A tale of two Regge limits

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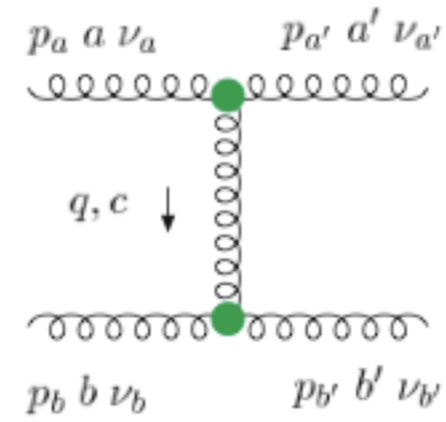
- In the last few years, a lot of progress has been made in understanding the analytic structure of multi-loop amplitudes

- we review what implications that progress has had on our understanding of:
— the Regge limit of QCD
— the Regge limit of N=4 Super Yang-Mills (SYM)

Regge limit of QCD

- In perturbative QCD, in the Regge limit $s \gg t$, any scattering process is dominated by gluon exchange in the t channel

- For a tree 4-gluon amplitude, we obtain



$$\mathcal{M}^{gg\to gg}_{aa'bb'}(s,t) = 2\, g_s^2 \left[(T^c)_{aa'} C_{\nu_a \nu_{a'}}(p_a, p_{a'}) \right] \frac{s}{t} \left[(T_c)_{bb'} C_{\nu_b \nu_{b'}}(p_b, p_{b'}) \right]$$

$C_{\nu_a \nu_{a'}}(p_a, p_{a'})$ are called *impact factors*

- we may break the amplitude into even/odd states under $s \Leftrightarrow u$ crossing

$$\mathcal{M}^{(\pm)}(s,t) = \frac{\mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t)}{2}$$

- we may decompose the amplitude into t-channel SU(3) representations. For gluon-gluon scattering, it is

$$\mathbf{8}_a \otimes \mathbf{8}_a = [\mathbf{1} \oplus \mathbf{8}_s \oplus \mathbf{27}] \oplus [\mathbf{8}_a \oplus \mathbf{10} \oplus \overline{\mathbf{10}}]$$

- at tree level, and at leading power in t/s, there is only $\mathbf{8}_a$ and only the odd amplitude under $s \Leftrightarrow u$ crossing

$$\mathcal{M}^{(0)}_{ij\to ij}(s,t) = \mathcal{M}^{(0,-)}_{ij\to ij}(s,t) \qquad \mathcal{M}^{(0,+)}_{ij\to ij}(s,t) = 0$$

LL accuracy

- At leading logarithmic (LL) accuracy in s/t, there is still only $\mathbf{8}_a$ and loops corrections are obtained by the substitution

$$\frac{1}{t} \to \frac{1}{t}\left(\frac{s}{-t}\right)^{\alpha(t)}$$

- $\alpha(t)$ is the Regge gluon trajectory, with infrared coefficients

$$\alpha(t) = \frac{\alpha_s(-t,\epsilon)}{4\pi}\,\alpha^{(1)} + \left(\frac{\alpha_s(-t,\epsilon)}{4\pi}\right)^2 \alpha^{(2)} + \mathcal{O}\left(\alpha_s^3\right) \qquad \alpha_s(-t,\epsilon) = \left(\frac{\mu^2}{-t}\right)^{\epsilon} \alpha_s(\mu^2)$$

$$\alpha^{(1)} = C_A \frac{\widehat{\gamma}_K^{(1)}}{\epsilon} = C_A \frac{2}{\epsilon} \qquad \alpha^{(2)} = \frac{b_0}{\epsilon^2} C_A + \frac{2\gamma_K^{(2)}}{\epsilon} C_A + \left(\frac{404}{27} - 2\zeta_3\right) C_A^2 - \frac{56}{27} C_A n_f + \mathcal{O}(\epsilon)$$

- the exponentiation through a Regge trajectory is called Reggeisation

- in Mellin space, the amplitude displays a (Regge) pole

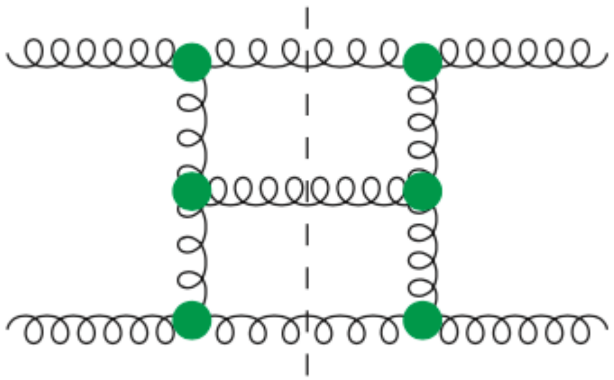
$$f_\ell^{(8_a)}(t) \propto \frac{\alpha(t)}{\ell - 1 - \alpha(t)}$$

- the Regge gluon trajectory is universal, i.e. process independent

Building blocks of BFKL at LL accuracy

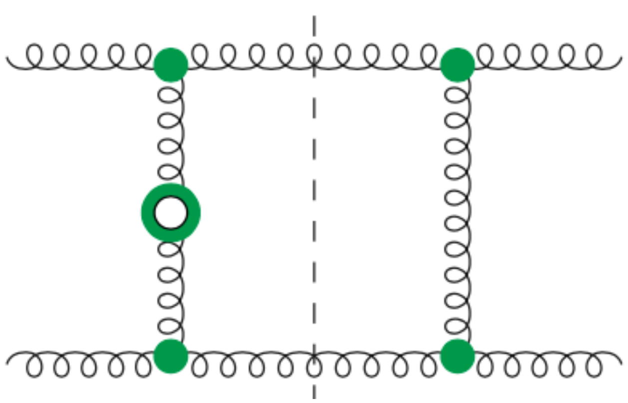
The building blocks of the BFKL equation at LL accuracy are

- real: the emission of a gluon along the ladder



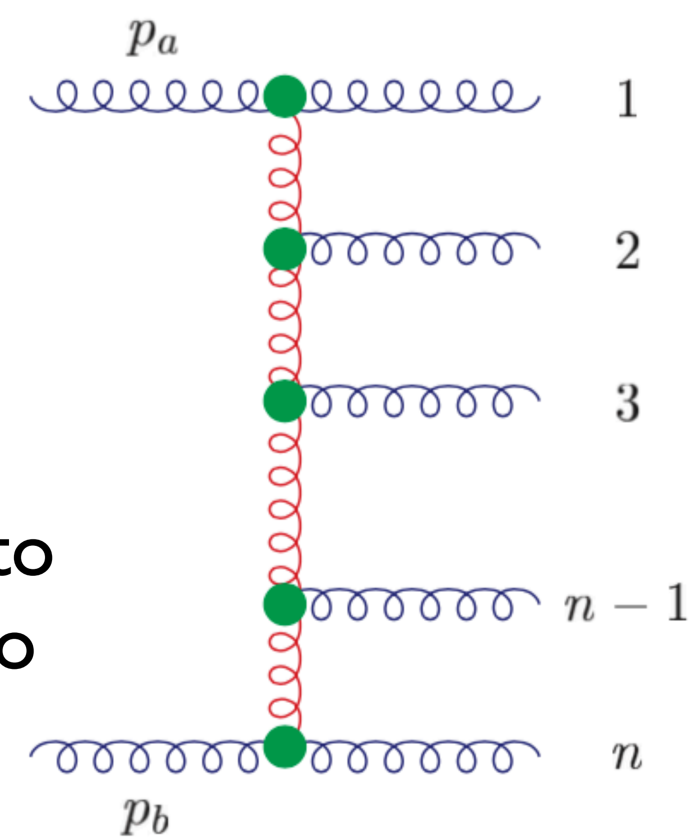
$$\mathcal{M}^{(0)}_{gg\to ggg}(s,t) = 2s\left[g_s f^{ad_1c_1} C_{\nu_a\nu_1}(p_a,p_1)\right] \times \frac{1}{t_1}\left[g_s f^{c_1d_2c_2} C_{\nu_2}(p_2)\right] \times \frac{1}{t_2}\left[g_s f^{bd_3c_2} C_{\nu_b\nu_3}(p_b,p_3)\right]$$

- virtual: the one-loop Regge trajectory



$$\mathcal{M}^{(1)}_{gg\to gg}(s,t) = \frac{\alpha_s}{4\pi}\left(\frac{\mu^2}{-t}\right)^{\epsilon}\frac{2C_A}{\epsilon}\ln\frac{s}{-t}\mathcal{M}^{(0)}_{gg\to gg}(s,t)$$

BFKL resummation



- BFKL is a resummation of multiple gluon radiation out of the gluon exchanged in the *t* channel
- the LL (Balitski Fadin Kuraev Lipatov 1976-77) and Next-to-Leading Logarithmic (Fadin-Lipatov 1998) contributions in $\log(s/|t|)$ of the radiative corrections to the gluon propagator in the *t* channel are resummed to all orders in α_s
- the resummation yields an integral (BFKL) equation for the evolution of the gluon propagator in 2-dim transverse momentum space
- the BFKL equation is obtained in the limit of strong rapidity ordering of the emitted gluons, with no ordering in transverse momentum - *multi-Regge kinematics* (MRK)
- the solution is a Green's function of the momenta flowing in and out of the gluon ladder exchanged in the *t* channel

BFKL theory

- the BFKL equation describes the evolution of the gluon propagator in 2-dim transverse momentum space

$$\omega\, f_\omega(q_1,q_2) = \frac{1}{2}\,\delta^{(2)}(q_1-q_2) + \int d^2k\, K(q_1,k)\, f_\omega(k,q_2)$$

- the solution is given in terms of eigenfunctions $\Phi_{\nu n}$ and an eigenvalue $\omega_{\nu n}$

$$f_\omega(q_1,q_2) = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu\, \frac{1}{\omega-\omega_{\nu n}}\, \Phi_{\nu n}(q_1)\, \Phi^*_{\nu n}(q_2)$$

as a function of rapidity, the solution is

$$f(q_1,q_2,y) = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu\, \Phi_{\nu n}(q_1)\, \Phi^*_{\nu n}(q_2)\, e^{y\,\omega_{\nu n}}$$

- we expand kernel K, eigenfunctions $\Phi_{\nu n}$ and eigenvalue $\omega_{\nu n}$ in powers of $\bar{\alpha}_\mu = \frac{N_C}{\pi}\alpha_S(\mu^2)$

$$K(q_1,q_2) = \bar{\alpha}_\mu \sum_{l=0}^{\infty} \bar{\alpha}_\mu^l\, K^{(l)}(q_1,q_2) \qquad \omega_{\nu n} = \bar{\alpha}_\mu \sum_{l=0}^{\infty} \bar{\alpha}_\mu^l\, \omega_{\nu n}^{(l)} \qquad \Phi_{\nu n}(q) = \sum_{l=0}^{\infty} \bar{\alpha}_\mu^l\, \Phi_{\nu n}^{(l)}(q)$$

- At LL accuracy

$$\omega_{\nu n}^{(0)} = -2\gamma_E - \psi\left(\frac{|n|+1}{2}+i\nu\right) - \psi\left(\frac{|n|+1}{2}-i\nu\right) \qquad \Phi_{\nu n}^{(0)}(q) = \frac{1}{2\pi}\,(q^2)^{-1/2+i\nu}\, e^{in\theta}$$

note that in N=4 SYM the eigenfunctions and the eigenvalue are the same

Regge-pole factorisation

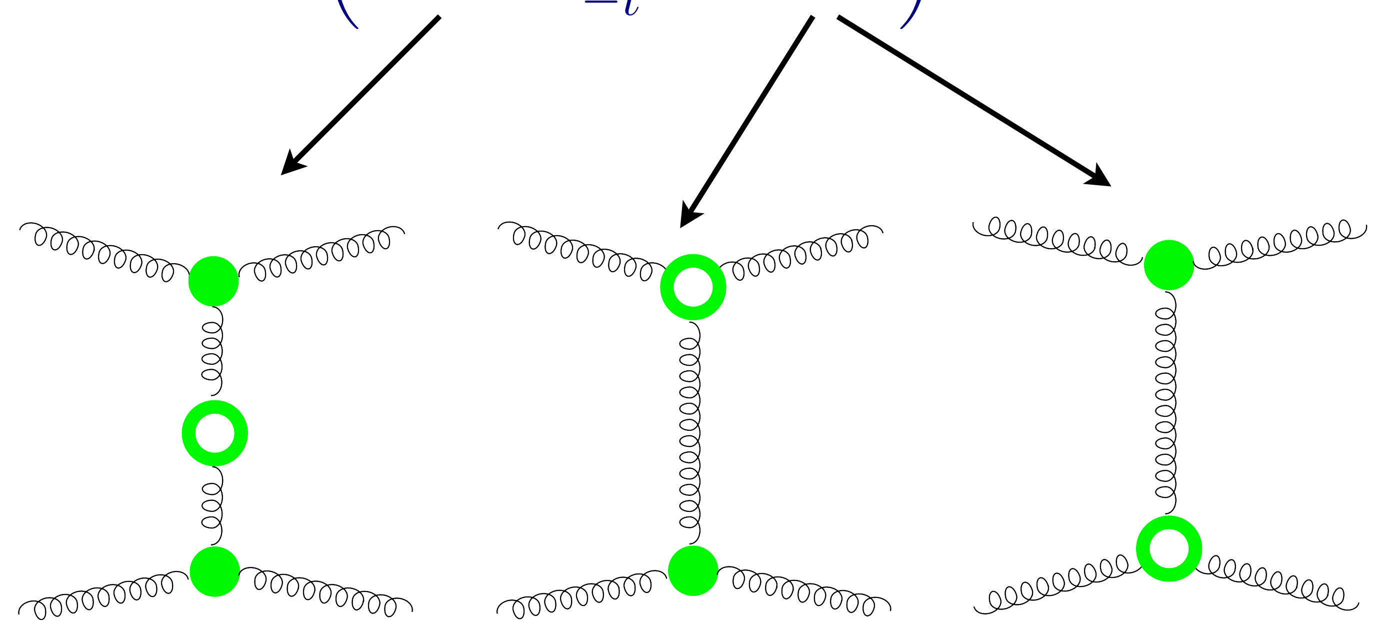
- gluon-gluon (odd) amplitude for $\mathbf{8}_a$

$$\mathcal{M}^{gg\to gg}_{aa'bb'}(s,t) = 2\,g_s^2\,\frac{s}{t}\left[(T^c)_{aa'}C_{\nu_a\nu_{a'}}(p_a,p_{a'})\right]\left[\left(\frac{s}{-t}\right)^{\alpha(t)}+\left(\frac{-s}{-t}\right)^{\alpha(t)}\right]\left[(T_c)_{bb'}C_{\nu_b\nu_{b'}}(p_b,p_{b'})\right]$$

