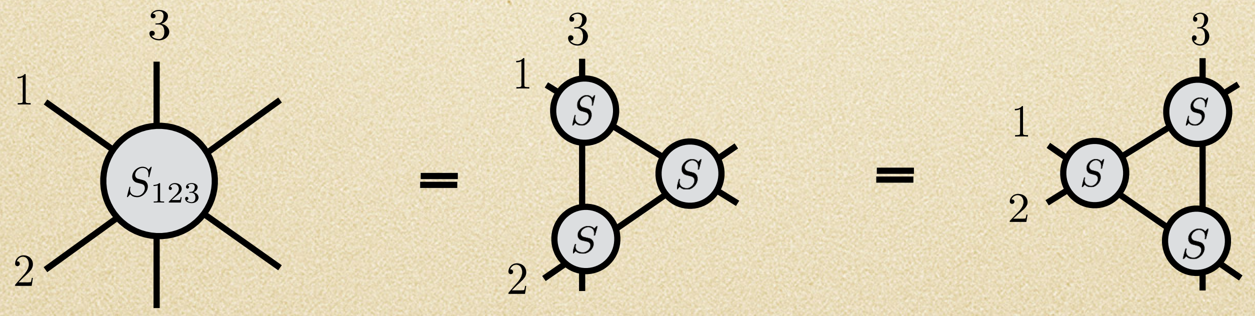
$$\gamma(\{s_i\}) = \underbrace{\ln(s_1+\cdots+s_N)\Gamma_{\text{cusp}}(g)}_{\text{flux-tube vacuum}} + \underbrace{E_N(\psi)}_{\text{flux-tube excitations}} + \ldots$$



- ✔ Excitations exhibit diffractionless scattering

$$E_N(\psi) = E(u_1) + E(u_2) + \cdots + E(u_N)$$
$$p_N(\psi) = p(u_1) + p(u_2) + \cdots + p(u_N)$$

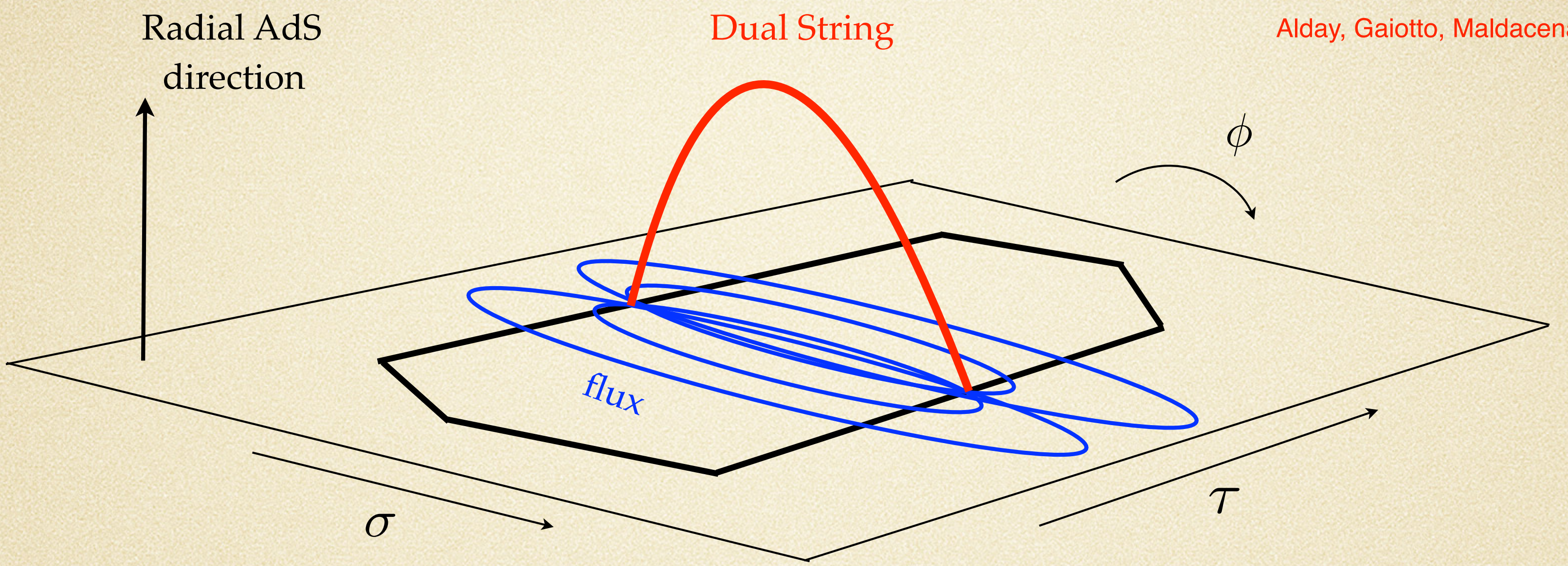
- ✔ S-matrices factorize in terms of two-particle ones



