Simplicity in AdS Perturbative Dynamics

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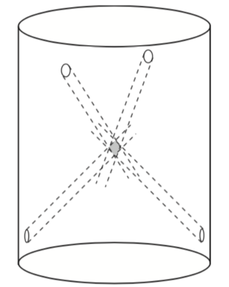
Why AdS?

Simplest curved space where a scattering problem can be well-defined.

$$\text{CFT} \xleftrightarrow{\text{holography}} \textbf{AdS} \xrightarrow{R_{\mathrm{AdS}} \to \infty} \text{Minkowski}$$



[fig: Heemskerk et al, '09]

Perturbative computation organized by Witten diagrams.

Very recently: ideas from conformal bootstrap [see Agnese' and James' talks]

Aim of the talk

- Efficient and systematic computational methods.
- Detailed understanding of the analytic structure.

Scalar EFTs in pure AdS

$$\sum_{p=3}^{\infty} \frac{g_p}{p!} \phi^p$$

but we allow arbitrary species of particles with arbitrary masses, i.e., arbitrary scalar diagrams.

Review

Preliminary

Mellin amplitude [Mack,'09]

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_{\text{Mellin}} [\mathrm{d}\delta]\, \mathcal{M}[\delta] \prod_{i<j} \frac{\Gamma[\delta_{i\,j}]}{(x_i - x_j)^{2\delta_{i\,j}}}.$$

- $[\mathrm{d}\delta] \equiv \prod \frac{\mathrm{d}\delta_{i\,j}}{2\pi i}$ for any collection of independent $\delta_{i\,j}$'s ($\frac{n(n-3)}{2}$).

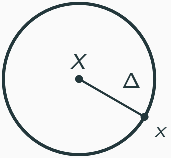
$$\delta_{i\,j} = \delta_{j\,i}, \quad \delta_{i\,i} = -\Delta_i, \quad \sum_{j=1}^{n} \delta_{i\,j} = 0.$$

- δ analogous to $k \cdot k \implies$ "Mandelstam variables"

$$\text{e.g.,} \quad s_A = \sum_{i \in A} \Delta_i - 2 \sum_{i<j \in A} \delta_{i\,j}.$$

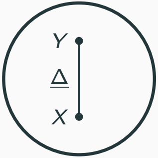
Propagators

- **Bulk-to-boundary propagator** $(m^2 = \Delta(\Delta - d))$

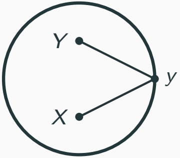


$$G^{\Delta}_{\mathrm{b}\partial}[X, x] = \frac{\mathcal{C}_{\Delta}}{(-2X \cdot x)^{\Delta}}$$

- **Bulk-to-bulk propagator** $([\mathrm{d}c]_{\underline{\Delta}} = \mathrm{d}c/((\underline{\Delta} - h)^2 - c^2),\ h = d/2)$



$$= \int_{\text{Mellin}} [\mathrm{d}c]_{\underline{\Delta}} \frac{N_c}{\mathcal{C}_{h\pm c}} \int_{\partial\text{AdS}} \mathrm{d}y$$



$$G^{\underline{\Delta}}_{\mathrm{bb}}[X, Y] \qquad G^{h+c}_{\mathrm{b}\partial}[X, y]\, G^{h-c}_{\mathrm{b}\partial}[Y, y]$$

split representation [Penedones,'10].

Mellin pre-amplitudes

$$\underbrace{\mathcal{M}[\delta, \{\Delta, \underline{\Delta}\}]}_{\text{amplitude}} = \int \mathcal{N} \underbrace{M[\delta, \{\Delta, c\}]}_{\text{pre-amplitude}},$$

$$\mathcal{N} = \frac{\pi^{(1-L)h}}{2^{2V+L-1}} \prod_{i=1}^{n} \frac{\mathcal{C}_{\Delta_i}}{\Gamma[\Delta_i]} \prod_a \frac{[\mathrm{d}c_a]_{\underline{\Delta}_a}}{\Gamma[\pm c_a]}.$$

Contact diagram

$$M_{\text{contact}} = \Gamma[\tfrac{\sum_i \Delta_i}{2} - h].$$

Tree Level

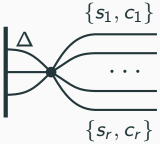
Pre-amplitudes at tree level

M_{tree} **factorizes**: [EYY,'18]

- Each bulk-to-bulk (tree) propagator a

$$\Gamma\left[\tfrac{h\pm c_a - s_a}{2}\right] \equiv \Gamma\left[\tfrac{h+c_a-s_a}{2}\right]\Gamma\left[\tfrac{h-c_a-s_a}{2}\right]$$

- Each bulk vertex A ($\Delta_A \equiv \sum \Delta_i$)



$$\int \frac{\prod_{a=1}^{r} \mathrm{d}w_a}{(2\pi i)^r}\, \Gamma\left[\tfrac{\Delta_A+(r-2)h+\sum_{a=1}^{r}(c_a+2w_a)}{2}\right] \times \prod_{a=1}^{r} \frac{\Gamma[-w_a]\,\Gamma[-c_a-w_a]\,\Gamma\left[\tfrac{h+c_a-s_a}{2}+w_a\right]}{\Gamma\left[\tfrac{h\pm c_a-s_a}{2}\right]}$$