Fadin Lipatov 1993

strip colour off & expand at one loop

$$\mathrm{Re}[\mathcal{M}^{(1,-)[8_a]}_{gg\to gg}(s,t)] = \mathcal{M}^{(0)}_{gg\to gg}(s,t)\left(\alpha^{(1)}(t)\ln\frac{s}{-t}+2C^{(1)}_{gg}(t)\right)$$

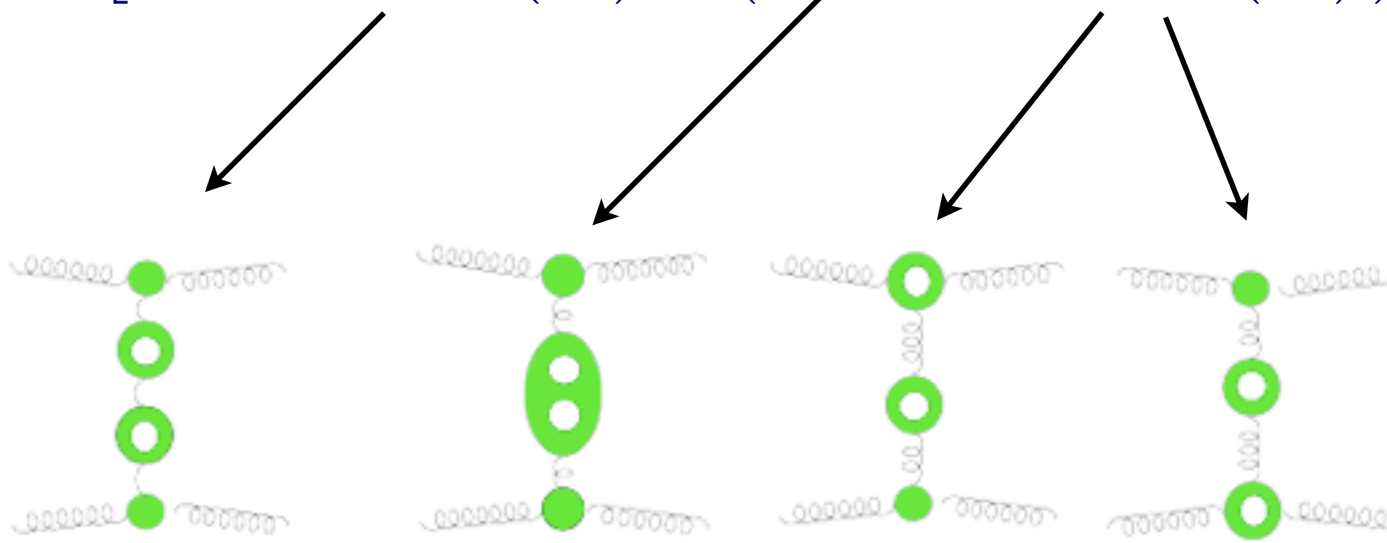


- the Regge gluon trajectory is universal, i.e. process independent
- the one-loop gluon impact factor $C^{(1)}_{gg}(t)$ is a polynomial in t, ϵ
- perform the Regge limit of the quark-quark amplitude
  → get one-loop quark impact factor $C^{(1)}_{qq}(t)$
- if factorisation holds, one can obtain the one-loop quark-gluon amplitude
  by assembling the Regge trajectory and the gluon and quark impact factors
  the result should match the quark-gluon amplitude in the high-energy limit: it does

Regge-pole factorisation at NLL accuracy

- in the Regge limit, the two-loop expansion of the gluon-gluon (odd) amplitude for $\mathbf{8}_a$ is

$$\mathrm{Re}[\mathcal{M}^{(2,-)[8_a]}_{gg\to gg}(s,t)] = \left[\frac{1}{2}\left(\alpha^{(1)}(t)\right)^2 \ln^2\left(\frac{s}{-t}\right) + \left(\alpha^{(2)}(t) + 2C^{(1)}_{gg}(t)\ln\left(\frac{s}{-t}\right)\right)\right] \mathcal{M}^{(0)}_{gg\to gg}(s,t)$$

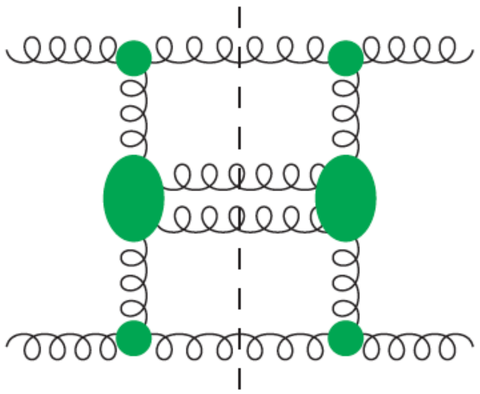


- for the real part of the amplitude, at NLL accuracy in *s/t* there is still only $\mathbf{8}_a$ which yields the 2-loop gluon trajectory
- gluon Reggeisation has been proven at NLL accuracy   Fadin Fiore Kozlov Reznichenko 2006
- the two-loop Regge gluon trajectory is universal

Building blocks of BFKL at NLL accuracy

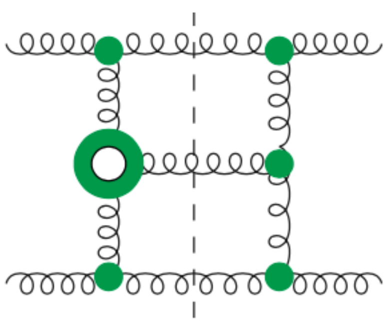
The building blocks of the BFKL equation at NLL accuracy are

- RR: the emission of two gluons, or a qq pair, along the ladder



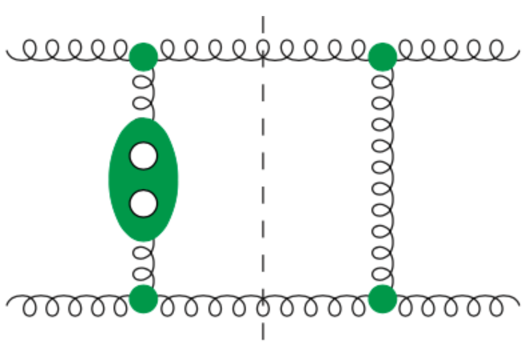
Fadin Lipatov 1989
VDD 1996; Fadin Lipatov 1996

- RV: the one-loop correction to the emission of a gluon along the ladder



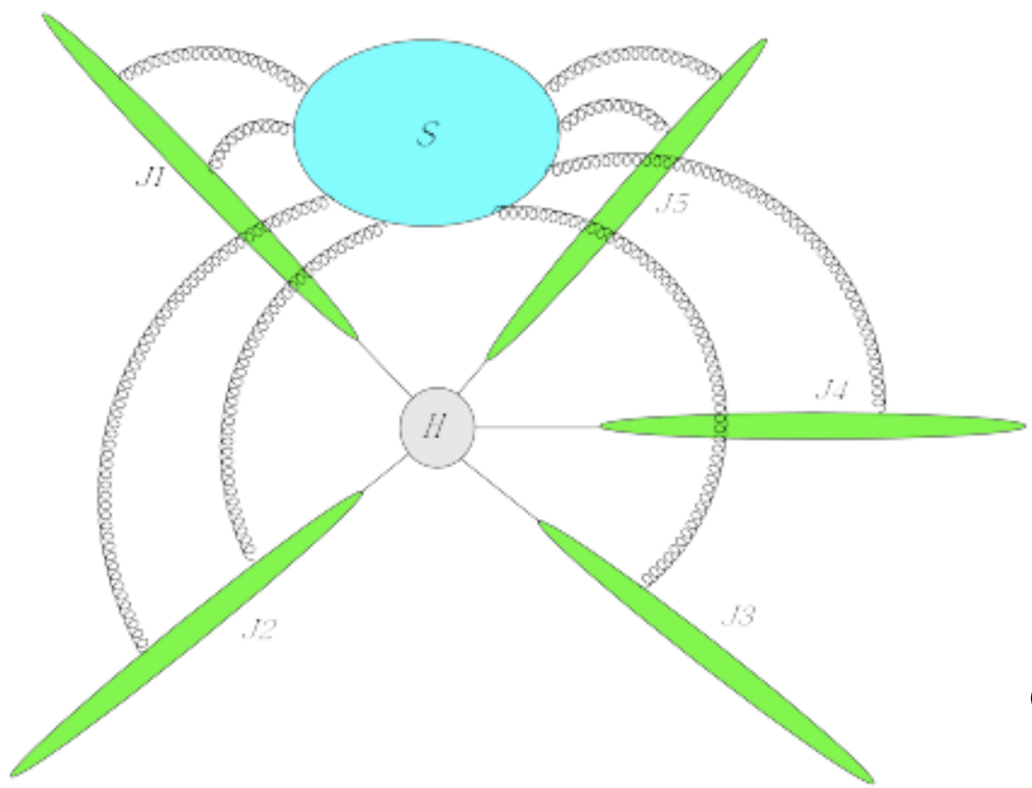
Fadin Lipatov 1993
Fadin Fiore Quartarolo 1994
Fadin Fiore Kotsky 1996
VDD Schmidt 1998

- VV: the two-loop Regge trajectory



Fadin Fiore Quartarolo 1995
Fadin Fiore Kotsky 1995, 1996
VDD Glover 2001

Infrared factorisation



$$\mathcal{M}_n(\{p_i\},\alpha_s) = Z_n(\{p_i\},\alpha_s,\mu)\,\mathcal{H}_n(\{p_i\},\alpha_s,\mu)$$

Z_n is solution to the RGE equation

$$Z_n = \mathrm{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\Gamma_n(\{p_i\},\lambda,\alpha_s(\lambda^2))\right\}$$

$\mathbf{\Gamma}_n$ is the soft anomalous dimension

$$\Gamma_n(\{p_i\},\lambda,\alpha_s) = \Gamma_n^{\mathrm{dip}}(\{p_i\},\lambda,\alpha_s) + \Delta_n(\{\rho_{ijkl}\},\alpha_s)$$

dipole form

$$\Gamma_n^{\mathrm{dip}}(\{p_i\},\lambda,\alpha_s) = -\frac{1}{2}\hat{\gamma}_K(\alpha_s)\sum_{i<j}\log\left(\frac{-s_{ij}}{\lambda^2}\right)\mathbf{T_i}\cdot\mathbf{T_j} + \sum_{\mathbf{i=1}}^{\mathbf{n}}\gamma_{\mathbf{J_i}}(\alpha_\mathbf{s})$$

$$\rho_{ijkl} = \frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}$$

Becher Neubert; Gardi Magnea 2009

At 2 loops, $\Delta^{(2)} = 0$, $\mathbf{\Gamma}_2$: Catani 1998; Aybat Dixon Sterman 2006

At 3 loops,

$$\Delta_4^{(3)}(\rho_{1234},\rho_{1432},\alpha_s) = 16\mathbf{T}_1^{a_1}\mathbf{T}_2^{a_2}\mathbf{T}_3^{a_3}\mathbf{T}_4^{a_4}\left\{f^{a_1a_2b}f^{a_3a_4b}\left[F\left(1-\frac{1}{z}\right)-F\left(\frac{1}{z}\right)\right]\right.$$
$$\left. + f^{a_1a_3b}f^{a_4a_2b}\left[F(z)-F(1-z)\right] + f^{a_1a_4b}f^{a_2a_3b}\left[F\left(\frac{1}{1-z}\right)-F\left(\frac{z}{z-1}\right)\right]\right\}$$

$\rho_{1234} = z\bar{z}$ $\quad\rho_{1432} = (1-z)(1-\bar{z})$ $\qquad F(z) = \mathcal{L}_{10101}(z) + 2\zeta_2[\mathcal{L}_{001}(z)+\mathcal{L}_{100}(z)] + 6\zeta_4\mathcal{L}_1(z)$

is given in terms of SVHPLs

Almelid Duhr Gardi 2015

Infrared factorisation in the Regge limit

- we introduce the colour operators

$$\mathbf{T}_s = \mathbf{T}_a + \mathbf{T}_b\,, \quad \mathbf{T}_t = \mathbf{T}_a + \mathbf{T}_{a'}\,, \quad \mathbf{T}_u = \mathbf{T}_a + \mathbf{T}_{b'}$$

$$\mathbf{T}_a + \mathbf{T}_b + \mathbf{T}_{a'} + \mathbf{T}_{b'} = 0$$

$$\mathbf{T}_s^2 + \mathbf{T}_t^2 + \mathbf{T}_u^2 = \sum_{i=1}^{4} C_i = \mathcal{C}_{\rm tot}$$

- in the limit $s \gg t$, the dipole operator Z becomes

$$Z(\{p_i\}, \alpha_s(\mu^2), \mu) = \tilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \mu\right) Z_i(t, \alpha_s(\mu^2), \mu)\, Z_j(t, \alpha_s(\mu^2), \mu)$$

VDD Duhr Gardi Magnea White 2011
VDD Falcioni Magnea Vernazza 2014
Caron-Huot Gardi Vernazza 2017