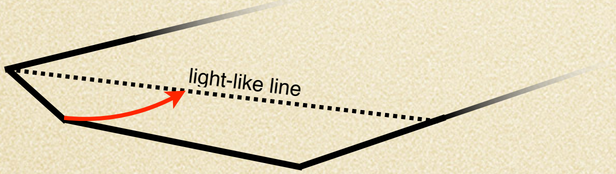
Yang-Baxter relation

Wilson loop at finite coupling

Alday, Gaiotto, Maldacena, Sever, Vieira '09

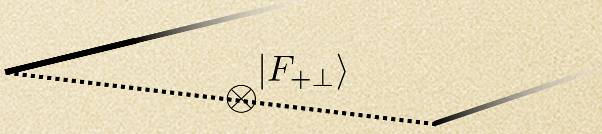


- ✔ Flux-tube is sourced by two light-like lines, i.e., chromo-charged particles propagating with the speed of light
- ✔ Bottom/Top caps excite the flux-tube from its ground state, i.e., create/absorb flux-tube excitations. E.g.,

$C_{\rm bot} =$ 

(curvature expansion)

$=$ 

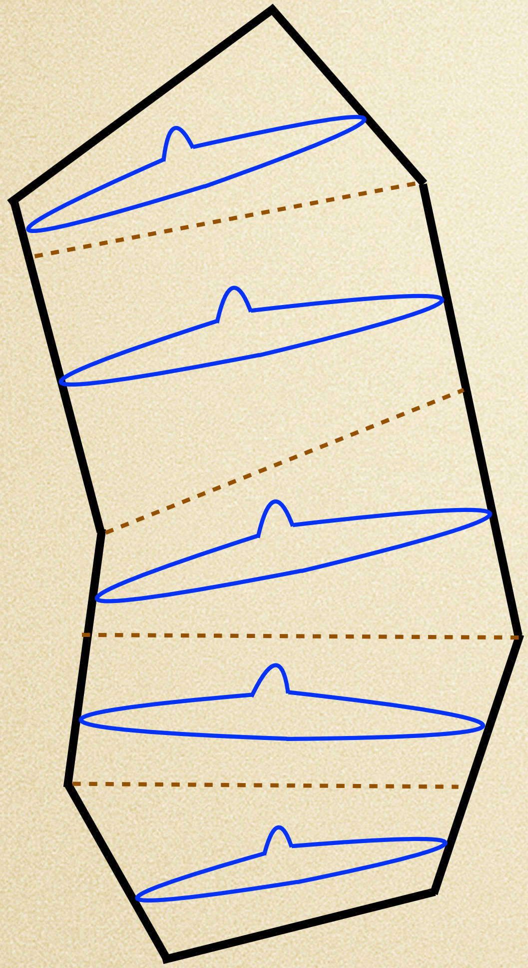
$+\ \mathrm{e}^{-\tau}$ 

$+\dots$

- ✔ The sum over all flux-tube states determines the Wilson loop/Scattering amplitude

$$\mathcal{W} = \sum_{\psi} C_{\rm bot}(\psi) \times \mathrm{e}^{-\tau E(\psi) + i\sigma p(\psi) + i\phi m(\psi)} C_{\rm top}(\psi)$$

Pentagon paradigm



- ✔ Tesselate a polygon into squares with light-like lines:

  For n-side polygon, (n-3) middle squares (matches the number of external kinematical (geometrical) data)

  (This is equivalent to multichannel OPE for correlators)

- ✔ Two squares overlap in a pentagon, (n-4) of them

$$= \sum_{\psi_i} \left[\mathrm{e}^{-\tau_i E(\psi_i) + i\sigma_i p(\psi_i) + i m_i \phi(\psi_i)} \right]$$

$$\times P(0|\psi_1) P(\psi_i|\psi_2) P(\psi_2|\psi_3) P(\psi_3|0)$$

To calculate any amplitude at any value of the coupling on needs

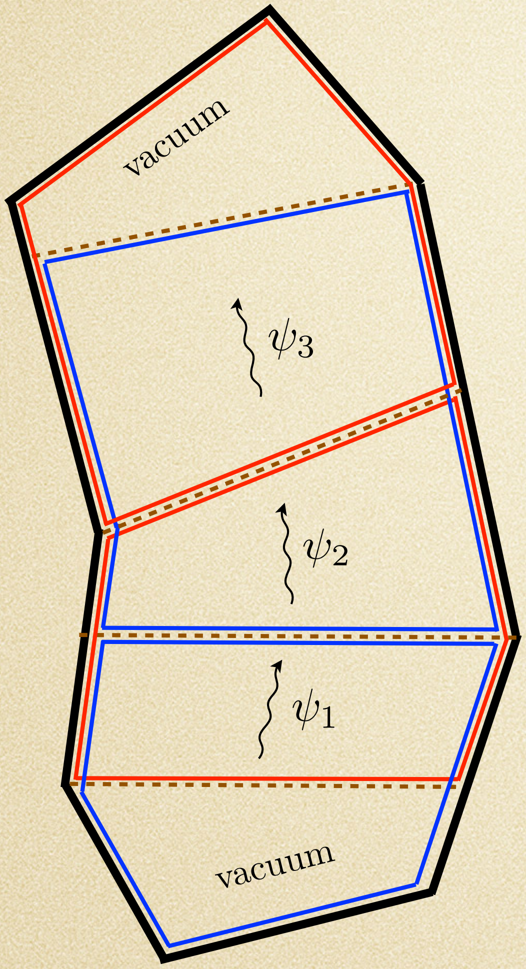
- ✔ The spectrum of flux-tube excitations (E, p)

- ✔ Pentagon transitions between all states

$$P(\psi_i|\psi_j) = \langle \psi_j | \widehat{\mathcal{P}} | \psi_i \rangle$$

This is the dynamical input! Geometry enters trivially through (τ, σ, ϕ)