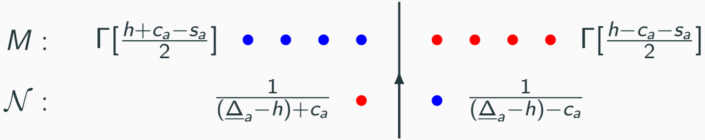
Denote this as (C is free of poles)

$$V\left[\Delta_A; {}^{c_1,\dots,c_r}_{s_1,\dots,s_r}\right] \equiv \Gamma\left[\tfrac{\Delta_A+(r-2)h\pm c_1\pm\cdots\pm c_r}{2}\right] C\left[\Delta_A; {}^{c_1,\dots,c_r}_{s_1,\dots,s_r}\right]$$

(c.f., [Paulos,'11], [Fitzpatrick,Kaplan,'11], [Nandan et al,'11])

Amplitudes at tree level

Mellin contour: $\int_{-i\infty}^{+i\infty}$, right to all left poles, left to all right poles.



$$\Gamma[\tfrac{h\pm c_a-s_a}{2}]\,\frac{1}{(\underline{\Delta}_a-h)\mp c_a}\cdots \xrightarrow{\int \mathrm{d}c_a} \underbrace{\gamma[\tfrac{\underline{\Delta}_a-s_a}{2}]}_{\text{poles of }\mathcal{M}}\cdots .$$

($\gamma[x]$: a family of poles at $x+m=0$, $m\in\mathbb{N}$.)

To study the residue at, e.g., the leading pole, replace $\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}c_a}{2\pi i}$ to

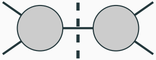
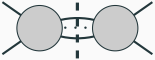
$$\operatorname*{Res}_{s_a=\underline{\Delta}_a}\ \operatorname*{Res}_{c_a=h-\underline{\Delta}_a} \quad \text{or} \quad -\operatorname*{Res}_{s_a=\underline{\Delta}_a}\ \operatorname*{Res}_{c_a=h-s_a}$$

plus a similar contour for the other pinching.

AdS vs Minkowski

$$\text{tree propagagtor } a \Longleftrightarrow \Gamma[\tfrac{\Delta_a - s_a}{2}] \Longleftrightarrow \mathcal{O}_{\underline{a}} + \text{descendents}$$

Many intuitions from Minkowski are expected to carry over

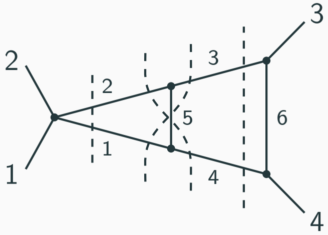
		
Minkowski	pole	branch cut
AdS	pole family	

Mellin amplitudes are expected to be **meromorphic** at all loops.

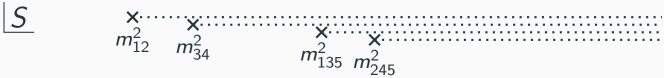
A better chance for a precise understanding of loop-level dynamics?

Some diagrammatic intuitions

Cut: diagram $\longrightarrow$ two *connected* diagrams.



Scattering amplitude (e.g., $m_{12} \equiv m_1 + m_2$):



[Educated guess] Mellin amplitude (e.g., $\underline{\Delta}_{12} \equiv \underline{\Delta}_1 + \underline{\Delta}_2$):

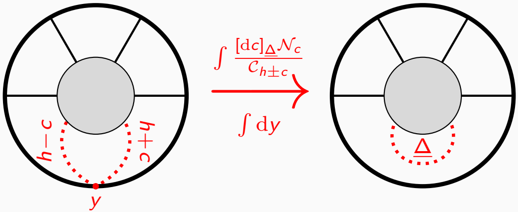
$$\Gamma\left[\tfrac{\underline{\Delta}_{12}-S}{2}\right]\Gamma\left[\tfrac{\underline{\Delta}_{34}-S}{2}\right]\Gamma\left[\tfrac{\underline{\Delta}_{135}-S}{2}\right]\Gamma\left[\tfrac{\underline{\Delta}_{245}-S}{2}\right].$$

Construction

Basic idea & strategy

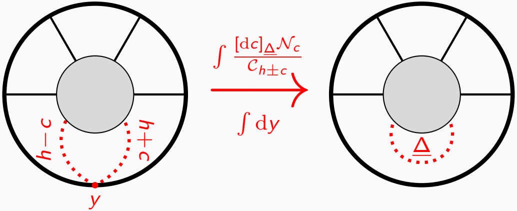
- In general, $M_{\text{loop}} \Longrightarrow \mathcal{M}_{\text{loop}}$.
  M is much simpler, but still keeps a lot of analytic features.
- **Recursion.**
  Utilize simpler diagrams in the computation of more complicated diagrams, in particular, from lower loops to higher loops.
- **Mellin space.**
  Carry out the computation fully in Mellin space.
  Advantage: unify with the spectrum integrals later on.
- **Meromophicity.**
  The (pre-)amplitude is effectively tree-like at any loop level.

Creating a new loop



- Assume the complete knowledge about the original diagram.
- Assign $\Delta_0 = h - c$, $\Delta_{n+1} = h + c$; identify $x_0 \equiv x_{n+1} \equiv y$.
- Integrate over the boundary $\int_{\partial \mathrm{AdS}} \mathrm{d}y$.
- For the amplitude, further integrate the spectrum variable c.