- Z_i are scalar factors which define the impact factors in terms of cusp and collinear anomalous dimensions

$$Z_n = \mathrm{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\left[\frac{\gamma_K(\alpha_s(\lambda^2))}{4}C_i \ln\frac{-t}{\lambda^2}\right] + \gamma_i(\alpha_s(\lambda^2))\right\}$$

- colour and ln(s/t) dependence are in the operator $\tilde{Z}$

$$\tilde{Z} = \exp\left\{K(\alpha_s(\mu^2))\left[\left(\ln\left(\frac{s}{-t}\right) - i\frac{\pi}{2}\right)\mathbf{T}_t^2 + i\pi\mathbf{T}_{s-u}^2\right] + Q_\Delta^{(3)}\right\}$$

which is determined by the cusp anomalous dimension and by Q, through

$$K\Big(\alpha_s(\mu^2), \epsilon\Big) = -\frac{1}{4}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\widehat{\gamma}_K\left(\alpha_s(\lambda^2, \epsilon)\right)\,,$$

$$Q_\Delta^{(3)} = -\frac{\Delta^{(3)}}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\left(\frac{\alpha_s(\lambda^2)}{\pi}\right)^3$$

Caron-Huot Gardi Vernazza 2017

comparing infrared and Regge factorizations

- the pole terms of the Regge trajectory are fixed by the operator K and thus by the cusp anomalous dimension Korchemskaya Korchemsky 1994

- the pole terms of the (one-loop) impact factor are fixed by the cusp and collinear anomalous dimensions

  VDD Falcioni Magnea Vernazza 2014

- in infrared factorisation, gluon Reggeisation at LL and NLL accuracy is due to the operator $\tilde{Z}$ being diagonal in the t-channel colour basis

  VDD Duhr Gardi Magnea White 2011

a mysterious relation …

- in infrared factorisation, we have a precise knowledge of how the infrared poles in ε occur in the impact factors and in the Regge trajectory.
  Their finite parts, though, are treated as free parameters

- the Regge limit is an expansion in ln(*s*/*t*) and is valid to all orders of ε

- the one-loop gluon impact factor $C_{gg}^{(1)}(\epsilon)$ is known, in CDR/HV, to all orders of ε

$$C_{gg}^{(1)}(\epsilon) = -\frac{\gamma_K^{(1)}}{\epsilon^2} C_A + \frac{4\gamma_g^{(1)}}{\epsilon} + \frac{b_0}{2\epsilon} + \left(3\zeta_2 - \frac{67}{18}\right) C_A + \frac{5}{9} n_f$$
$$+ \left[\left(\zeta_3 - \frac{202}{27}\right) C_A + \frac{28}{27} n_f\right] \epsilon + \mathcal{O}(\epsilon^2)$$

Fadin Fiore 1992
Fadin Lipatov 1993
Bern VDD Schmidt 1998

- the two-loop Regge trajectory is

$$\alpha^{(2)}(\epsilon) = \frac{b_0}{\epsilon^2} C_A + \frac{2\gamma_K^{(2)}}{\epsilon} C_A + \left(\frac{404}{27} - 2\zeta_3\right) C_A^2 - \frac{56}{27} C_A n_f + \mathcal{O}(\epsilon)$$

- the O(ε) term of the one-loop gluon impact factor predicts the O(ε^0) term of the two-loop Regge trajectory

VDD 2017

- What is the O(ε^0) term of the two-loop Regge trajectory?

In planar N=4 SYM, the BDS ansatz fixes the O(ε^0) term of the Regge trajectory to be the gluon collinear anomalous dimension

$$\alpha^{(l)}_{N=4}(\epsilon) = 2^{l-1}\,\alpha^{(1)}(l\epsilon)\,\left(f_0^{(l)} + \epsilon f_1^{(l)}\right) + \mathcal{O}(\epsilon)$$

Drummond Korchemsky Sokatchev 2007
Naculich Schnitzer 2007
VDD Glover 2008

In QCD, the eikonal anomalous dimension, the collinear anomalous dimension on a polygonal Wilson loop, the threshold soft anomalous dimension in SCET, the *D* term in threshold resummation, all include a block which up to the ζ_3 term coincide with the O(ε^0) term of the Regge trajectory

the ``eikonal'' anomalous dimension of Erdogan Sterman 2011 agrees with the O(ε^0) term of the Regge trajectory

- Why does the O(ε) term of the one-loop gluon impact factor know the O(ε^0) term of the two-loop Regge trajectory?

it hints at more structure in infrared factorisation than we currently know (perhaps related to this being a two-hard-scale problem)

Regge-pole factorisation breaks at NNLO

- at LL accuracy for the amplitude, and at NLL accuracy for the real part of the amplitude, Regge-pole factorisation is based on the t-channel exchange of $\mathbf{8}_a$ only as one Reggeised gluon

- one can see in 3 ways that this is not correct at NNLO:

  — if pole factorisation holds, one can obtain the two-loop quark-gluon amplitude by assembling the two-loop Regge trajectory and gluon and quark impact factors. The result should match the quark-gluon amplitude in the high-energy limit. It doesn't by an N_c-subleading π^2/ε^2 factor

  VDD Glover 2001

  — in infrared factorisation at NNLL accuracy, the operator $\tilde{Z}$ is non-diagonal in the t-channel colour basis

  VDD Duhr Gardi Magnea White 2011
  VDD Falcioni Magnea Vernazza 2014

  — at NNLO, the picture based on one Reggeised-gluon exchange breaks down. Using the Balitsky-JIMWLK rapidity evolution equation, or a direct computation, one can see that a N_c-subleading 3-Reggeised-gluons exchange occurs at NNLO and NNLL accuracy

  Caron-Huot Gardi Vernazza 2017
  Fadin Lipatov 2017

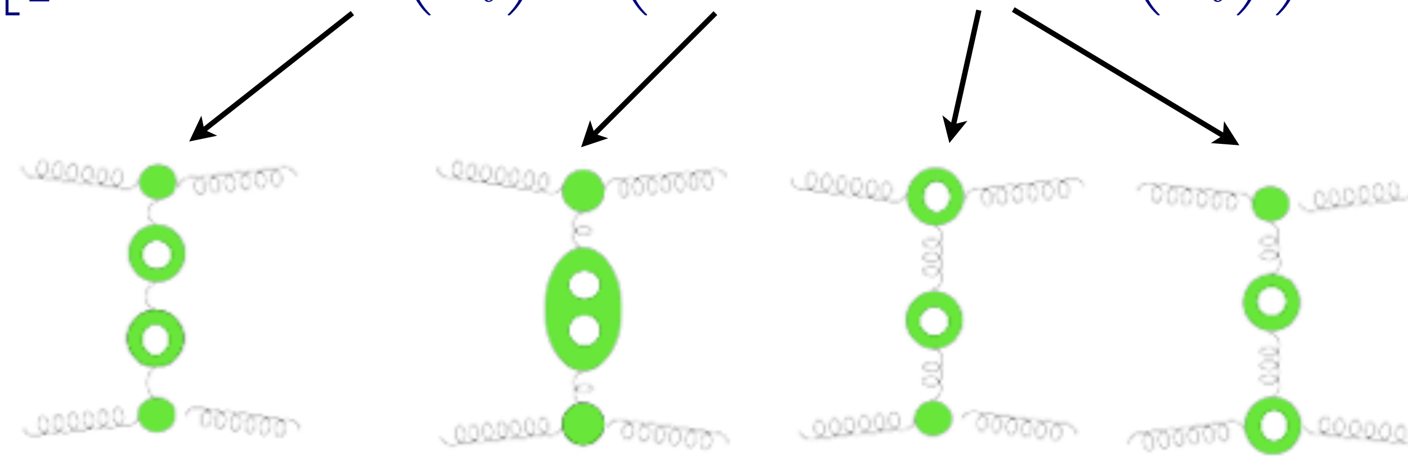
- It is still possible, though, to define a 2-loop impact factor, based on one Reggeised-gluon exchange

  VDD Falcioni Magnea Vernazza 2014
  Caron-Huot Gardi Vernazza 2017

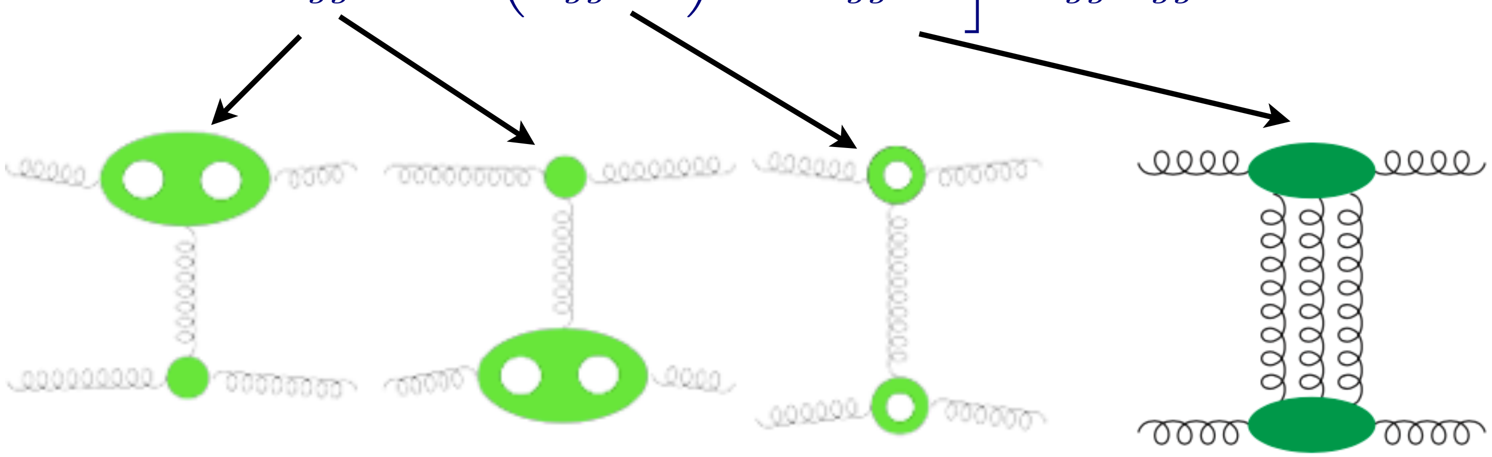
Regge factorisation at 2 loops

- in the Regge limit, the two-loop expansion of the gluon-gluon (odd) amplitude for **8**$_a$ is

$$\mathrm{Re}[\mathcal{M}^{(2,-)[8_a]}_{gg\to gg}(s,t)] = \left[\frac{1}{2}\left(\alpha^{(1)}(t)\right)^2 \ln^2\left(\frac{s}{-t}\right) + \left(\alpha^{(2)}(t) + 2\,C^{(1)}_{gg}(t)\ln\left(\frac{s}{-t}\right)\right)\right.$$



$$\left. + 2\,C^{(2)}_{gg}(t) + \left(C^{(1)}_{gg}(t)\right)^2 + R^{(2)}_{gg}(t)\right] \mathcal{M}^{(0)}_{gg\to gg}(s,t)$$



Regge factorisation at NNLL accuracy

- $\mathcal{M}^{(2,0,-)}$: **8**$_a$, Regge pole, one Reggeised gluon
  8$_a$, Regge cut, three Reggeised gluons (N_c-subleading)

- $\mathcal{M}^{(3,1,-)}$: **8**$_a$, Regge pole, one Reggeised gluon
  8$_a$, Regge cut, three Reggeised gluons
  $\mathbf{10} \oplus \overline{\mathbf{10}}$, Regge cut, three Reggeised gluons Caron-Huot Gardi Vernazza 2017

  the N_c-subleading pole-factorisation violation (**8**$_a$, Regge cut, three Reggeised gluons)
  predicted for $\mathcal{M}^{(3,1,-)}$ in VDD Falcioni Magnea Vernazza 2014
  confirmed by the 3-loop 4-pt amplitude computation in full *N*=4 SYM Henn Mistlberger 2016

- one must also consider the imaginary parts at NLL accuracy,
  since their squares would be relevant to resummations at NNLL accuracy

  $\mathcal{M}^{(1,0,+)}$: **8**$_s$, Regge pole, one Reggeised gluon
  1 and **27**, Regge cut, two Reggeised gluons

  $\mathcal{M}^{(2,1,+)}$: **1** and **27**, Regge cut, two Reggeised gluons

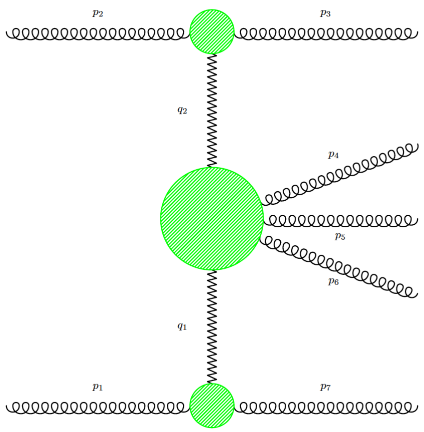
  Caron-Huot Gardi Reichel Vernazza 2017

- finally, we may ignore $Q_{\Delta}^{(3)}$ since it contributes to the imaginary parts at NNLL accuracy,
  and to the real parts at N^3LL accuracy Caron-Huot Gardi Vernazza 2017

Building blocks of BFKL at NNLL accuracy

The building blocks of a would-be BFKL ladder at NNLL accuracy

- RRR: the emission of three partons along the ladder



VDD Frizzo Maltoni 1999

- VVV: the three-loop Regge trajectory

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- still unknown

RRV: the one-loop correction to the emission of two gluons, or a qq pair, along the ladder