Pentagon paradigm



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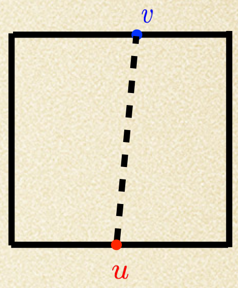
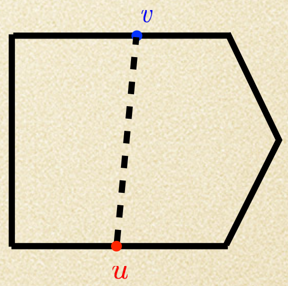
Pentagon transition (g=0)

✔ Single-particle square transition (orthogonality condition)



$$\langle \psi_v | \psi_u \rangle \sim \delta(u-v)$$

✔ Single-particle pentagon transition



$$P(u|v) = \langle \psi_v | T | \psi_u \rangle = \frac{\Gamma(2s)\Gamma(iu-iv)}{\Gamma(s+iu)\Gamma(s-iv)}$$

conformal transformation

✔ Obeys a "bootstrap" equation (not a Watson equation!)

$$\frac{P(u|v)}{P(v|u)} = S(u,v)$$

✔ The flux-tube scattering matrix

$$\psi \overset{\sigma_2 \gg \sigma_1}{=} \mathrm{e}^{2iu\sigma_1}\mathrm{e}^{2iv\sigma_2} + S(u,v)\mathrm{e}^{2iv\sigma_1}\mathrm{e}^{2iu\sigma_2}$$

$$S(u,v) = \frac{\Gamma(iu-iv)\Gamma(s-iu)\Gamma(s+iv)}{\Gamma(iv-iu)\Gamma(s-iv)\Gamma(s+iu)}$$

✔ Consistency with Watson equation? Move excitations from one side of the polygon to another with mirror transformations. The analytic continuation in rapidity γ is non-perturbative in its origin since to reach the mirror kinematics one has to go through the cuts which open up only at finite coupling.

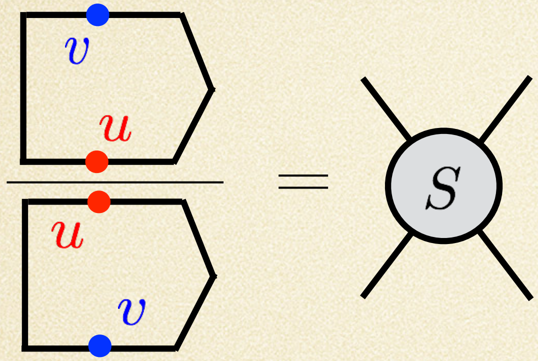
$$E(u^\gamma) = ip(u) \qquad p(u^\gamma) = iE(u)$$

AB, Basso '11

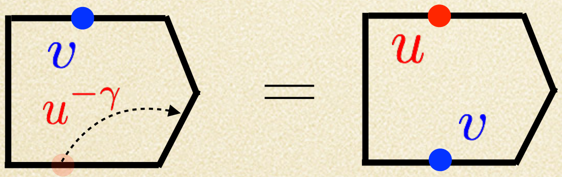
Pentagon bootstrap @ any coupling

Basso, Sever, Vieira '13

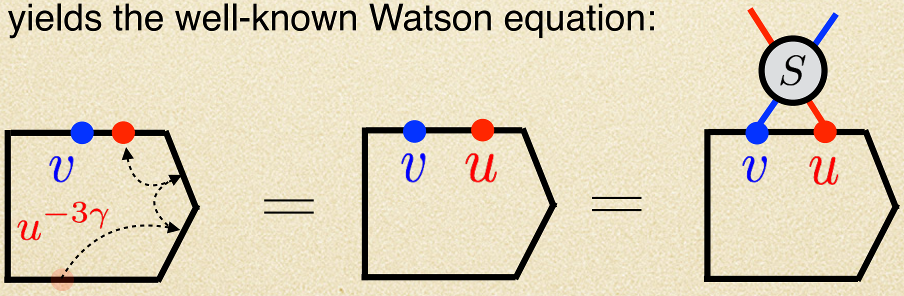
✔ Watson-like equation:



✔ Mirror equation:



Consistency condition yields the well-known Watson equation:



Basso, Sever, Vieira, Caetano, Cordova '13-15
AB '13-14

✔ Solution to bootstrap equations

$$P(u|v) = w(u,v)\left[\frac{S(u,v)}{S(u^{\gamma},v)}\right]^{1/2}$$

The most uncertain part of the solutions! Huge ambiguities! Data availability (from hexagon bootstrap) was crucial for success!

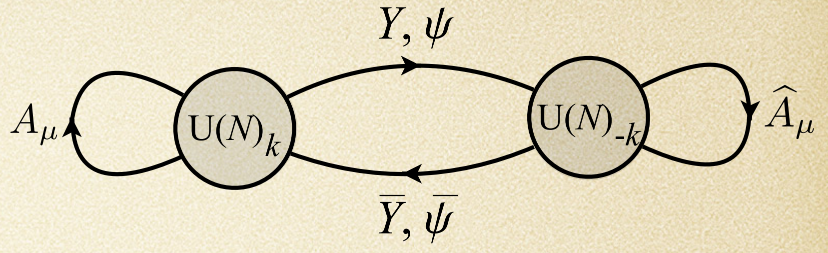
Dixon, Drummond, Henn, von Hippel, McLeod, Pennington '11-'16
Bourjaily, Caron-Huot, Trnka '13

Verification

- ✔ Bootstrap program explicitly verified against explicit calculations to four loops order by Dixon et al. (in fact the latter were indispensable for fixing ambiguities in solution of functional equations!)