Creating a new loop



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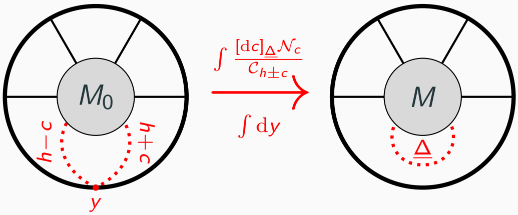
- Integrate over the boundary $\int_{\partial\mathrm{AdS}} \mathrm{d}y$.
- For the amplitude, further integrate the spectrum variable c.

Implement this fully in Mellin space.

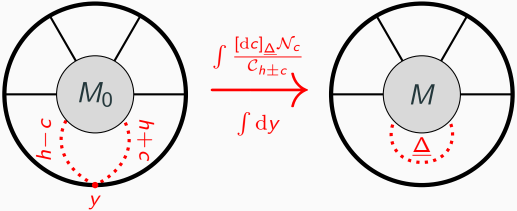
Creating a new loop



$$M[s] = \int_{\text{Mellin}} [\mathrm{d}\Xi]\, M_0[\Xi]\, K'[\Xi; s].$$

- s: Mandelstam variables for the new space.
- Ξ: Mandelstam variables in the original space.

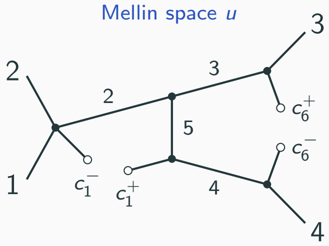
Creating a new loop



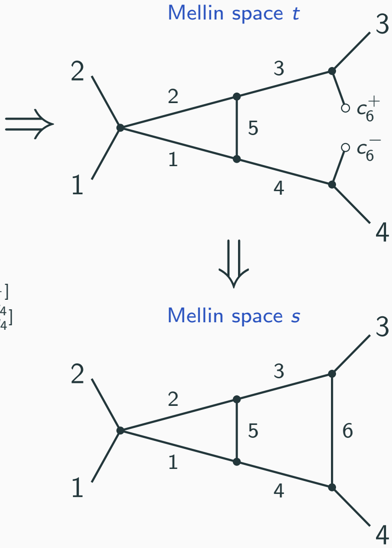
$$M[s] = \int_{\text{Mellin}} [\mathrm{d}\Xi\mathrm{d}\xi]\, M_0[\Xi]\, K[\Xi, \xi; s].$$

- s: Mandelstam variables for the new space.
- Ξ: Mandelstam variables in the original space.
- ξ: extra Mandelstam variables in the original space.

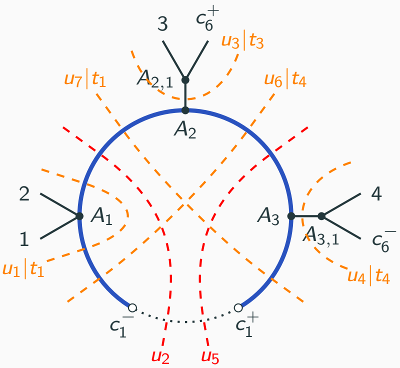
Recursive construction

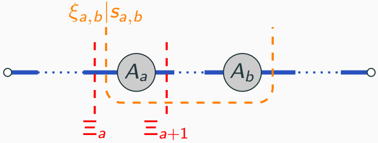


$$\Gamma[\tfrac{h\pm c_2-u_2}{2}]\Gamma[\tfrac{h\pm c_3-u_3}{2}]\Gamma[\tfrac{h\pm c_4-u_4}{2}]\Gamma[\tfrac{h\pm c_5-u_5}{2}]$$
$$V[\Delta_{12}+h-c_1;{}^{c_2}_{u_2}]\; V[\Delta_3+h+c_6;{}^{c_3}_{u_3}]\; V[\Delta_4+h-c_6;{}^{c_4}_{u_4}]$$
$$V[0;{}^{c_2,c_3,c_5}_{u_2,u_3,u_5}]\; V[h+c_1;{}^{c_4,c_5}_{u_4,u_5}]$$

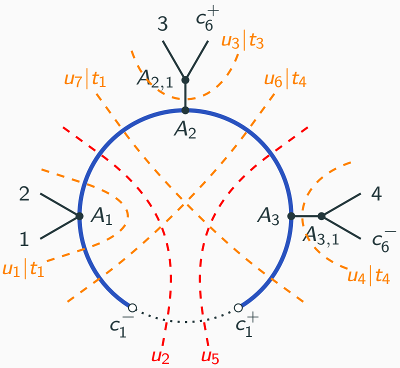


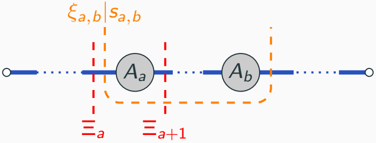
Integral kernel K_0





Integral kernel K_0





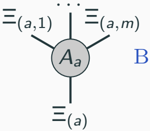
Each a ($\mathrm{B}\left[{x \atop y}\right] \equiv \frac{\Gamma[x]\Gamma[y-x]}{\Gamma[y]}$):

$$\mathrm{B}_a\left[{\frac{(\xi_{a,r}-\xi_{a+1,r})-(\Xi_a-\Xi_{a+1})}{2} \atop \frac{(\xi_{a,r}-\xi_{a+1,r})-(\xi_{1,a-1}-\xi_{1,a})}{2}}\right]$$

Each (a, b):