RVV: the two-loop correction to the emission of a gluon along the ladder

Planar N=4 Super Yang Mills

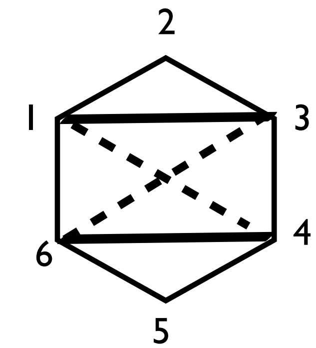
- In the last years, a huge progress has been made in understanding the analytic structure of the *S*-matrix of planar N=4 SYM
- Besides the ordinary conformal symmetry, in the planar limit the *S*-matrix exhibits a dual conformal symmetry
  Drummond Henn Smirnov Sokatchev 2006
- Accordingly, the analytic structure of the scattering amplitudes is highly constraint
- 4- and 5-point amplitudes are fixed to all loops by the symmetries in terms of the one-loop amplitudes and the cusp anomalous dimension
  Anastasiou Bern Dixon Kosower 2003, Bern Dixon Smirnov 2005
  Drummond Henn Korchemsky Sokatchev 2007
- Beyond 5 points, the finite part of the amplitudes is given in terms of a remainder function R. The symmetries only fix the variables of R (some conformally invariant cross ratios) but not the analytic dependence of R on them

- for $n = 6$, the conformally invariant cross ratios are

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$$

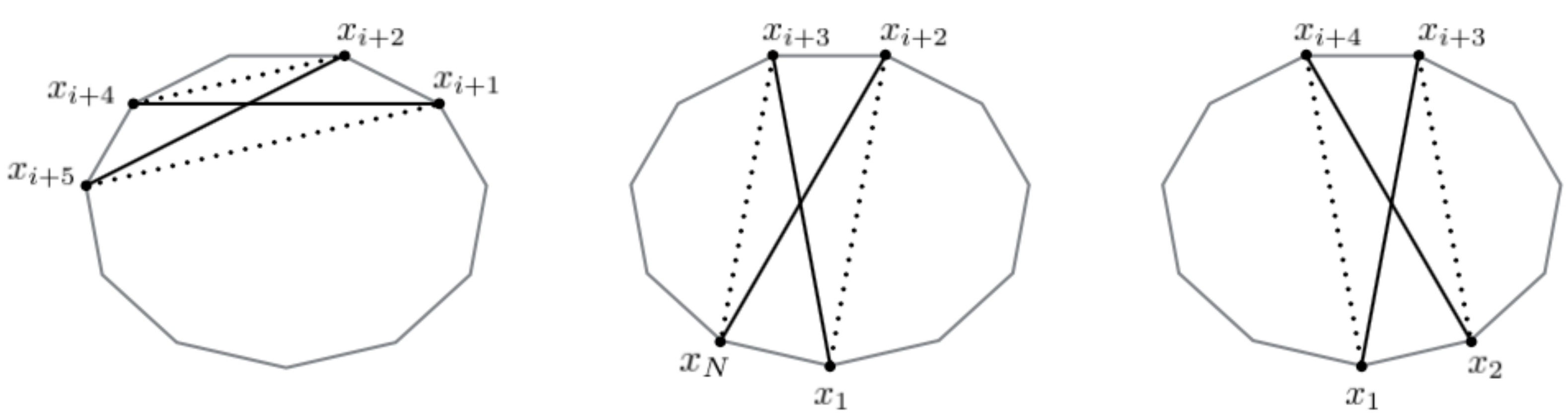
x_i are variables in a dual space s.t. $p_i = x_i - x_{i+1}$

thus $x_{k,k+r}^2 = (p_k + \ldots + p_{k+r-1})^2$



- for n points, dual conformal invariance implies dependence on $3n$-15 independent cross ratios

$$u_{1i} = \frac{x_{i+1,i+5}^2 x_{i+2,i+4}^2}{x_{i+1,i+4}^2 x_{i+2,i+5}^2}, \qquad u_{2i} = \frac{x_{N,i+3}^2 x_{1,i+2}^2}{x_{N,i+2}^2 x_{1,i+3}^2}, \qquad u_{3i} = \frac{x_{1\,i+4}^2 x_{2,i+3}^2}{x_{1,i+3}^2 x_{2,i+4}^2}$$



u_{1i} u_{2i} u_{3i}

Multi-Regge kinematics in planar $\mathcal{N}=4$ SYM

- Amplitudes in multi-Regge kinematics (MRK) at LL accuracy factorise in terms of building blocks, which are expressed through Regge poles and can be determined through the 4-pt and 5-pt amplitudes

- In planar $\mathcal{N}=4$ SYM, the symmetries (BDS ansatz) fix the 4-pt and 5-pt amplitudes to all orders. Thus, it comes as no surprise that (in the Euclidean region) the remainder functions R vanish at all points

  Brower Nastase Schnitzer Tan; Bartels Lipatov Sabio-Vera; VDD Duhr Glover 2008

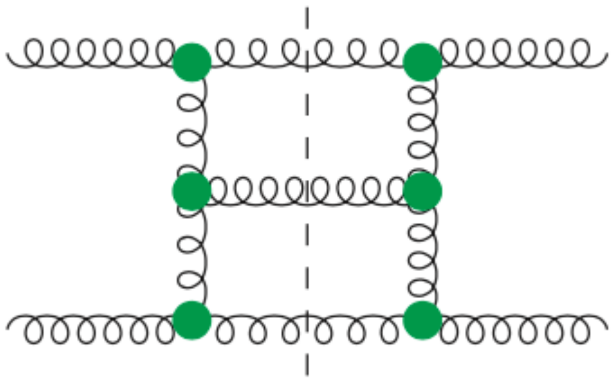
- If, before taking the multi-Regge limit, we analytically continue to regions of the Minkowski space where some Mandelstam invariants may pick up a phase, the amplitude may develop cuts, due to 2-Reggeon exchange.
  The discontinuity of the amplitude is described by a dispersion relation for the adjoint, which is similar to the singlet BFKL equation in QCD

  Bartels Lipatov Sabio-Vera 2008

Building blocks of BFKL at LL accuracy

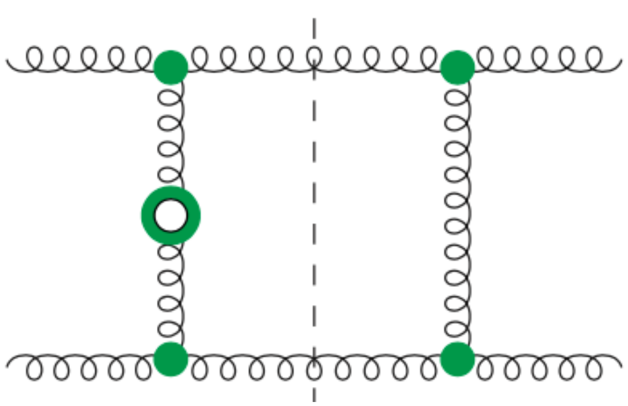
The building blocks of the BFKL equation at LL accuracy are

- real: the emission of a gluon along the ladder



$$\mathcal{M}^{(0)}_{gg\to ggg}(s,t) = 2s\left[g_s f^{ad_1c_1} C_{\nu_a\nu_1}(p_a,p_1)\right] \times \frac{1}{t_1}\left[g_s f^{c_1d_2c_2} C_{\nu_2}(p_2)\right] \times \frac{1}{t_2}\left[g_s f^{bd_3c_2} C_{\nu_b\nu_3}(p_b,p_3)\right]$$

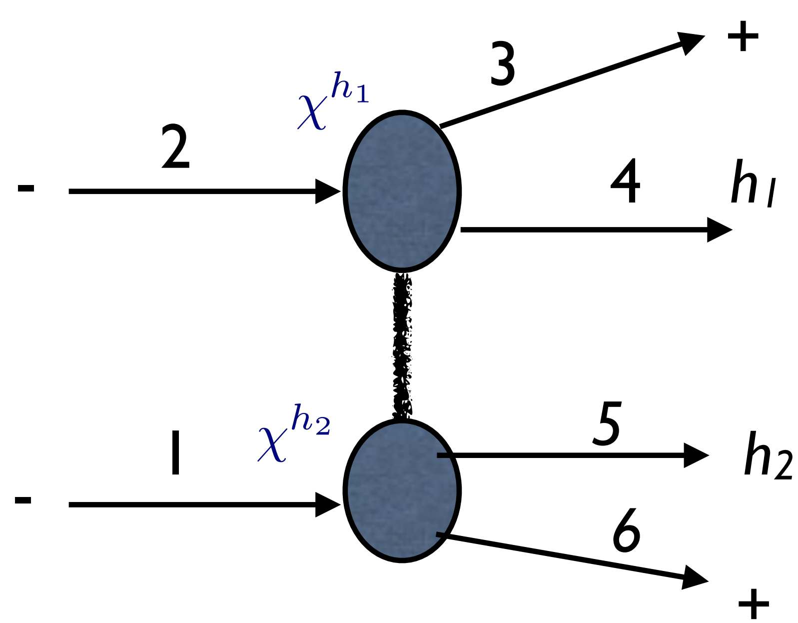
- virtual: the one-loop Regge trajectory



$$\mathcal{M}^{(1)}_{gg\to gg}(s,t) = \frac{\alpha_s}{4\pi}\left(\frac{\mu^2}{-t}\right)^{\epsilon} \frac{2C_A}{\epsilon} \ln\frac{s}{-t} \mathcal{M}^{(0)}_{gg\to gg}(s,t)$$

Discontinuity of the amplitude in MRK

6-pt amplitude



continue to a Minkowski region

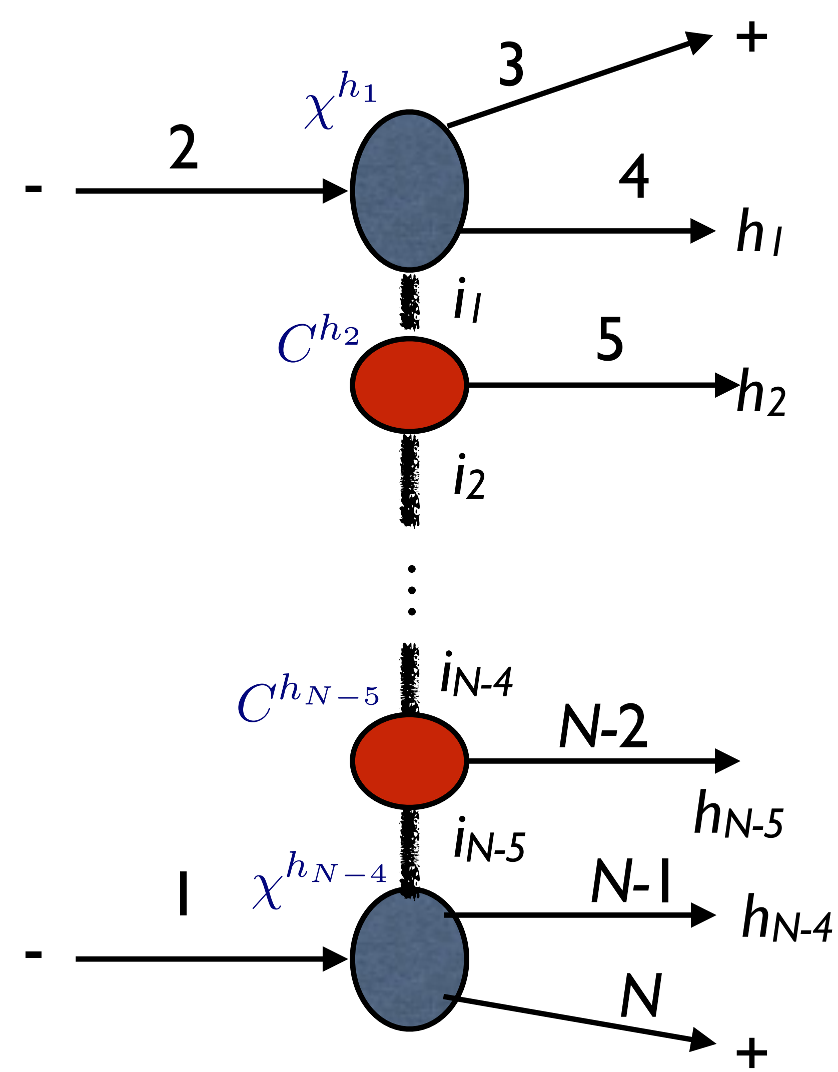
$s_{34},\, s_{56} < 0 \quad s,\, s_{45} > 0$

one cross ratio picks up a phase

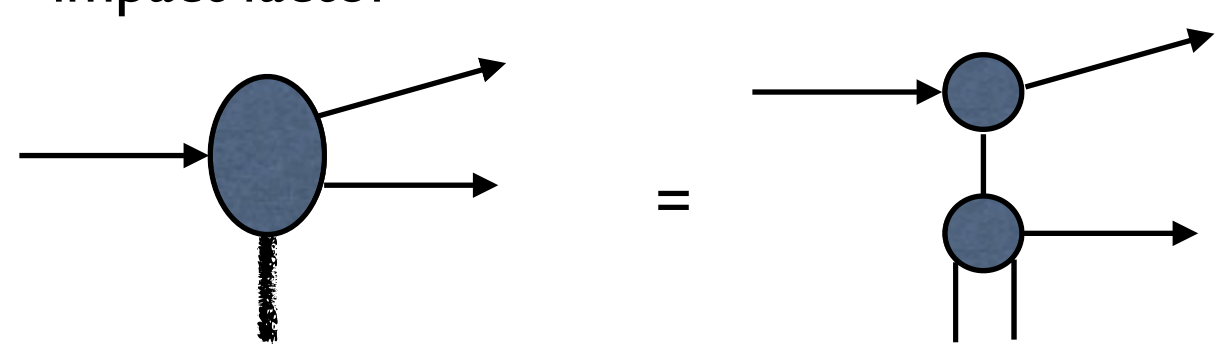
$$u_1 = \frac{s_{12}s_{45}}{s_{345}s_{456}} \to |u_1|\, e^{-2\pi i}$$

compute $\quad \mathrm{Disc}(\mathcal{M})|_{s_{45}}$

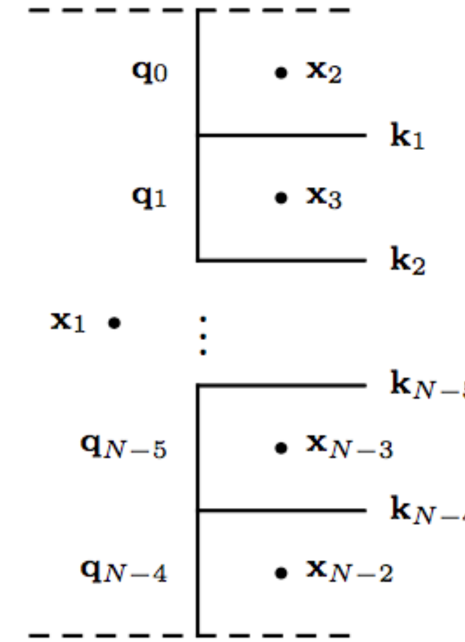
n-pt amplitude



impact factor



Moduli space of Riemann spheres



- in MRK, there is no ordering in transverse momentum, i.e. only the *n*-2 transverse momenta are non-trivial