- ✔ The OPE series can be resummed to recover the exact kinematics
- ✔ At strong coupling it reproduces TBA and provides results beyond the minimal area approximation

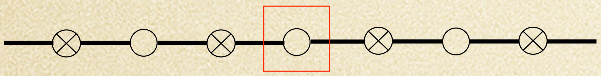
ABJM flux tube



- ✔ 3D U(N) x U(N) Chern-Simons theory with bi-fundamental hypermultiplet $(Y, \psi)^A$ matter in the fundamental of SU(4)
- ✔ Chern-Simons fields are not propagating, so only matter can emerge in asymptotic states
- ✔ The spectral problem of ABJM is integrable (like in $N = 4$ SYM, but there are differences, obviously)

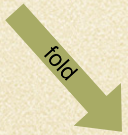
Minahan, Zarembo '08
Gromov, Vieira '08

$N = 4$ SYM (SU(2,2|4) chain)

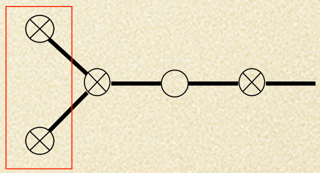


minimal sector is SL(2)

momentum carrying Bethe roots



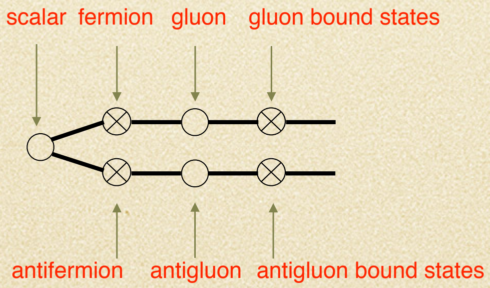
$N = 6$ CS (OSp(2,2|6) chain)

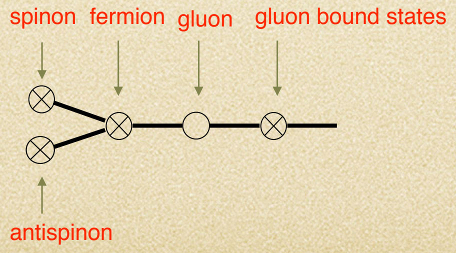


minimal sector is SL(2|1)

- ✔ Flux-tube excitations

Basso, Rej '12





- ✗ Scalar ϕ^{AB} (twist-1):

$$E_\phi(u)$$

- ✗ (Anti)fermion $(\bar{\psi}^A)\psi_A$ (twist-1):

$$E_\psi(u)$$

- ✗ (Anti)gluon $(\bar{F})F$ (twist-1):

$$E_F(u)$$

- ✗ (Anti)spinon $(\bar{Y}^A)\ Y_A$ (twist-1/2):

$$E_Z(u) = E_{\bar{Z}}(u) = \frac{1}{2} E_\phi^{N=4}(u)$$

- ✗ Fermion $\Psi_{AB} \sim Y_{[A}\psi_{B]}$ (twist-1):

$$E_\Psi(u) = E_\Psi^{N=4}(u) = E_{\bar{\Psi}}^{N=4}(u)$$

- ✗ Gluon $F \sim Y_A \bar{Y}^A$ (twist-1):

$$E_F(u) = E_F^{N=4}(u) = E_{\bar{F}}^{N=4}(u)$$

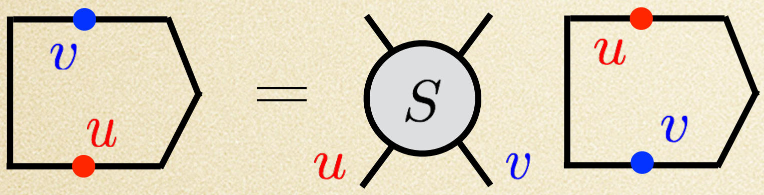
ABJM S-matrices and pentagons

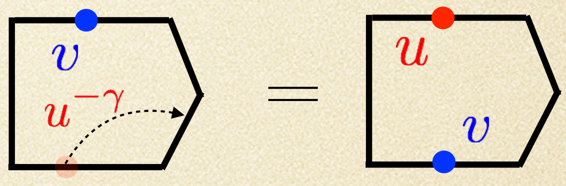
✔ Twist-1— Twist-1 excitations (square of $N = 4$ SYM)

$$S_{FF}(u,v) = S^{N=4}_{FF}(u,v) \times S^{N=4}_{F\bar{F}}(u,v) \qquad P_{F|F}(u|v) = P^{N=4}_{F|F}(u|v) \times P^{N=4}_{F|\bar{F}}(u|v)$$

$$S_{\Psi\Psi}(u,v) = S^{N=4}_{\Psi\Psi}(u,v) \times S^{N=4}_{\Psi\bar{\Psi}}(u,v) \qquad P_{\Psi|\Psi}(u|v) = P^{N=4}_{\Psi|\Psi}(u|v) \times P^{N=4}_{\Psi|\bar{\Psi}}(u|v)$$

obey both the fundamental and mirror axioms (as a consequence of their validity in $N = 4$ SYM)