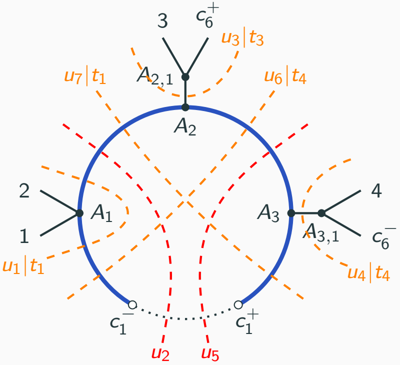
$$\mathrm{B}_{a,b}\left[{\frac{\xi_{a,b-1}+\xi_{a+1,b}-\xi_{a+1,b-1}-\xi_{a,b}}{2} \atop \frac{s_{a,b-1}+s_{a+1,b}-s_{a+1,b-1}-s_{a,b}}{2}}\right]$$

Each vertex in each branch:



$$\mathrm{B}_{(a)}\left[{\frac{\Delta_{A_a}-\Xi_{(a)}+\sum_m \Xi_{(a,m)}}{2} \atop \frac{\Delta_{A_a}-s_{(a)}+\sum_m s_{(a,m)}}{2}}\right]$$

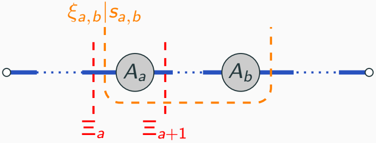
Integral kernel K_0



$$\mathrm{B}_1\left[\begin{matrix}\frac{h+c_1+u_2-u_7}{2}\\ \frac{2h+u_1-u_7}{2}\end{matrix}\right]\mathrm{B}_2\left[\begin{matrix}\frac{u_5+u_7-u_2-u_4}{2}\\ \frac{u_6+u_7-u_1-u_4}{2}\end{matrix}\right]\mathrm{B}_3\left[\begin{matrix}\frac{h+c_1+u_4-u_5}{2}\\ \frac{2h+u_4-u_6}{2}\end{matrix}\right]$$

$$\mathrm{B}_{1,2}\left[\begin{matrix}\frac{u_1+u_3-u_6}{2}\\ \frac{t_1+t_3-t_4}{2}\end{matrix}\right]\mathrm{B}_{1,3}\left[\begin{matrix}\frac{u_6+u_7-u_3-2h}{2}\\ \frac{t_1-t_3+t_4}{2}\end{matrix}\right]\mathrm{B}_{2,3}\left[\begin{matrix}\frac{u_3+u_4-u_7}{2}\\ \frac{t_3+t_4-t_1}{2}\end{matrix}\right]$$

$$\mathrm{B}_{(1)}\left[\begin{matrix}\frac{\Delta_{12}-u_1}{2}\\ \frac{\Delta_{12}-t_1}{2}\end{matrix}\right]\mathrm{B}_{(2,1)}\left[\begin{matrix}\frac{\Delta_3+h+c_6-u_3}{2}\\ \frac{\Delta_3+h+c_6-t_3}{2}\end{matrix}\right]\mathrm{B}_{(3,1)}\left[\begin{matrix}\frac{\Delta_4+h-c_6-u_4}{2}\\ \frac{\Delta_4+h-c_6-t_4}{2}\end{matrix}\right]$$



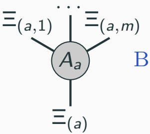
Each a $\left(\mathrm{B}\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]\equiv\frac{\Gamma[x]\Gamma[y-x]}{\Gamma[y]}\right)$:

$$\mathrm{B}_a\left[\begin{matrix}\frac{(\xi_{a,r}-\xi_{a+1,r})-(\Xi_a-\Xi_{a+1})}{2}\\ \frac{(\xi_{a,r}-\xi_{a+1,r})-(\xi_{1,a-1}-\xi_{1,a})}{2}\end{matrix}\right]$$

Each (a,b):

$$\mathrm{B}_{a,b}\left[\begin{matrix}\frac{\xi_{a,b-1}+\xi_{a+1,b}-\xi_{a+1,b-1}-\xi_{a,b}}{2}\\ \frac{s_{a,b-1}+s_{a+1,b}-s_{a+1,b-1}-s_{a,b}}{2}\end{matrix}\right]$$

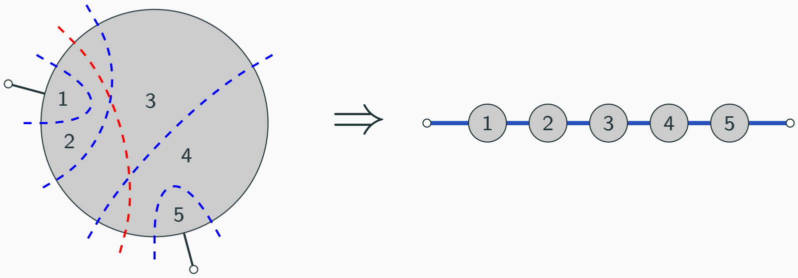
Each vertex in each branch:



$$\mathrm{B}_{(a)}\left[\begin{matrix}\frac{\Delta_{A_a}-\Xi_{(a)}+\sum_m\Xi_{(a,m)}}{2}\\ \frac{\Delta_{A_a}-s_{(a)}+\sum_m s_{(a,m)}}{2}\end{matrix}\right]$$

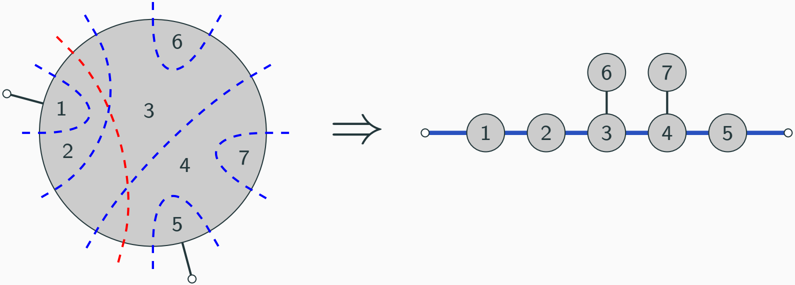
Meromorphicity: effective trees

1. Collect a maximal set of channels separating the two points to be glued, such that OPEs can be performed sequentially.
   (In practice this is indicated in the poles of M_0.)
   This effectively induces a chain diagram.