- dual conformal invariance in transverse momentum space implies dependence on *n*-5 cross ratios of the transverse momenta

$$z_i = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})\,(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+1})\,(\mathbf{x}_{i+2} - \mathbf{x}_{i+3})} = -\frac{\mathbf{q}_{i+1}\,\mathbf{k}_i}{\mathbf{q}_{i-1}\,\mathbf{k}_{i+1}} \qquad i = 1, \dots, n-5$$

- $\mathcal{M}_{0,p}$ = space of configurations of *p* points on the Riemann sphere
  Because we can fix 3 points at 0, 1, ∞, its dimension is dim($\mathcal{M}_{0,p}$)= *p*-3

  see Y. Geyer's talk

- $\mathcal{M}_{0,n-2}$ is the space of the n-pt amplitudes in MRK, with dim($\mathcal{M}_{0,n-2}$) = *n*-5

  Its coordinates can be chosen to be the z_i's,
  i.e. the cross ratios of the transverse momenta

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- on $\mathcal{M}_{0,n-2}$, the singularities are associated to degenerate configurations when two points merge $x_i \to x_{i+1}$
  i.e. when momentum p_i becomes soft $p_i \to 0$

Iterated integrals on $\mathcal{M}_{0,n-2}$

- iterated integrals on $\mathcal{M}_{0,p}$ can be written as multiple polylogarithms (MPL)

→ amplitudes in MRK can be written in terms of MPLs

Brown 2006

- unitarity implies that for massless amplitudes

  $\Delta(M) = \ln(s_{ij}) \otimes \ldots$

- dual conformal invariance requires that the first entry be a cross ratio
  in particular, for amplitudes in MRK $\Delta(M) = \ln|\mathbf{x}_i - \mathbf{x}_j|^2 \otimes \ldots$

- except for the soft limit $p_i \to 0$, in MRK the transverse momenta never vanish

  $|\mathbf{x}_i - \mathbf{x}_j|^2 \neq 0$ → single-valued functions

  thus, *n*-point amplitudes in MRK of planar *N*=4 SYM can be written
  in terms of single-valued iterated integrals on $\mathcal{M}_{0,n-2}$

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- for n=6, iterated integrals on $\mathcal{M}_{0,4}$ are harmonic polylogarithms
  so, 6-point amplitudes in MRK of can be written in terms of
  single-valued harmonic polylogarithms (SVHPL)

Dixon Duhr Pennington 2012

Unitarity on massless amplitudes

- analytic structure of amplitudes is constrained by unitarity $\mathrm{Disc}(M) = iMM^\dagger$

  massless amplitudes may have branch points when Mandelstam invariants vanish $s_{ij} \to 0$ or become infinite $s_{ij} \to \infty$

  discontinuity acts in the first entry of the coproduct $\Delta\mathrm{Disc} = (\mathrm{Disc} \otimes \mathrm{id})\Delta$

  Duhr 2012

  then the coproduct of an amplitude is related to unitarity,
  and for massless amplitudes $\Delta(M) = \ln(s_{ij}) \otimes \ldots$

  ⇒ see A. McLeod's talk

MRK at LL accuracy

- In MRK, 6-pt MHV and NMHV amplitudes are known at any number of loops

Lipatov Prygarin 2010-2011
Dixon Duhr Pennington 2012
Lipatov Prygarin Schnitzer 2012

- knowing the space of functions of the n-point amplitudes in MRK,
(i.e. that is made of single-valued iterated integrals on $\mathcal{M}_{0,n-2}$)
allowed us to compute all MHV amplitudes at ℓ loops in LL accuracy
in terms of amplitudes with up to (ℓ+4) points, in practice up to 5 loops,
and all non-MHV amplitudes in LL accuracy up 8 points and 4 loops

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for MHV amplitudes in MRK at LL accuracy at:

- at 2 loop, the n-pt remainder function $R_n^{(2)}$ can be written as a sum of 2-loop 6-pt remainder functions $R_6^{(2)}$
- …
- …
- at 5 loops, the n-pt remainder function $R_n^{(5)}$ can be written as a sum of 5-loop 6-, 7-, 8- and 9-pt amplitudes

Prygarin Spradlin Vergu Volovich 2011
Bartels Kormilitzin Lipatov Prygarin 2011
Bargheer Papathanasiou Schomerus 2015

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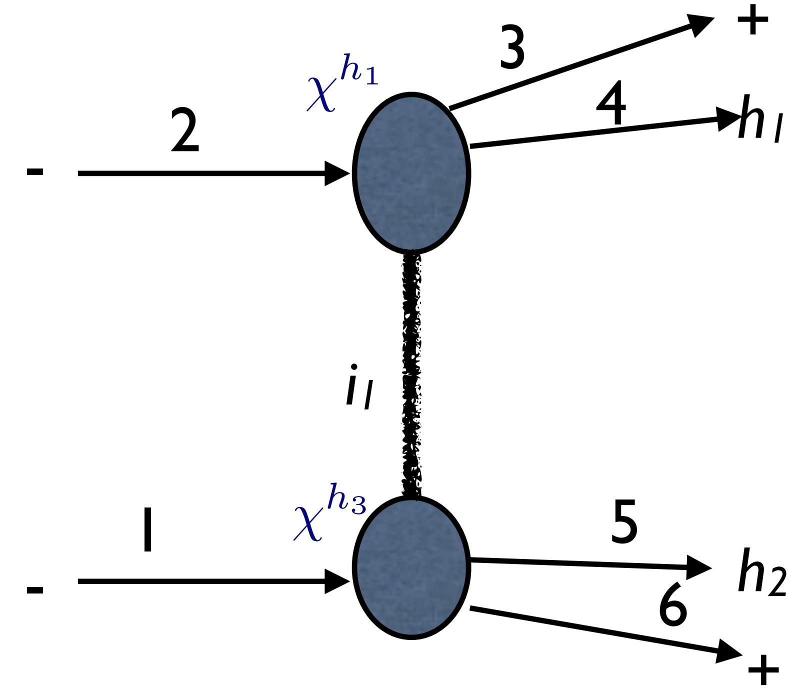
- MRK factorisation works also for non-MHV amplitudes,
however at each loop the number of building blocks is infinite

Beyond the LL accuracy

The building blocks of 6-pt amplitudes:
impact factors and 2-Reggeon exchange,
have been determined at finite coupling

Basso Caron-Huot Sever 2014



Beyond 6 points, the only additional building block is the central-emission vertex.
That has been determined at NLO, which allows for computing the 7-pt amplitudes at NLL accuracy

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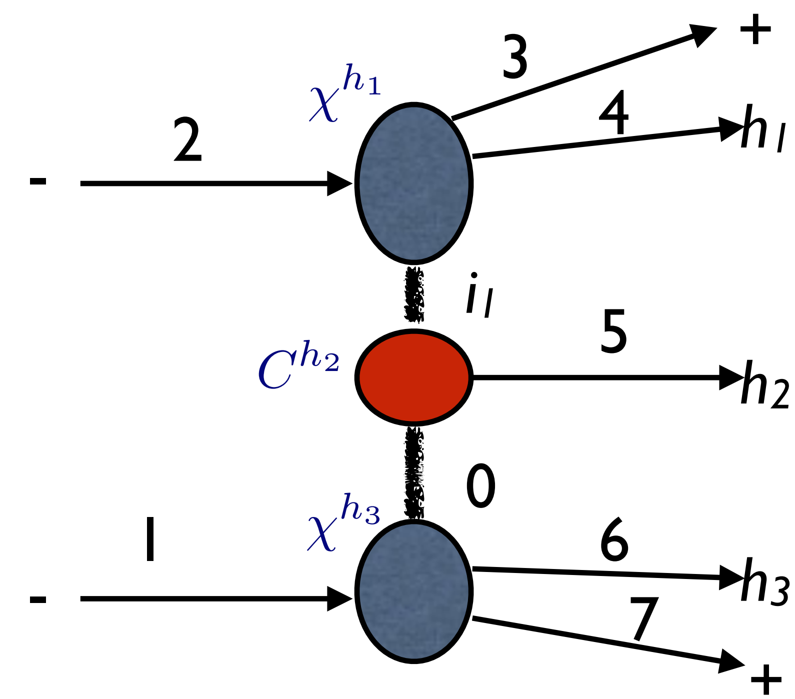

BFKL eigenvalue at LL accuracy in QCD

- The singlet LL BFKL ladder in QCD, and thus the dijet cross section in the high-energy limit, can also be expressed in terms of SVHPLs, i.e. in terms of single-valued iterated integrals on $\mathscr{M}_{0,4}$

  VDD Dixon Duhr Pennington 2013

- Mueller & Navelet evaluated analytically the inclusive dijet cross section up to 5 loops. We evaluated it analytically up to 13 loops

- Also, we could evaluate analytically the dijet cross section differential in the jet transverse energies or the azimuthal angle between the jets (up to 6 loops)

BFKL eigenvalue at NLL accuracy in QCD

- At NLL accuracy in QCD and in *N*=4 SYM, the eigenvalue is

$$\omega^{(1)}_{\nu n} = \frac{1}{4}\,\delta^{(1)}_{\nu n} + \frac{1}{4}\,\delta^{(2)}_{\nu n} + \frac{1}{4}\,\delta^{(3)}_{\nu n} + \gamma_K^{(2)}\chi_{\nu n} - \frac{1}{8}\beta_0\chi^2_{\nu n} + \frac{3}{2}\zeta_3$$

Fadin Lipatov 1998
Kotikov Lipatov 2000, 2002

with one-loop beta function and two-loop cusp anomalous dimension

$$\beta_0 = \frac{11}{3} - \frac{2N_f}{3N_c} \qquad \gamma_K^{(2)} = \frac{1}{4}\left(\frac{64}{9} - \frac{10N_f}{9N_c}\right) - \frac{\zeta_2}{2}$$

and with

$$\delta^{(1)}_{\nu n} = \partial^2_\nu \chi_{\nu n} \qquad \chi_{\nu n} = \omega^{(0)}_{\nu n}$$

$$\delta^{(2)}_{\nu n} = -2\Phi(n,\gamma) - 2\Phi(n,1-\gamma)$$

$$\delta^{(3)}_{\nu n} = -\frac{\Gamma(\frac{1}{2}+i\nu)\Gamma(\frac{1}{2}-i\nu)}{2i\nu}\left[\psi\left(\frac{1}{2}+i\nu\right) - \psi\left(\frac{1}{2}-i\nu\right)\right]$$
$$\times\left[\delta_{n0}\left(3 + \left(1+\frac{N_f}{N_c^3}\right)\frac{2+3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)}\right) - \delta_{|n|2}\left(\left(1+\frac{N_f}{N_c^3}\right)\frac{\gamma(1-\gamma)}{2(3-2\gamma)(1+2\gamma)}\right)\right]$$

$\Phi(n,\gamma)$ is a sum over linear combinations of ψ functions
and γ is a shorthand $\gamma = 1/2 + i\nu$

In blue we labeled the terms which occur only in QCD,
in red the ones which occur in QCD and in *N*=4 SYM

Fourier-Mellin transform

- At NLL accuracy, the BFKL ladder is

$$f^{NLL}(q_1, q_2, \eta_{s_0}) = \frac{1}{2\pi\sqrt{q_1^2 q_2^2}} \sum_{k=1}^{\infty} \frac{\eta_{s_0}^k}{k!} f_{k+1}^{NLL}(z) \qquad \eta_{s_0} = \overline{\alpha}_S(s_0)\, y$$

with coefficients given by the Fourier-Mellin transform

$$f_k^{NLL}(z) = \mathcal{F}\left[\omega_{\nu n}^{(1)} \chi_{\nu n}^{k-2}\right] = \sum_{n=-\infty}^{+\infty} \left(\frac{z}{\bar{z}}\right)^{n/2} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} |z|^{2i\nu} \omega_{\nu n}^{(1)} \chi_{\nu n}^{k-2} \qquad \chi_{\nu n} = \omega_{\nu n}^{(0)}$$

using the explicit form of the eigenvalue

$$\omega_{\nu n}^{(1)} = \frac{1}{4}\delta_{\nu n}^{(1)} + \frac{1}{4}\delta_{\nu n}^{(2)} + \frac{1}{4}\delta_{\nu n}^{(3)} + \gamma_K^{(2)} \chi_{\nu n} - \frac{1}{8}\beta_0 \chi_{\nu n}^2 + \frac{3}{2}\zeta_3$$

the coefficients can be written as

$$f_k^{NLL}(z) = \frac{1}{4} C_k^{(1)}(z) + \frac{1}{4} C_k^{(2)}(z) + \frac{1}{4} C_k^{(3)}(z) + \gamma_K^{(2)} f_{k-1}^{LL}(z) - \frac{1}{8}\beta_0 f_k^{LL}(z) + \frac{3}{2}\zeta_3 f_{k-2}^{LL}(z)$$

with $\quad C_k^{(i)}(z) = \mathcal{F}\left[\delta_{\nu n}^{(i)} \chi_{\nu n}^{k-2}\right]$

the weight of f^{NLL}_k is

weight(f^{NLL}_k)= $\quad k \quad\quad k \quad\quad 0 \le w \le k \quad\quad k-2 \le w \le k \quad\quad k-1 \quad\quad k$