✔ Twist-1/2 — Twist-1 excitations (same as $N = 4$ SYM)

$$S_{Z|F}(u|v) = S_{\bar{Z}|F}(u|v) = S^{N=4}_{\phi|F}(u|v) \qquad P_{Z|F}(u|v) = P_{\bar{Z}|F}(u|v) = P^{N=4}_{\phi|F}(u|v)$$

$$S_{Z|\Psi}(u|v) = S_{\bar{Z}|\Psi}(u|v) = S^{N=4}_{\phi|\Psi}(u|v) \qquad P_{Z|\Psi}(u|v) = P_{\bar{Z}|\Psi}(u|v) = P^{N=4}_{\phi|\Psi}(u|v)$$

✔ Twist-1/2 — Twist-1/2 excitations (square root of $N = 4$ SYM)

$$S_{Z|Z}(u|v) \times S_{\bar{Z}|Z}(u|v) = S^{N=4}_{\phi|\phi}(u|v) \qquad P_{Z|Z}(u|v) \times P_{\bar{Z}|Z}(u|v) = P_{\phi|\phi}(u|v)$$

$$S_{Z|Z}(u|v)/S_{\bar{Z}|Z}(u|v) = S^{\mathrm{SU}(2)}(u-v)$$

Solved up to an unknown function $f(u,v)$:

$$P_{Z|Z}(u|v)^2 = f(u,v) \times P^{N=4}_{\phi|\phi}(u|v)\,, \qquad P_{\bar{Z}|Z}(u|v)^2 = \frac{1}{f(u,v)} \times P^{N=4}_{\phi|\phi}(u|v)$$

ABJM NPWL

A Wilson loop (there are other options as well) for ABJM theory

$$W_n = \frac{1}{2N}\left\langle \mathrm{tr}_{\mathrm{SU_k(N)}} \exp\left(\int_{C_n} dy_\mu A^\mu\right) + \mathrm{tr}_{\mathrm{SU_{-k}(N)}} \exp\left(\int_{C_n} dy_\mu \widehat{A}^\mu\right)\right\rangle$$

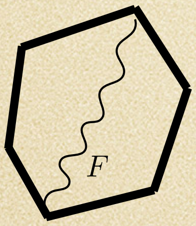
Henn, Plefka, Wiegandt '10
Bianchi, Gilbert, Leoni, Penati '11

- ✔ Any number of edges
- ✔ Only even powers in 't Hooft coupling *N*/*k*
- ✔ Computed at two loops and found to be given by one-loop BDS ansatz of *N*=4 SYM (i.e., they obey anomalous conformal Ward Identities of dual conformal boost!) for for *n*=4 and by BDS along with *N*=4 remainder for *n*=6

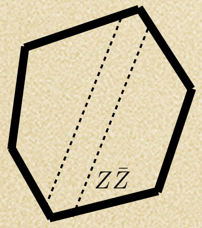
Properly subtracted *n*=6 NPWL

$$W_6^R = \exp\left(\Gamma_{\mathrm{cusp}}(N/k)[2\zeta_2 - \ln(1-u_2)\ln\frac{u_1 u_2}{(1-u_2)u_3} - \ln u_1 \ln u_3 - \sum_{i=1}^{3}\mathrm{Li}_2(1-u_i)]\right)$$

Within pentagon OPE, the leading twist contribution comes from an exchange of a gluon and spinon-antispinon pair



$$\begin{aligned} I_F &= \int \frac{du}{2\pi}\mu_F(u)e^{-\tau E_F(u)+i\sigma p_F(u)} \\ &= -(N/k)^2 e^{-\tau}\frac{\sigma}{\sinh\sigma} \end{aligned}$$



$$\begin{aligned} I_{Z\bar{Z}} &= \int \frac{du_1 du_2}{(2\pi)^2}\frac{4\mu_Z(u_1)\mu_{\bar{Z}}(u_2)}{[(u_1-u_2)^2+4]|P_{Z|\bar{Z}}(u_1|u_2)|^2}e^{-\tau(E_Z(u_1)+E_Z(u_2))+i\sigma(p_Z(u_1)+p_Z(u_2))} \\ &= -(N/k)^2 e^{-\tau}\left[2\cosh\sigma\log\left(1+e^{-2\sigma}\right) - \frac{\sigma e^{-2\sigma}}{\sinh\sigma}\right] \end{aligned}$$

- ✔ Note that spurious poles cancel only in the sum of the two reproduces perturbative calculation!
- ✔ The applicability of pentagon OPE to null polygonal Wilson loops is not surprising at all.

Scattering amplitudes

$\mathcal{N}=4$ SYM | ABJM

✔ Asymptotic states:

Packed in to a single CPT self-conjugate superfield

$$\Phi = F + \theta^A\psi_A + \frac{1}{2}\phi_{AB} + \frac{1}{3!}\varepsilon_{ABCD}\theta^A\theta^B\theta^C\bar{\psi}^D + \frac{1}{4!}\varepsilon_{ABCD}\theta^A\theta^B\theta^C\theta^D\bar{F}$$

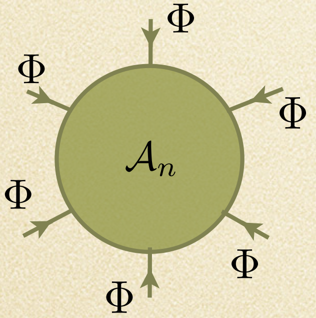
Two $\mathcal{N}=3$ superfields (only U(3) of SU(4) is manifest)