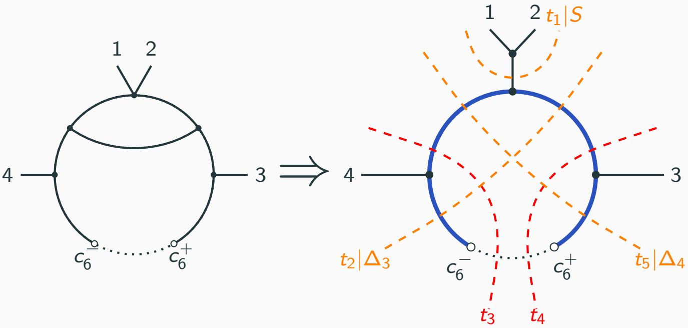


Meromorphicity: effective trees

1. Collect a maximal set of channels separating the two points to be glued, such that OPEs can be performed sequentially.
   (In practice this is indicated in the poles of M_0.)
   This effectively induces a chain diagram.
2. The scattering may allow further OPEs, but they only lead to additional branches.



Integral kernel K_1



$$\mathrm{B}_1\begin{bmatrix}\frac{h+c_6+t_3-t_5}{2}\\ \frac{2h-t_5+\Delta_4}{2}\end{bmatrix}\mathrm{B}_2\begin{bmatrix}\frac{t_4+t_5-t_3-\Delta_3}{2}\\ \frac{t_2+t_5-\Delta_{34}}{2}\end{bmatrix}\mathrm{B}_3\begin{bmatrix}\frac{h+c_6-t_4+\Delta_3}{2}\\ \frac{2h-t_2+\Delta_3}{2}\end{bmatrix}$$

$$\mathrm{B}_{1,2}\begin{bmatrix}\frac{t_1-t_2+\Delta_4}{2}\\ \frac{S-\Delta_3+\Delta_4}{2}\end{bmatrix}\mathrm{B}_{1,3}\begin{bmatrix}\frac{t_2+t_5-t_1-2h}{2}\\ \frac{\Delta_{34}-S}{2}\end{bmatrix}\mathrm{B}_{2,3}\begin{bmatrix}\frac{t_1-t_5+\Delta_3}{2}\\ \frac{S+\Delta_3-\Delta_4}{2}\end{bmatrix}$$

$$\mathrm{B}_{(2)}\begin{bmatrix}\frac{\Delta_{12}-t_1}{2}\\ \frac{\Delta_{12}-S}{2}\end{bmatrix}$$

Mellin integral representation for (pre-)amplitudes

- 7 u, 5 t, and 6 c integrals. All are Mellin integrals.

$$M_1[t] = \int \mathrm{d}u_1 \cdots \mathrm{d}u_7 \, M_0[u] \, K_0[u,t],$$

$$M_2[s] = \int \mathrm{d}t_1 \cdots \mathrm{d}t_5 \, M_1[t] \, K_1[t,s],$$

$$\mathcal{M}_2[s] = \int [\mathrm{d}c_1]_{\underline{\Delta}_1} \cdots [\mathrm{d}c_6]_{\underline{\Delta}_6} \, M_2[s].$$

  These are directly read from the diagram **without** computation.
- Numerical integration can in principle be done efficiently.
- Hard to obtain analytic answer in terms of familiar functions (though in specific cases it reduces to ${}_pF_q$, e.g., [Aharony et al,'16]).

Mellin integral representation for (pre-)amplitudes

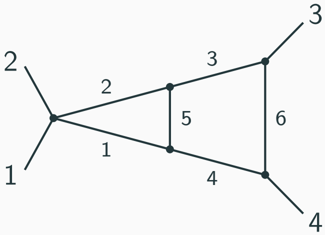
- It is guaranteed by this integral representation that both M and $\mathcal{M}$ are always **meromophic** functions.
- Hence we are interested in
  - determining the entire pole structure (i.e., existence, location);
  - estimating the order of each specific pole;
  - computing the residue at each pole
    (again in terms of Mellin integrals but usually much simpler).
- There is a systematic method for answering all these questions, but I will not develop this point in this talk.
- I will mainly describe the pole structure resulting from this analysis, which turns out to be **universal** to all (scalar) diagrams.