SV functions

- $C_k^{(1)}(z)$ are SVHPLs of uniform weight k with singularities at z=0 and z=1

  $C_k^{(3)}(z)$ are MPLs of type $G(a_1, \ldots, a_n; |z|)$ with $a_k \in \{-i, 0, i\}$

  they are SV functions of z because they have no branch cut on the positive real axis, and have weight $0 \leq w \leq k$

  For $C_k^{(2)}(z)$ one needs Schnetz' generalised SVMPLs with singularities at

  $$z = \frac{\alpha\,\bar{z} + \beta}{\gamma\,\bar{z} + \delta}\,, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{C}$$

  Schnetz 2016

  then one can show that $C_k^{(2)}(z)$ are Schnetz' generalised SVMPLs

  $\mathcal{G}(a_1, \ldots, a_n; z)$ with singularities at $a_i \in \{-1, 0, 1, -1/\bar{z}\}$

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- In moment space, the maximal weight of the BFKL eigenvalue and of the anomalous dimensions of the leading twist operators which control the Bjorken scaling violations in QCD is the same as the corresponding quantities in N=4 SYM (Principle of Maximal Transcendentality)

  Kotikov Lipatov 2000, 2002
  Kotikov Lipatov Velizhanin 2003

  → see B. Penante's talk

  Interestingly, in transverse momentum space at NLL accuracy, the maximal weight of the BFKL ladder in QCD is *not* the same as the one of the ladder in *N*=4 SYM

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BFKL ladder in a generic SU(N$_c$) gauge theory

- one can consider the BFKL eigenvalue at NLL accuracy in a SU(N$_c$) gauge theory with scalar or fermionic matter in arbitrary representations

$$\omega_{\nu n}^{(1)} = \frac{1}{4}\delta_{\nu n}^{(1)} + \frac{1}{4}\delta_{\nu n}^{(2)} + \frac{1}{4}\delta_{\nu n}^{(3)}(\tilde{N}_f,\tilde{N}_s) + \frac{3}{2}\zeta_3 + \gamma^{(2)}(\tilde{n}_f,\tilde{n}_s)\,\chi_{\nu n} - \frac{1}{8}\beta_0(\tilde{n}_f,\tilde{n}_s)\,\chi_{\nu n}^2$$

Kotikov Lipatov 2000

with $\beta_0(\tilde{n}_f,\tilde{n}_s) = \frac{11}{3} - \frac{2\tilde{n}_f}{3N_c} - \frac{\tilde{n}_s}{6N_c}$ $\qquad \gamma^{(2)}(\tilde{n}_f,\tilde{n}_s) = \frac{1}{4}\left(\frac{64}{9} - \frac{10\,\tilde{n}_f}{9N_c} - \frac{4\,\tilde{n}_s}{9N_c}\right) - \frac{\zeta_2}{2}$

$$\tilde{n}_f = \sum_R n_f^R T_R \qquad \tilde{n}_s = \sum_R n_s^R T_R \qquad \mathrm{Tr}(T_R^a T_R^b) = T_R\,\delta^{ab} \qquad T_F = \frac{1}{2}$$

$\tilde{n}_s(\tilde{n}_f) =$ number of scalars (Weyl fermions) in the representation *R*

$$\delta_{\nu n}^{(3)}(\tilde{N}_f,\tilde{N}_s) = \delta_{\nu n}^{(3,1)}(\tilde{N}_f,\tilde{N}_s) + \delta_{\nu n}^{(3,2)}(\tilde{N}_f,\tilde{N}_s)$$

with $\tilde{N}_x = \frac{1}{2}\sum_R n_x^R\, T_R\,(2C_R - N_c)\,, \qquad x = f, s$

- Necessary and sufficient conditions for a SU(N$_c$) gauge theory to have a BFKL ladder of maximal weight are:

— the one-loop beta function must vanish
— the two-loop cusp AD must be proportional to ζ_2
— $\delta_{\nu n}^{(3,2)}$ must vanish $\rightarrow$ $2\tilde{N}_f = N_c^2 + \tilde{N}_s$

➡ There is no theory whose BFKL ladder has uniform maximal weight which agrees with the maximal weight terms of QCD

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Matter in the fundamental and in the adjoint

- We solve the conditions above for matter in the fundamental *F* and in the adjoint *A* representations. We obtain:

$$2\,n_f^F = n_s^F \qquad\qquad 2\,n_f^A = 2 + n_s^A$$

which describes the spectrum of a gauge theory with *N* supersymmetries and $n^F = n_f^F$ chiral multiplets in *F* and $n^A = n_f^A - N$ chiral multiplets in *A*

- There are four solutions to those conditions

$\mathcal{N}$	4	2	1	1
n_A	0	0	0	2
n_F	0	$4N_c$	$6N_c$	$2N_c$

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— the first is *N*=4 SYM
— the second is N=2 superconformal QCD with $N_f = 2N_c$ hypermultiplets
— the third is *N*=1 superconf. QCD

- because the one-loop beta function is fixed by matter loops in gluon self-energies, we are only sensitive to the matter content of a theory, and not to its details (like scalar potential or Yukawa couplings)

Lipatov large N_c picture

- In MRK, amplitudes of QCD in the large N_c limit and amplitudes of planar N=4 SYM are described by similar (BFKL-like) Hamiltonians, corresponding to the *t*-channel exchange of *n* Reggeons

Lipatov 1993 - 2009

$$H = h + h^* \qquad h = \sum_{i=1}^{n} h_{i,i+1} \qquad h_{12} = \ln(p_1 p_2) + \frac{1}{p_1}\ln(\rho_{12})p_1 + \frac{1}{p_2}\ln(\rho_{12})p_2 - 2\psi(1)$$

$$\rho_{12} = \rho_1 - \rho_2 \qquad \rho_k = x_k + i y_k \qquad p_k = i\frac{\partial}{\partial \rho_k}$$

- those Hamiltonians coincide with the Hamiltonian of an integrable Heisenberg spin chain

Lipatov 1994
Faddeev Korchemsky 1995

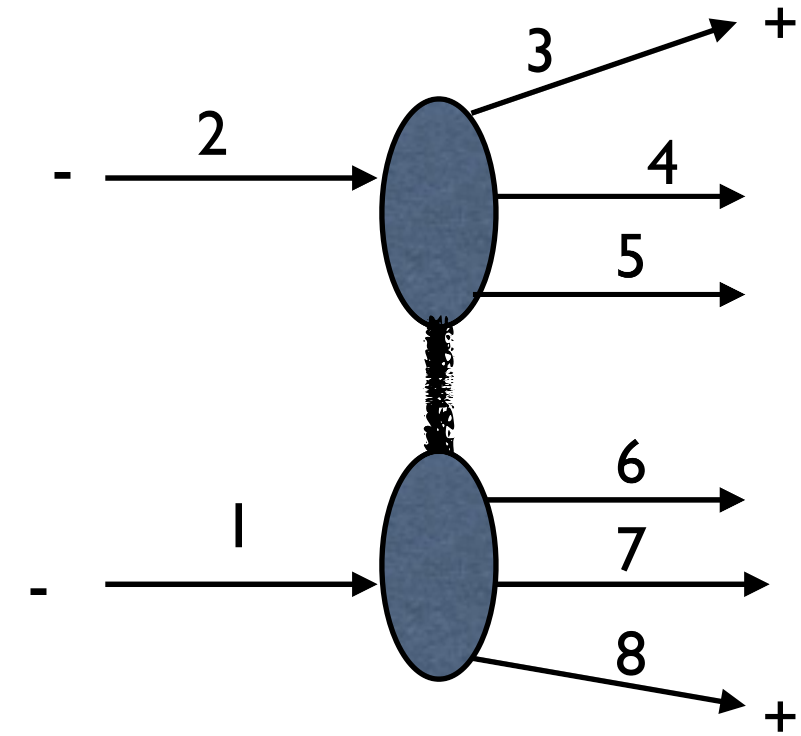
- the Hamiltonians differ only by the boundary conditions, which one chooses for the *t*-channel exchange of an adjoint (→ open spin chain) in planar N=4 SYM, or of a singlet (→ closed spin chain) in large N_c QCD

  singlet $\quad h_{n,1} \to \ln\frac{p_1 p_n}{q^2} \quad$ adjoint

- the simplest case is the *t*-channel exchange of two Reggeons (→ two links on the spin chain), which corresponds to the BFKL equation in QCD and to the 6-pt amplitude in planar N=4 SYM

Double discontinuity of the amplitude in MRK

- in planar *N*=4 SYM, 3-Reggeon exchange starts occurring with the 8-pt amplitude. We need take the double discontinuity



8-pt amplitude

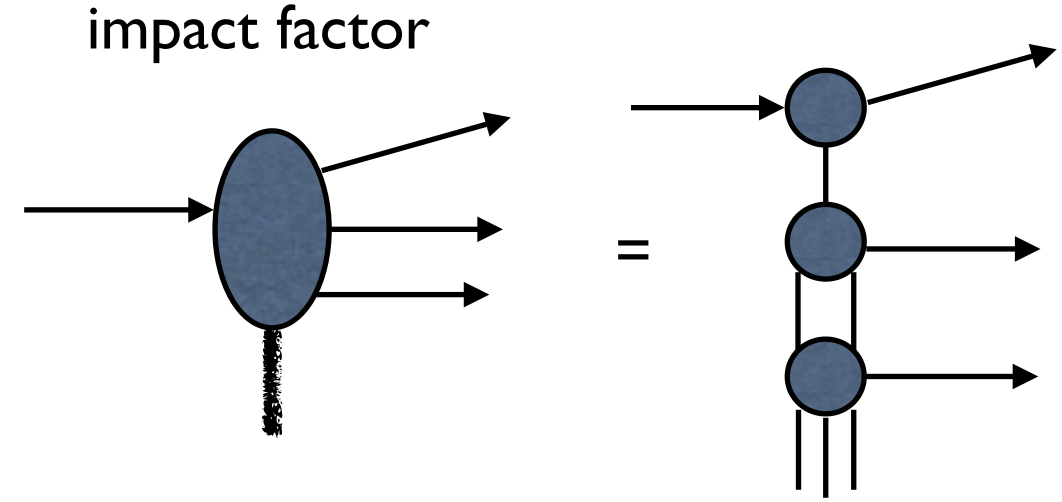
continue to a Minkowski region

$s, s_{4567}, s_{56} > 0$

$s_{34}, s_{45}, s_{67}, s_{78} < 0$

- we examined the double discontinuity of two-loop amplitudes, and found that it is determined to any number of points by building blocks which appear through 9 points.

impact factor



Caron-Huot VDD Duhr Dulat Penante, in preparation

This is consistent with the picture above: the building blocks of double Disc are impact factors and 3-Reggeon exchange. Beyond 8 points, the only additional building block is the central-emission vertex, occurring at 9 points

Conclusions

- In QCD, amplitudes in the Regge limit features one-Reggeon exchange through NLL accuracy (for the real part, and 2-Reggeon exchange for the imaginary part)

  3-Reggeon exchange appears in N_c-subleading pieces at NNLL accuracy

  Although we are far from having a BFKL ladder,
  we understand the NNLL context in which it would arise

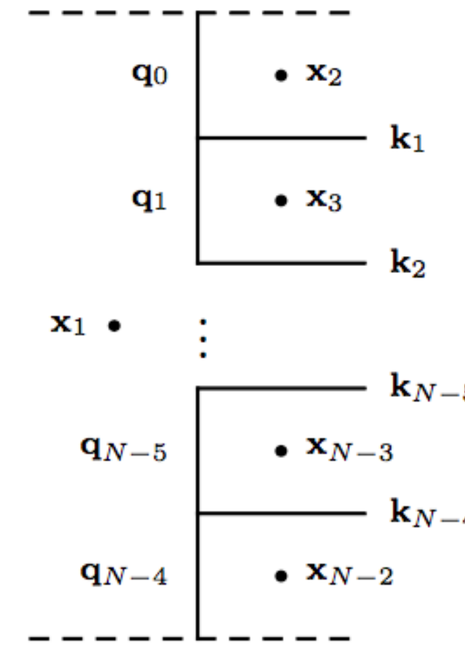
- In analogy to planar N=4 SYM, the functions which characterise the BFKL ladder in QCD are single-valued functions, specifically (generalised) SVMPLs

- In planar N=4 SYM, 2-Reggeon exchange is understood, even at finite coupling (where we just miss the central-emission vertex). At weak coupling, we know amplitudes at LL and NLL accuracy, in terms of SVMPLs

- We have just begun exploring 3-Reggeon exchange

Back-up slides

Factorisation in MRK at LL accuracy



- Factorisation in MRK at LL accuracy implies that the building blocks are: the impact factors, the 2-Reggeon exchange, and the central-emission vertex