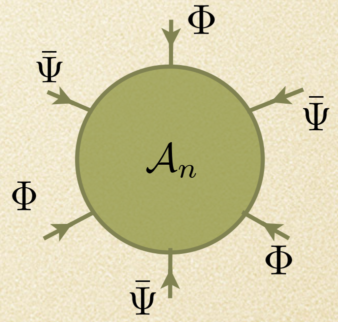
$$\Phi = Y^4 + \theta^a\psi_a + \frac{1}{2}\varepsilon_{abc}\theta^a\theta^a Y^c + \frac{1}{6}\varepsilon_{abc}\theta^a\theta^b\theta^c\psi_4$$

$$\bar{\Psi} = \bar{\psi}^4 + \theta^a\bar{Y}_a + \frac{1}{2}\varepsilon_{abc}\theta^a\theta^b\bar{\psi}^c + \frac{1}{6}\varepsilon_{abc}\theta^a\theta^b\theta^c\bar{Y}_4$$

Agarwal, Beisert, McLoughlin '09

✔ Superamplitudes:



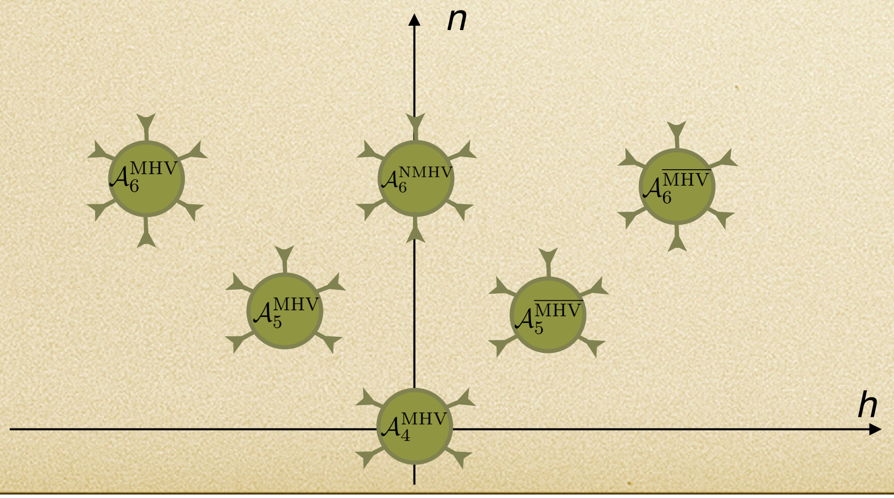


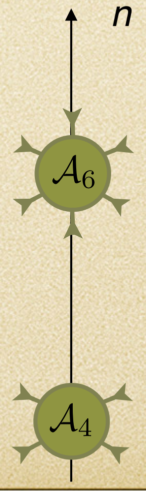
✔ Nomenclature:

$$\mathcal{A}_n = \mathcal{A}_n^{\mathrm{MHV}} + \mathcal{A}_n^{\mathrm{NMHV}} + \cdots + \mathcal{A}_n^{\mathrm{N}^{n-4}\mathrm{MHV}}$$