Mellin Pre-amplitudes

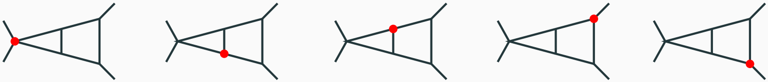
Diagrammatic rules for M's pole structure

- **Vertex rule.** For each bulk vertex A (data: $\{\Delta_A, c_1, \ldots, c_r\}$)

$$\gamma\left[\frac{\Delta_A+(r-2)h\pm c_1\pm\cdots\pm c_r}{2}\right].$$



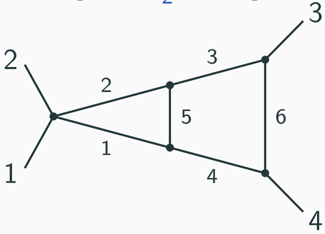
$$\underbrace{\gamma\left[\frac{\Delta_{12}\pm c_1\pm c_2}{2}\right]}\underbrace{\gamma\left[\frac{h\pm c_1\pm c_4\pm c_5}{2}\right]}\underbrace{\gamma\left[\frac{h\pm c_2\pm c_3\pm c_5}{2}\right]}\underbrace{\gamma\left[\frac{\Delta_3\pm c_3\pm c_6}{2}\right]}\underbrace{\gamma\left[\frac{\Delta_4\pm c_4\pm c_6}{2}\right]}.$$



Diagrammatic rules for M's pole structure

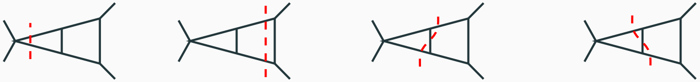
- **Channel rule.** For each cut in each channel S (data: $\{c_1, \ldots, c_r, S\}$)

$$\gamma\left[\frac{rh \pm c_1 \pm \cdots \pm c_r - S}{2}\right].$$



S channel:

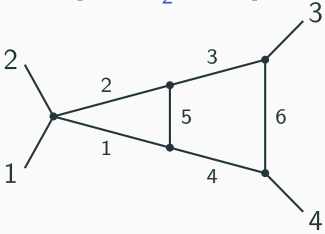
$$\underbrace{\gamma\left[\frac{2h \pm c_1 \pm c_2 - S}{2}\right]}\underbrace{\gamma\left[\frac{2h \pm c_3 \pm c_4 - S}{2}\right]}\underbrace{\gamma\left[\frac{3h \pm c_1 \pm c_3 \pm c_5 - S}{2}\right]}\underbrace{\gamma\left[\frac{3h \pm c_2 \pm c_4 \pm c_5 - S}{2}\right]}.$$



Diagrammatic rules for M's pole structure

- **Channel rule.** For each cut in each channel S (data: $\{c_1, \ldots, c_r, S\}$)

$$\gamma\left[\frac{rh \pm c_1 \pm \cdots \pm c_r - S}{2}\right].$$

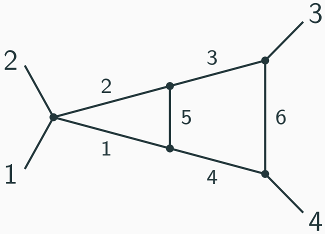


Trivial channels:

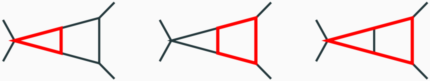
$$\underbrace{\gamma\left[\frac{2h\pm c_3\pm c_6-\Delta_3}{2}\right]}\underbrace{\gamma\left[\frac{3h\pm c_2\pm c_5\pm c_6-\Delta_3}{2}\right]}\underbrace{\gamma\left[\frac{2h\pm c_4\pm c_6-\Delta_4}{2}\right]}\underbrace{\gamma\left[\frac{3h\pm c_1\pm c_5\pm c_6-\Delta_4}{2}\right]}.$$

Diagrammatic rules for M's pole structure

- **Loop contraction rule.** For each new vertex emerged from contracting exisiting loops, apply the same vertex rule.



$$\underbrace{\gamma\left[\frac{\Delta_{12}\pm c_3 \pm c_4}{2}\right]}\,\underbrace{\gamma\left[\frac{\Delta_{34}\pm c_1 \pm c_2}{2}\right]}\,\underbrace{\gamma\left[\frac{\Delta_{1234}}{2}-h\right]}\,.$$



- **Generalized bubble rules.** (No need. And I will skip in this talk.)

Compositeness

Summary of M's pole structure:

$$\gamma[\tfrac{\Delta_{12}\pm c_1\pm c_2}{2}]\,\gamma[\tfrac{h\pm c_1\pm c_4\pm c_5}{2}]\,\gamma[\tfrac{h\pm c_2\pm c_3\pm c_5}{2}]\,\gamma[\tfrac{\Delta_3\pm c_3\pm c_6}{2}]\,\gamma[\tfrac{\Delta_4\pm c_4\pm c_6}{2}]$$
$$\gamma[\tfrac{2h\pm c_1\pm c_2-S}{2}]\,\gamma[\tfrac{2h\pm c_3\pm c_4-S}{2}]\,\gamma[\tfrac{3h\pm c_1\pm c_3\pm c_5-S}{2}]\,\gamma[\tfrac{3h\pm c_2\pm c_4\pm c_5-S}{2}]$$
$$\gamma[\tfrac{2h\pm c_3\pm c_6-\Delta_3}{2}]\,\gamma[\tfrac{3h\pm c_2\pm c_5\pm c_6-\Delta_3}{2}]\,\gamma[\tfrac{2h\pm c_4\pm c_6-\Delta_4}{2}]\,\gamma[\tfrac{3h\pm c_1\pm c_5\pm c_6-\Delta_4}{2}]$$
$$\gamma[\tfrac{\Delta_{12}\pm c_3\pm c_4}{2}]\,\gamma[\tfrac{\Delta_{34}\pm c_1\pm c_2}{2}]\,\gamma[\tfrac{\Delta_{1234}}{2}-h].$$

Imagine that we further perform integrals on c's or Δ's.