- For the helicities $h_1, \ldots, h_{N-4}$ define the ratio

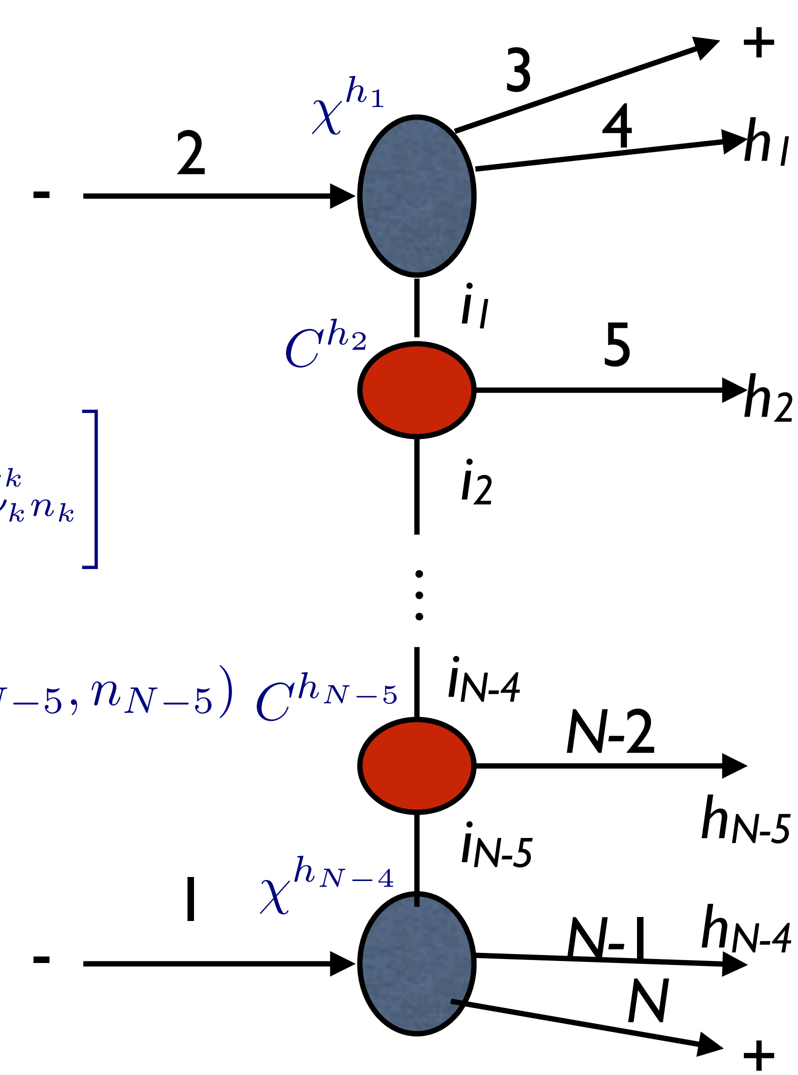
$$\mathcal{R}_{h_1,\ldots,h_{N-4}} = \left[\frac{A_N(-,+,h_1,\ldots,h_{N-4},+,-)}{A_N^{\mathrm{BDS}}(-,+,\ldots,+,-)} \right]_{|\mathrm{MRK,\ LLA}}$$

- factorisation in MRK at LL accuracy

$$\mathcal{R}_{h_1,\ldots,h_{N-4}}(\tau_1, z_1, \ldots, \tau_{N-5}, z_{N-5})$$
$$\approx 2\pi i \sum_{i=2}^{\infty} \sum_{i_1+\ldots+i_{N-5}=i-1} a^i \left(\prod_{k=1}^{N-5} \frac{1}{i_k!} \ln^{i_k} \tau_k \right) g_{h_1,\ldots,h_{N-4}}^{(i_1,\ldots,i_{N-5})}(z_1,\ldots,z_{N-5})$$

with τ_k = function of cross ratios, and with coefficients

$$g_{h_1,\ldots,h_{N-4}}^{(i_1,\ldots,i_{N-5})}(z_1,\ldots,z_{N-5}) = \frac{(-1)^{N+1}}{2} \left[\prod_{k=1}^{N-5} \sum_{n_k=-\infty}^{+\infty} \left(\frac{z_k}{\bar{z}_k} \right)^{n_k/2} \int_{-\infty}^{+\infty} \frac{d\nu_k}{2\pi} |z_k|^{2i\nu_k} E_{\nu_k n_k}^{i_k} \right]$$
$$\times \chi^{h_1}(\nu_1, n_1) \left[\prod_{j=2}^{N-5} C^{h_j}(\nu_{j-1}, n_{j-1}, \nu_j, n_j) \right] \chi^{-h_{N-4}}(\nu_{N-5}, n_{N-5})$$



where:
the χ's are the 2 impact factors,
the C's are the *N*-6 central-emission vertices
the E's are the *N*-5 BFKL-like eigenvalues for octet exchange

Convolutions

- we use the Fouries-Mellin (FM) transform

$$\mathcal{F}[F(\nu,n)] = \sum_{n=-\infty}^{\infty} \left(\frac{z}{\bar{z}}\right)^{n/2} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} \, |z|^{2i\nu} \, F(\nu,n)$$

which maps products into convolutions

$$\mathcal{F}[F \cdot G] = \mathcal{F}[F] * \mathcal{F}[G] = (f * g)(z) = \frac{1}{\pi} \int \frac{d^2w}{|w|^2} \, f(w) \, g\left(\frac{z}{w}\right)$$

- we compute the integral through the residue formula

$$\int \frac{d^2z}{\pi} \, f(z) = \mathrm{Res}_{z=\infty} F(z) - \sum_i \mathrm{Res}_{z=a_i} F(z)$$

Schnetz 2013

where F is the antiholomorphic primitive of f $\quad \bar{\partial}_z F = f$

Convolutions and factorization

- through the FM transform of the BFKL eigenvalue

  $\mathcal{E}(z) = \mathcal{F}\left[E_{\nu n}\right]$

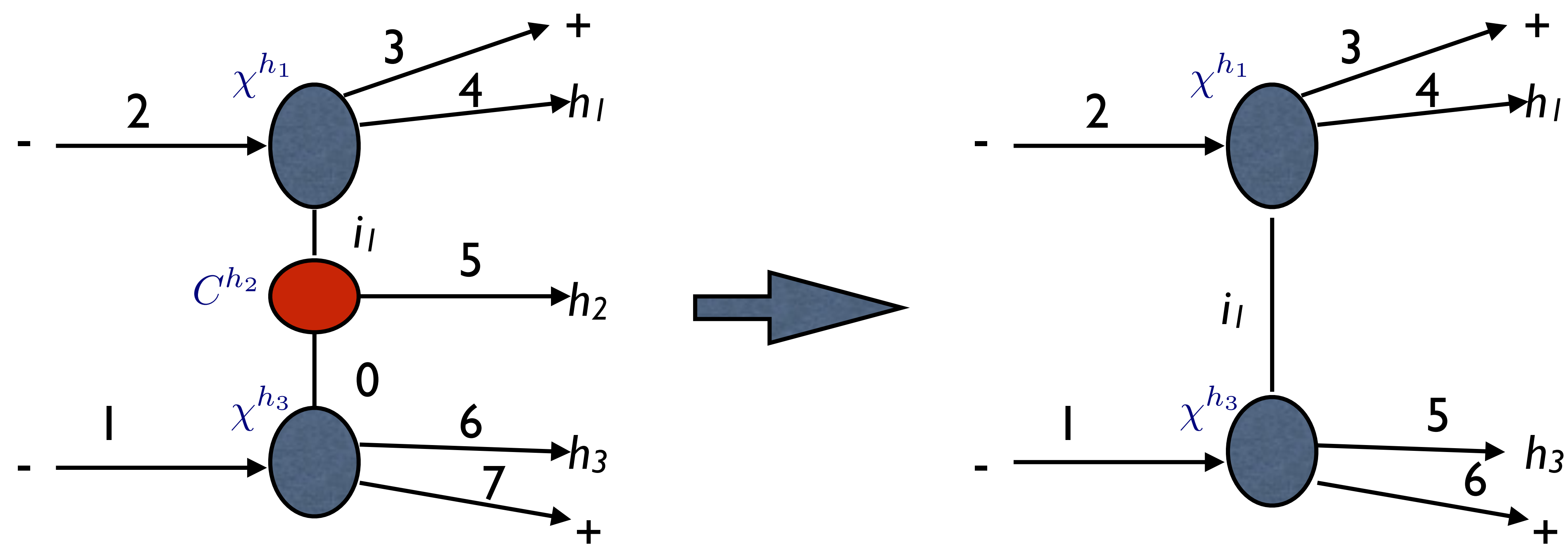
  we can write the recursion

  $$g_{+\ldots+}^{(i_1,\ldots,i_k+1,\ldots,i_{N-5})}(z_1,\ldots,z_{N-5}) = \mathcal{E}(z_k) * g_{+\ldots+}^{(i_1,\ldots,i_{N-5})}(z_1,\ldots,z_{N-5})$$

  which implies that we can drop all the propagators without a log

  $$g_{+\ldots+}^{(0,\ldots,0,i_{a_1},0,\ldots,0,i_{a_2},0,\ldots,0,i_{a_k},0,\ldots,0)}(\rho_1,\ldots,\rho_{N-5}) = g_{+\ldots+}^{(i_{a_1},i_{a_2},\ldots,i_{a_k})}(\rho_{i_{a_1}},\rho_{i_{a_2}},\ldots,\rho_{i_{a_k}})$$

example for N=7, with $h_l = h_2$



which connects amplitudes with a different number of legs in fact, if all indices are zero except for one

$$g_{+\ldots+}^{(0,\ldots,0,i_a,0,\ldots,0)}(\rho_1,\ldots,\rho_{N-5}) = g_{++}^{(i_a)}(\rho_a)$$

- which implies that

$$\mathcal{R}_{+\ldots+}^{(2)} = \sum_{1\le i\le N-5} \ln\tau_i\, g_{++}^{(1)}(\rho_i)$$

with

$$g_{++}^{(1)}(\rho_1) = -\frac{1}{4}\mathcal{G}_{0,1}(\rho_1) - \frac{1}{4}\mathcal{G}_{1,0}(\rho_1) + \frac{1}{2}\mathcal{G}_{1,1}(\rho_1)$$

which shows, as previously stated, that in MRK at LLA, the 2-loop *n*-pt remainder function $R_n^{(2)}$ can be written as a sum of 2-loop 6-pt amplitudes, in terms of SVHPLs

- At 3 loops, the n-pt remainder function $R_n^{(3)}$ can be written as a sum of 3-loop 6-pt and 7-pt amplitudes

$$\mathcal{R}^{(3)}_{+\ldots+} = \frac{1}{2} \sum_{1\leq i\leq N-5} \ln^2 \tau_i \, g^{(2)}_{++}(\rho_i) + \sum_{1\leq i<j\leq N-5} \ln \tau_i \, \ln \tau_j \, g^{(1,1)}_{+++}(\rho_i, \rho_j)$$

with

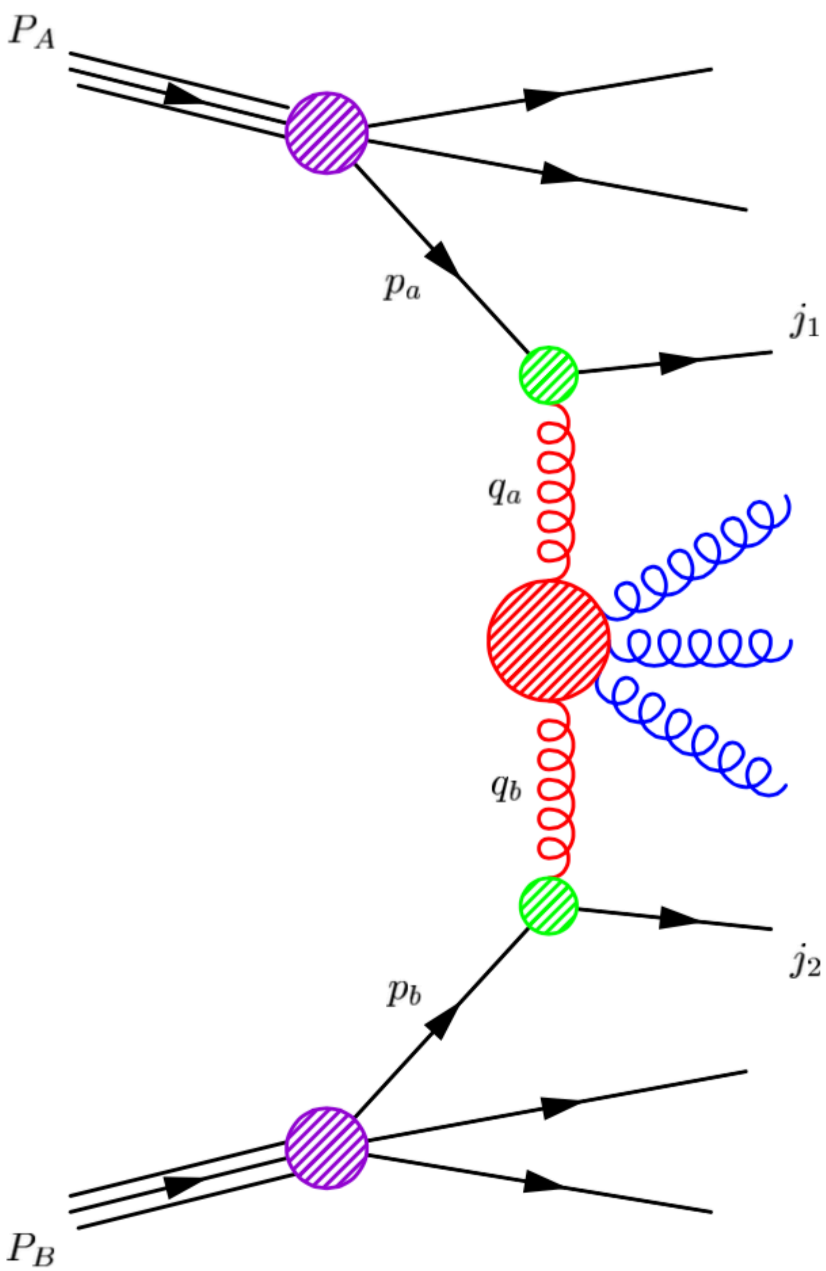
$$\begin{aligned} g^{(2)}_{++}(\rho_1) = & -\frac{1}{8}\mathcal{G}_{0,0,1}(\rho_1) - \frac{1}{4}\mathcal{G}_{0,1,0}(\rho_1) + \frac{1}{2}\mathcal{G}_{0,1,1}(\rho_1) - \frac{1}{8}\mathcal{G}_{1,0,0}(\rho_1) \\ & + \frac{1}{2}\mathcal{G}_{1,0,1}(\rho_1) + \frac{1}{2}\mathcal{G}_{1,1,0}(\rho_1) - \mathcal{G}_{1,1,1}(\rho_1) \end{aligned}$$