$$\mathcal{A}_n^{\mathrm{N}^k\mathrm{MHV}} \sim \theta^{8+4k}$$

$$\mathcal{A}_n \sim \theta^{3n/2}$$



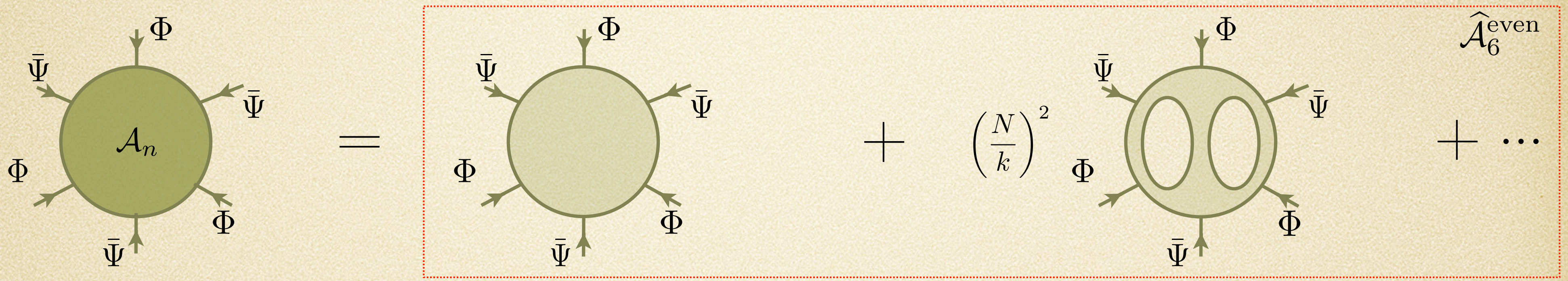


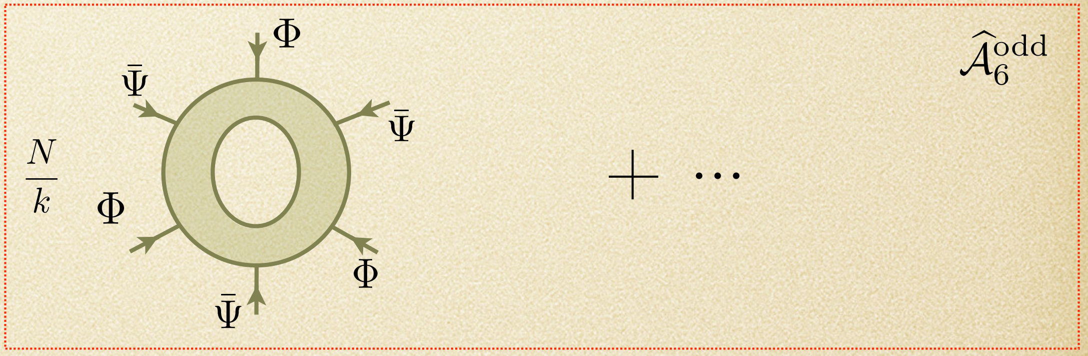
NMHV in 4D language

MHV in 4D language

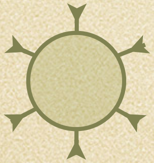
Six-leg amplitude

$$\langle \bar\Psi_1 \Phi_2 \bar\Psi_3 \Phi_4 \bar\Psi_5 \Phi_6 \rangle_{\text{amp}} = \frac{4\pi}{k} \frac{\delta^{3|6}\left(\sum_{j=1}^{6} \mathcal{P}_j\right)}{\sqrt{\langle 12\rangle\langle 23\rangle \ldots \langle 61\rangle}} \widehat{\mathcal{A}}_6$$

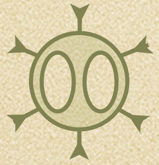
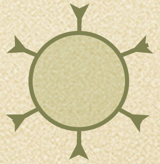
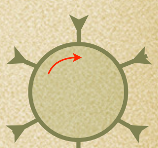




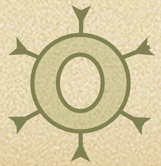
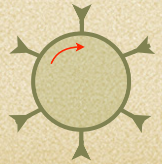
Dual superconformal symmetry is manifest:

✔ $\widehat{\mathcal{A}}_6^{\text{even}}$:  $= \mathrm{e}^{\tau+\sigma}(\mathcal{Y}_1 + \mathcal{Y}_2)$

Gang,Huang, Koh, Lee, Lipstein '11

 $=\left\{\frac{1}{2}\mathrm{BDS}_6 + \frac{1}{2}\prod_{j=1}^{3}\left[-2\pi^2 + \mathrm{Li}_2(1-u_j) + \frac{1}{2}\ln u_j \ln(1-u_{j+1}) + \arccos^2 u_j\right]\right\}$  $- \frac{1}{2}\prod_{j=1}^{3} \arccos u_j \ln \frac{u_{j+1}}{u_{j+2}}$ 

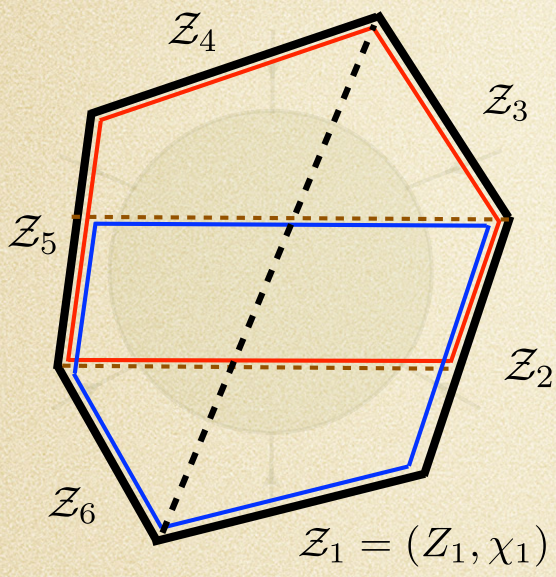
Caron-Huot, Huang '12

✔ $\widehat{\mathcal{A}}_6^{\text{odd}}$:  $= \frac{\pi}{2}$  $= \frac{\pi}{2}\mathrm{e}^{\tau+\sigma}(\mathcal{Y}_1 - \mathcal{Y}_2)$

Huang '12
Bianchi, Leoni, Mauri, Penati, Santambrogio '12
Bargheer, Beisert, Loebbert, McLoughlin '12

Flux-tube for ABJM amplitudes

- Tree level: contribution from leading-twist spinons



- only two independent components:

$$= \mathrm{e}^{-\tau/2}\left[\chi_1^3 \frac{2\mathrm{e}^{\sigma/2}}{1+\mathrm{e}^{2\sigma}} + \chi_4^3 \frac{2\mathrm{e}^{3\sigma/2}}{1+\mathrm{e}^{2\sigma}}\right]$$

$$= \mathrm{e}^{-\tau/2}\int \frac{du}{2\pi}\mathrm{e}^{i\sigma u}\left[\chi_1^3\,\mu_Z^{(0)}(u) + \chi_4^3\,\mu_{\bar{Z}}^{(0)}(u)\right]$$

- "doubling" of spinon measures:

$$\mu_Z^{(0)}(u) = \Gamma\left(\frac{3}{4}+\frac{i}{2}u\right)\Gamma\left(\frac{1}{4}-\frac{i}{2}u\right), \qquad \mu_{\bar{Z}}^{(0)}(u) = \Gamma\left(\frac{3}{4}-\frac{i}{2}u\right)\Gamma\left(\frac{1}{4}+\frac{i}{2}u\right)$$