Example

$$\gamma[\tfrac{2h+c_1+c_2}{2}]\,\gamma[\tfrac{h-c_2+c_3+c_5}{2}] \xrightarrow{\int \mathrm{d}c_2} \gamma[\tfrac{3h+c_1+c_3+c_5-S}{2}],$$
$$\gamma[\tfrac{2h+c_3+c_4}{2}]\,\gamma[\tfrac{h+c_1-c_4+c_5}{2}] \xrightarrow{\int \mathrm{d}c_4} \gamma[\tfrac{3h+c_1+c_3+c_5-S}{2}].$$

We call these poles "**composite**".

Conjecture on Mellin pre-amplitudes

For an arbitrary scalar Witten diagram, we read off all the poles following

1. *the vertex rule,*
2. *the channel rule,*
3. *the loop contraction rule,*
4. *the generalized bubble rule.*

After eliminating the composite poles, the remaining families exactly constitute all the genuine poles of the pre-amplitude of the diagram.

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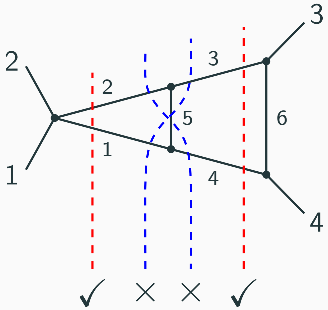
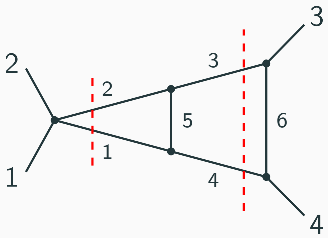
The channel rule is ultimately responsible for
the singularities of $\mathcal{M}$ in the "Mandelstam" variables.

Conjecture on Mellin pre-amplitudes



Mellin Amplitudes

Minimal/non-minimal cuts

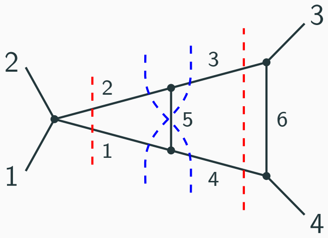


- **Minimal:** whose corresponding poles are present in M.

$$\left.\begin{matrix} \gamma\left[\frac{2h+c_1+c_2-S}{2}\right] \\ \frac{1}{(\underline{\Delta}_1-h)-c_1} \; \frac{1}{(\underline{\Delta}_2-h)-c_2} \end{matrix}\right\} \xrightarrow{\int \mathrm{d}c_1 \mathrm{d}c_2} \gamma\left[\frac{\underline{\Delta}_{12}-S}{2}\right].$$

The corresponding poles in $\mathcal{M}$ emerges from the spectrum integrals in the minimal way: only those associated to propagators in the cut are necessary (analogous to tree diagrams).

Minimal/non-minimal cuts



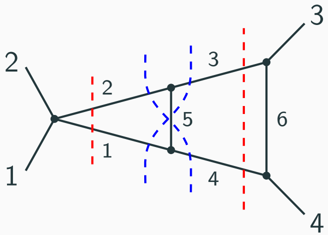
- **Non-minimal:** whose corresponding poles are absent from M.

$$\gamma\left[\frac{3h+c_1+c_3+c_5-S}{2}\right]$$

$$\frac{1}{(\underline{\Delta}_1-h)-c_1}\ \frac{1}{(\underline{\Delta}_3-h)-c_3}\ \frac{1}{(\underline{\Delta}_6-h)-c_5}$$

$$\xrightarrow{\int \mathrm{d}c_1\mathrm{d}c_3\mathrm{d}c_5} \gamma\left[\frac{\underline{\underline{\Delta}}_{135}-S}{2}\right].$$

Minimal/non-minimal cuts



- **Non-minimal:** whose corresponding poles are absent from M.

$$
\left.\begin{array}{l}
\text{or}\;\; \gamma\left[\frac{2h+c_1+c_2-S}{2}\right]\gamma\left[\frac{h-c_2+c_3+c_5}{2}\right] \xrightarrow{\int \mathrm{d}c_2} \\
\gamma\left[\frac{2h+c_3+c_4-S}{2}\right]\gamma\left[\frac{h+c_1-c_4+c_5}{2}\right] \xrightarrow{\int \mathrm{d}c_4}
\end{array}\right\} \left.\begin{array}{l}
\gamma\left[\frac{3h+c_1+c_3+c_5-S}{2}\right] \\
\frac{1}{(\underline{\Delta}_1-h)-c_1}\;\frac{1}{(\underline{\Delta}_3-h)-c_3}\;\frac{1}{(\underline{\Delta}_6-h)-c_5}
\end{array}\right\}
$$

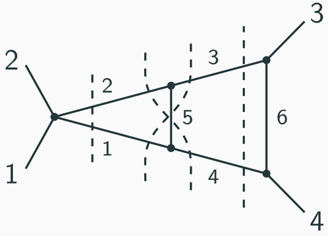
$$
\xrightarrow{\int \mathrm{d}c_1\mathrm{d}c_3\mathrm{d}c_5} \gamma\left[\frac{\underline{\underline{\Delta}}_{135}-S}{2}\right].
$$

Residue computation

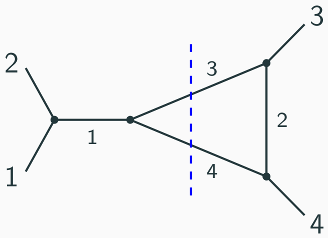
The above summary of the origin of poles also provide a guidance for the computation of the corresponding residues.