$$\begin{aligned} g^{(1,1)}_{+++}(\rho_1, \rho_2) = & -\frac{1}{8}\mathcal{G}_{0,1,\rho_2}(\rho_1) - \frac{1}{8}\mathcal{G}_{0,\rho_2,1}(\rho_1) + \frac{1}{8}\mathcal{G}_{1,1,\rho_2}(\rho_1) - \frac{1}{8}\mathcal{G}_{1,\rho_2,0}(\rho_1) \\ & - \frac{1}{8}\mathcal{G}_{\rho_2,1,0}(\rho_1) + \frac{1}{8}\mathcal{G}_{\rho_2,1,1}(\rho_1) + \frac{1}{4}\mathcal{G}_{1,\rho_2,1}(\rho_1) - \frac{1}{4}\mathcal{G}_{1}(\rho_2)\mathcal{G}_{1,\rho_2}(\rho_1) \\ & + \frac{1}{8}\mathcal{G}_{1}(\rho_1)\mathcal{G}_{0,0}(\rho_2) - \frac{1}{8}\mathcal{G}_{0}(\rho_2)\mathcal{G}_{0,1}(\rho_1) + \frac{1}{8}\mathcal{G}_{1}(\rho_2)\mathcal{G}_{0,1}(\rho_1) - \frac{1}{8}\mathcal{G}_{\rho_2}(\rho_1)\mathcal{G}_{0,1}(\rho_2) \\ & + \frac{1}{8}\mathcal{G}_{1}(\rho_2)\mathcal{G}_{0,\rho_2}(\rho_1) - \frac{1}{8}\mathcal{G}_{0}(\rho_2)\mathcal{G}_{1,0}(\rho_1) + \frac{1}{8}\mathcal{G}_{1}(\rho_2)\mathcal{G}_{1,0}(\rho_1) + \frac{1}{8}\mathcal{G}_{0}(\rho_2)\mathcal{G}_{1,1}(\rho_1) \\ & - \frac{1}{8}\mathcal{G}_{1}(\rho_2)\mathcal{G}_{1,1}(\rho_1) - \frac{1}{8}\mathcal{G}_{1}(\rho_1)\mathcal{G}_{1,1}(\rho_2) + \frac{1}{8}\mathcal{G}_{\rho_2}(\rho_1)\mathcal{G}_{1,1}(\rho_2) + \frac{1}{8}\mathcal{G}_{0}(\rho_2)\mathcal{G}_{1,\rho_2}(\rho_1) \\ & + \frac{1}{8}\mathcal{G}_{1}(\rho_2)\mathcal{G}_{\rho_2,0}(\rho_1) - \frac{1}{8}\mathcal{G}_{1}(\rho_2)\mathcal{G}_{\rho_2,1}(\rho_1) \end{aligned}$$

Note that $R_n^{(3)}$ cannot be written only in terms of SVHPLs, but SVMPLs are necessary

Mueller-Navelet jets

Mueller Navelet 1987



Dijet production cross section with two tagging jets in the forward and backward directions

$p_a = x_a P_A \quad p_b = x_b P_B$ incoming parton momenta

S: hadron centre-of-mass energy

$s = x_a x_b S$: parton centre-of-mass energy

E_{Tj}: jet transverse energies

$$\Delta y = |y_{j_1} - y_{j_2}| \simeq \log \frac{s}{E_{Tj_1} E_{Tj_2}}$$

is the rapidity interval between the tagging jets

gluon radiation is considered in MRK and resummed through the LL BFKL equation

Mueller-Navelet dijet cross section

- the cross section for dijet production at large rapidity intervals $\Delta y = y_1 - y_2 = \ln\left(\frac{\hat{s}}{-t}\right) \gg 1$

  with $\hat{s} = x_a x_b S\,, \quad t = -\sqrt{p_{1\perp}^2 p_{2\perp}^2}$

$$\frac{d\hat{\sigma}_{gg}}{dp_{1\perp}^2 dp_{2\perp}^2 d\phi_{jj}} = \frac{\pi}{2}\left[\frac{C_A\alpha_s}{p_{1\perp}^2}\right] f(\vec{q}_{1\perp}, \vec{q}_{2\perp}, \Delta y)\left[\frac{C_A\alpha_s}{p_{2\perp}^2}\right]$$

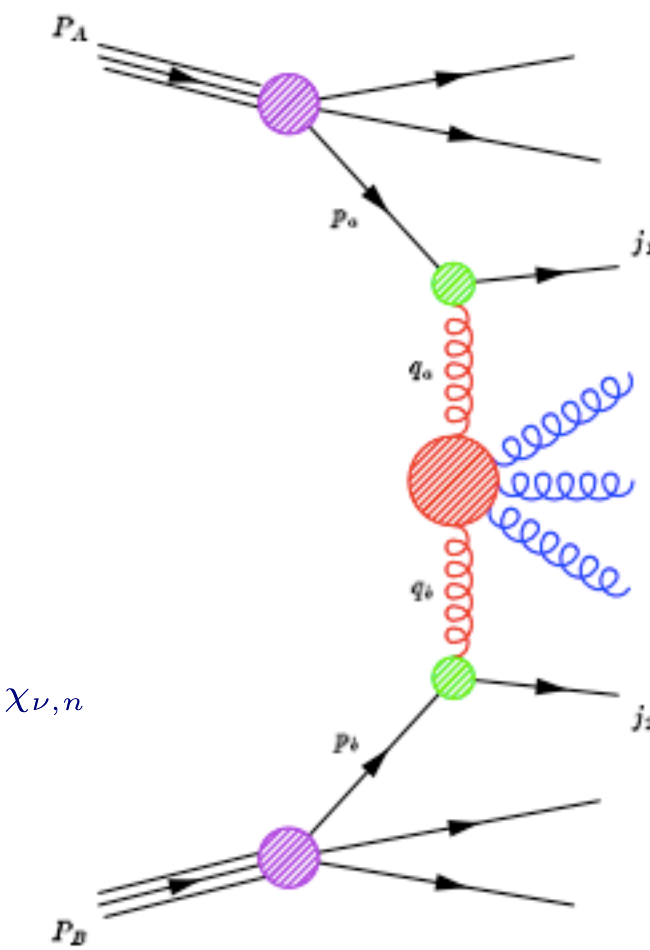
can be described through the BFKL Green's function

$$f(\vec{q}_{1\perp}, \vec{q}_{2\perp}, \Delta y) = \frac{1}{(2\pi)^2\sqrt{q_{1\perp}^2\, q_{2\perp}^2}}\sum_{n=-\infty}^{+\infty} e^{in\phi}\int_{-\infty}^{+\infty} d\nu\left(\frac{q_{1\perp}^2}{q_{2\perp}^2}\right)^{i\nu} e^{\eta\,\chi_{\nu,n}}$$

with $\eta \equiv \frac{C_A\,\alpha_s}{\pi}\Delta y$ and ϕ the angle between q_1^2 and q_2^2

and the LL BFKL eigenvalue

$$\chi_{\nu,n} = -2\gamma_E - \psi\left(\frac{1}{2} + \frac{|n|}{2} + i\nu\right) - \psi\left(\frac{1}{2} + \frac{|n|}{2} - i\nu\right)$$



Mueller-Navelet dijet cross section

- azimuthal angle distribution ($\phi_{jj} = \phi - \pi$)

$$\frac{d\hat{\sigma}_{gg}}{d\phi_{jj}} = \frac{\pi(C_A\alpha_s)^2}{2E_\perp^2}\left[\delta(\phi_{jj}-\pi) + \sum_{k=1}^{\infty}\left(\sum_{n=-\infty}^{\infty}\frac{e^{in\phi}}{2\pi}f_{n,k}\right)\eta^k\right]$$

with $$f_{n,k} = \frac{1}{2\pi}\frac{1}{k!}\int_{-\infty}^{\infty} d\nu\, \frac{\chi_{\nu,n}^k}{\nu^2+\frac{1}{4}}$$

- the dijet cross section is $$\hat{\sigma}_{gg} = \frac{\pi(C_A\alpha_s)^2}{2E_\perp^2}\sum_{k=0}^{\infty} f_{0,k}\,\eta^k$$ Mueller Navelet 1987

with
$$\begin{aligned}
f_{0,0} &= 1\,,\\
f_{0,1} &= 0\,,\\
f_{0,2} &= 2\zeta_2\,,\\
f_{0,3} &= -3\zeta_3\,,\\
f_{0,4} &= \frac{53}{6}\zeta_4\,,\\
f_{0,5} &= -\frac{1}{12}(115\zeta_5 + 48\zeta_2\zeta_3)
\end{aligned}$$

Mueller-Navelet evaluated the inclusive dijet cross section up to 5 loops

BFKL Green's function and single-valued functions

- use complex transverse momentum $\tilde{q}_k \equiv q_k^x + i q_k^y$

  and a complex variable $z \equiv \frac{\tilde{q}_1}{\tilde{q}_2}$

  the Green's function can be expanded into a power series in $\eta_\mu = \overline{\alpha}_\mu\, y$

$$f^{LL}(q_1, q_2, \eta_\mu) = \frac{1}{2}\delta^{(2)}(q_1 - q_2) + \frac{1}{2\pi\,\sqrt{q_1^2\, q_2^2}} \sum_{k=1}^{\infty} \frac{\eta_\mu^k}{k!}\, f_k^{LL}(z)$$

  where the coefficient functions f_k are given by the Fourier-Mellin transform

$$f_k^{LL}(z) = \mathcal{F}\left[\chi_{\nu n}^k\right] = \sum_{n=-\infty}^{+\infty} \left(\frac{z}{\bar{z}}\right)^{n/2} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi}\, |z|^{2i\nu}\, \chi_{\nu n}^k$$

  - the f_k have a unique, well-defined value for every ratio of the magnitudes of the two jet transverse momenta and angle between them.
    So, they are real-analytic functions of w

Mueller-Navelet dijet cross section reloaded

- the MN dijet cross section is $$\hat{\sigma}_{gg} = \frac{\pi (C_A \alpha_s)^2}{2E_\perp^2} \sum_{k=0}^{\infty} f_{0,k}\, \eta^k$$

the first 5 loops were computed by Mueller-Navelet.
We computed it through the 13 loops

VDD Dixon Duhr Pennington 2013

$$
\begin{aligned}
f_{0,6} &= \frac{13}{4}\zeta_3^2 + \frac{3737}{120}\zeta_6\,,\\
f_{0,7} &= -\frac{87}{5}\zeta_3\zeta_4 - \frac{116}{9}\zeta_2\zeta_5 - \frac{3983}{144}\zeta_7\,,\\
f_{0,8} &= -\frac{37}{75}\zeta_{5,3} + \frac{64}{15}\zeta_2\zeta_3^2 + \frac{369}{20}\zeta_5\zeta_3 + \frac{50606057}{453600}\zeta_8\,,\\
f_{0,9} &= -\frac{139}{60}\zeta_3^3 - \frac{15517}{252}\zeta_6\zeta_3 - \frac{3533}{63}\zeta_4\zeta_5 - \frac{557}{15}\zeta_2\zeta_7 - \frac{5215361}{60480}\zeta_9\,,\\
f_{0,10} &= -\frac{2488}{4725}\zeta_{5,3}\zeta_2 - \frac{94721}{211680}\zeta_{7,3} + \frac{1948}{105}\zeta_4\zeta_3^2 + \frac{2608}{105}\zeta_2\zeta_5\zeta_3 + \frac{12099}{224}\zeta_7\zeta_3 + \frac{1335931}{47040}\zeta_5^2 + \frac{25669936301}{63504000}\zeta_{10}\\
f_{0,11} &= \frac{62}{315}\zeta_{5,3}\zeta_3 + \frac{83}{120}\zeta_{5,3,3} - \frac{2872}{945}\zeta_2\zeta_3^3 - \frac{13211}{672}\zeta_5\zeta_3^2 - \frac{661411}{3024}\zeta_8\zeta_3\\
&\quad - \frac{242776937}{725760}\zeta_{11} - \frac{605321}{3024}\zeta_5\zeta_6 - \frac{2583643}{16200}\zeta_4\zeta_7 - \frac{28702763}{340200}\zeta_2\zeta_9\,,\\
f_{0,12} &= \frac{74711}{162000}\zeta_{5,3}\zeta_4 - \frac{13793}{7560}\zeta_{6,4,1,1} + \frac{3965011}{793800}\zeta_{7,3}\zeta_2 - \frac{33356851}{4082400}\zeta_{9,3}\\
&\quad + \frac{252163}{181440}\zeta_3^4 + \frac{620477}{10080}\zeta_6\zeta_3^2 + \frac{8101339}{75600}\zeta_4\zeta_5\zeta_3 + \frac{342869}{3780}\zeta_2\zeta_7\zeta_3\\
&\quad + \frac{101571047}{680400}\zeta_9\zeta_3 + \frac{71425871}{1587600}\zeta_2\zeta_5^2 + \frac{904497401571619}{620606448000}\zeta_{12} + \frac{484414571}{2721600}\zeta_5\zeta_7\,,\\
f_{0,13} &= \frac{4513}{1890}\zeta_{5,3}\zeta_5 + \frac{27248}{23625}\zeta_{5,3,3}\zeta_2 - \frac{97003}{235200}\zeta_{5,5,3} + \frac{13411}{75600}\zeta_{7,3}\zeta_3\\
&\quad + \frac{7997743}{12700800}\zeta_{7,3,3} - \frac{187318}{14175}\zeta_4\zeta_3^3 - \frac{125056}{4725}\zeta_2\zeta_5\zeta_3^2 - \frac{17411413}{302400}\zeta_7\zeta_3^2\\
&\quad - \frac{5724191}{100800}\zeta_5^2\zeta_3 - \frac{1874972477}{2376000}\zeta_{10}\zeta_3 - \frac{2418071698069}{2235340800}\zeta_{13}\\
&\quad - \frac{2379684877}{6048000}\zeta_{11}\zeta_2 - \frac{297666465053}{523908000}\zeta_6\zeta_7 - \frac{1770762319}{2494800}\zeta_5\zeta_8 - \frac{229717224973}{628689600}\zeta_4\zeta_9
\end{aligned}
$$