- squaring to $N = 4$ scalar measure:

$$\mu_Z^{(0)}(u)\mu_{\bar{Z}}^{(0)}(u) = \mu_\phi^{(0)N=4}(u)$$

- Two loops: contribution from leading-twist spinons

$$+\left(\frac{N}{k}\right)^2 \mathrm{e}^{-\tau/2}\int \frac{du}{2\pi}\mathrm{e}^{i\sigma u}\left[\chi_1^3\mu_Z^{(0)}(u) + \chi_4^3\mu_{\bar{Z}}^{(0)}(u)\right]\left[-\tau E_Z^{(2)}(u) + i\sigma p_Z^{(2)}(u) + \delta\mu_Z^{(2)}(u)\right]$$

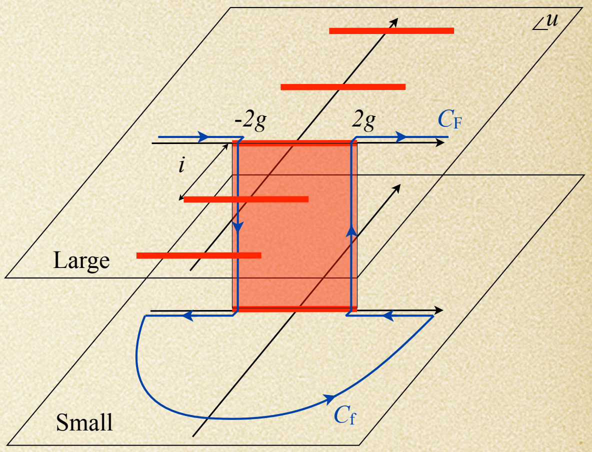
here

$$E_Z^{(2)}(u) = \frac{1}{2}E_\phi^{(2)N=4}(u)\,, \qquad p_Z^{(2)}(u) = \frac{1}{2}p_\phi^{(2)N=4}(u)$$

$$\delta\mu_Z^{(2)}(u) = \frac{1}{2}\delta\mu_\phi^{(2)N=4}(u) - \pi^2\mathrm{sech}^2(\pi u) - 2\zeta_2 \quad \Longrightarrow \quad f(u,u) = 1 + 2\left(\frac{N}{k}\right)^2(\pi^2\mathrm{sech}^2(\pi u) + 2\zeta_2) + \ldots$$

Not enough data (and imagination) to uncover $f(u,v)$!

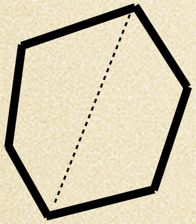
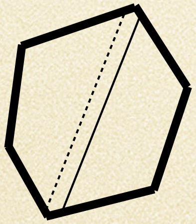
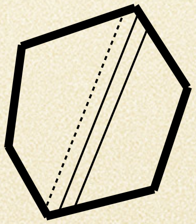
OPE resummation



$p_{\psi_{\rm s}}(u) \sim \Gamma_{\rm cusp}/(2u)$

✔ Small fermions (i.e., fermions living on the second sheet of the Riemann rapidity surface) act as supersymmetry generators (as their momentum goes to zero) and do not induce suppression in 't Hooft coupling!

✔ In fact the entire OPE series can be resummed:

$\mathrm{e}^{-\tau/2}$  $+\ \mathrm{e}^{-3\tau/2}$  $+\ \mathrm{e}^{-5\tau/2}$  $+ \dots$

Z^A $\qquad \bar{Z}_A \psi_{\rm s}^{AB}$ $\qquad Z_A \psi_{\rm s}^{BC} \psi_{{\rm s}BC}$

$$
= \mathrm{e}^{-\tau/2} \sum_{n=0}^{\infty} \mathrm{e}^{-2n\tau} \int \frac{du}{2\pi} \mathrm{e}^{i\sigma u} \left[\frac{(iu+3/2)_{2n}}{(2n)!} \mu_{Z/\bar{Z}}(u) + \mathrm{e}^{-\tau} \frac{(iu+3/2)_{2n+1}}{(2n+1)!} \mu_{\bar{Z}/Z}(u) \right]
$$

$$
\times \left\{ 1 + \left(\frac{N}{k} \right)^2 \left[-\tau E_Z^{(2)}(u) + \delta\mu_Z^{(2)}(u) + i\sigma p_Z^{(2)}(u) + i\sigma \sum_{j=1}^{n} p_{\psi_{\rm s}}^{(2)} \left(u - i(1/2+j)\right) \right. \right.
$$

$$
\left. \left. - \sum_{j=1}^{n} \frac{\pi \tanh(\pi u)}{u - i(1/2+j)} + \sum_{j>k=1}^{n} \frac{2}{\left(u - i(1/2+j)\right)\left(u - i(1/2+k)\right)} \right] \right\}
$$

✔ Complete agreement with available two-loop data!

Look forward

- ✔ The pentagon paradigm is generalized to ABJM.
- ✔ Natural (democratic) description of both Wilson loops and scattering amplitudes.
- ✔ Bootstrap equations for pure spinon sector were not solved to all orders. Currently, it is not even clear which of them are more fundamental.
- ✔ Odd and even order loop amplitudes require separate analyses.
- ✔ Wanted:
  - ✔ Two-loop octagon or four-loop hexagon
  - ✔ Three-loop hexagon

  These will allow us to bootstrap results to all orders in 't Hooft coupling.
- ✔ What is the unifying object accounting for both Wilson loops and scattering amplitudes? When addressed in the language of flux-tube physics, it seems to suggest that it does indeed exist.
- ✔ If it does not, the geometrization paradigm could be applicable in more general circumstances where dual description is lacking.


Pentagon rules the world?