Detailed computation verifies the existence of all the four families

$$\gamma[\tfrac{\Delta_{12}-S}{2}]\,\gamma[\tfrac{\Delta_{34}-S}{2}]\,\gamma[\tfrac{\Delta_{135}-S}{2}]\,\gamma[\tfrac{\Delta_{245}-S}{2}].$$



Example



Pinching pattern:

$$\left.\begin{array}{c} \gamma\big[\tfrac{h\pm c_1-S}{2}\big]\gamma\big[\tfrac{h\mp c_1\pm c_3\pm c_4}{2}\big] \xrightarrow{\int \mathrm{d}c_1} \color{blue}{\gamma\big[\tfrac{2h\pm c_3\pm c_4-S}{2}\big]} \\ \tfrac{1}{(\underline{\Delta}_3-h)\mp c_3}\;\tfrac{1}{(\underline{\Delta}_4-h)\mp c_4} \end{array}\right\} \xrightarrow{\int \mathrm{d}c_3\mathrm{d}c_4} \gamma\big[\tfrac{\underline{\underline{\Delta}}_{34}-S}{2}\big].$$

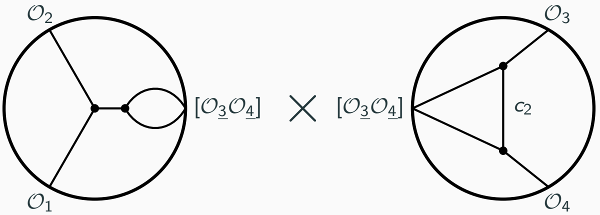
One contribution to the residue at the leading pole:

$$\gamma\big[\tfrac{h-c_1-S}{2}\big]\gamma\big[\tfrac{h+c_1-c_3-c_4}{2}\big]\frac{1}{(\underline{\Delta}_3-h)+c_3}\,\frac{1}{(\underline{\Delta}_4-h)+c_4}$$

$$\Longrightarrow \quad \operatorname*{Res}_{S=\underline{\underline{\Delta}}_{34}}\;\operatorname*{Res}_{c_4=h-\underline{\Delta}_4}\;\operatorname*{Res}_{c_3=h-\underline{\Delta}_3}\;\operatorname*{Res}_{c_1=c_3+c_4-h}.$$

Example

$$
\operatorname*{Res}_{s=\underline{\Delta}_{34}} \mathcal{M} = \frac{\pi^{2h}}{4} \prod_{i=1}^{4} \frac{\mathcal{C}_{\Delta_i}}{\Gamma[\Delta_i]} \frac{\mathcal{C}_{\underline{\Delta}_3}\mathcal{C}_{\underline{\Delta}_4}}{\Gamma[\underline{\Delta}_3]\Gamma[\underline{\Delta}_4]} \times \\
\underbrace{\frac{\Gamma[\frac{\Delta_{12}+\underline{\Delta}_{34}}{2} - h]}{(\underline{\Delta}_1 - \underline{\Delta}_{34})(\underline{\Delta}_{134} - 2h)}}_{\mathcal{O}_1\mathcal{O}_2[\mathcal{O}_{\underline{3}}\mathcal{O}_{\underline{4}}]} \times \underbrace{\int \frac{[\mathrm{d}c_2]_{\underline{\Delta}_2}}{\Gamma[\pm c_2]} \frac{\Gamma[\frac{\underline{\Delta}_3 \pm (h-\Delta_3) \pm c_2}{2}]\Gamma[\frac{\underline{\Delta}_4 \pm (h-\Delta_4) \pm c_2}{2}]}{\Gamma[h + \frac{\underline{\Delta}_{34} - \Delta_{34}}{2}]\Gamma[\frac{\underline{\Delta}_{34} \pm (\Delta_3 - \Delta_4)}{2}]}}_{[\mathcal{O}_{\underline{3}}\mathcal{O}_{\underline{4}}]\mathcal{O}_3\mathcal{O}_4} .
$$



Summary

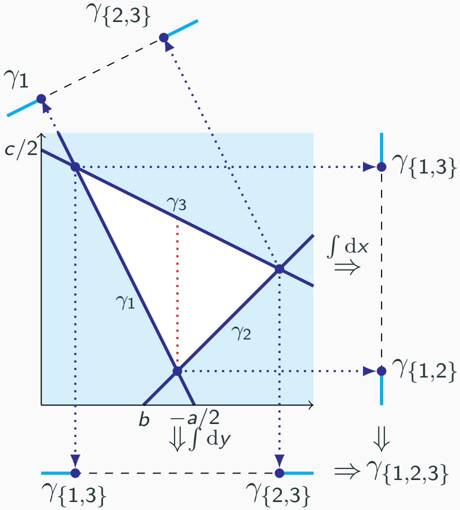
Summary

In this talk we have made some first investigations to perturbative dynamics to all loops.

- We designed a recursive construction that builds up arbitrary scalar Witten diagrams.
- This construction directly yields Mellin (pre-)amplitudes in terms of Mellin integrals, following simple diagrammatic rules.
- Analytic properties of (pre-)amplitudes can be extracted systematically using this integral representation
  - We conjectured that the pole structure of pre-amplitudes follows a set of diagrammatic rules.
  - Residues of Mellin amplitudes can be conveniently computed.

Many thanks
for your attention!

Singularities from multivariate Mellin integrals



Singularities from multivariate Mellin integrals