Cuts and coaction of Feynman integrals

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- There is a coaction on Feynman diagrams.

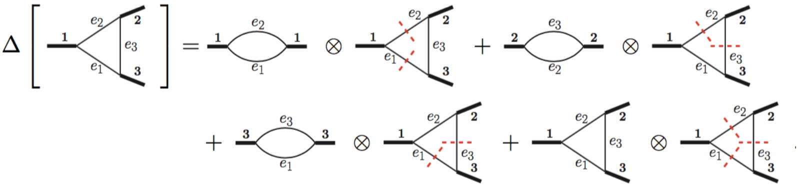
- At 1 loop, there is a basis for which the coaction is simply related to pinches and cuts of the original diagram and corresponds to Goncharov's Hopf algebra on MPLs.

- Natural interplay with discontinuities and differential equations is potentially useful for computation.

- Larger framework: coactions of the form

$$\Delta\left(\int_\gamma \omega\right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$$

Coaction operation



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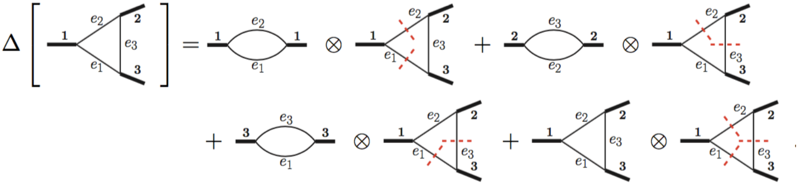

$$
\begin{aligned}
\Delta(\log z) &= 1 \otimes \log z + \log z \otimes 1 \\
\Delta(\log^2 z) &= 1 \otimes \log^2 z + 2 \log z \otimes \log z + \log^2 z \otimes 1 \\
\Delta(\mathrm{Li}_2(z)) &= 1 \otimes \mathrm{Li}_2(z) + \mathrm{Li}_2(z) \otimes 1 + \mathrm{Li}_1(z) \otimes \log z
\end{aligned}
$$

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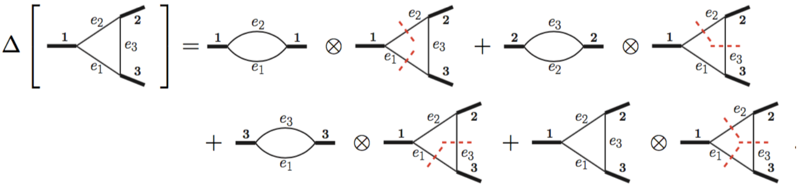

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\end{aligned}
$$

Discontinuities and cuts:

$$\Delta \, \mathrm{Disc} = (\mathrm{Disc} \otimes 1) \, \Delta$$

Differential operators:

$$\Delta \, \partial = (1 \otimes \partial) \, \Delta$$

Master formula for coaction on integrals

We conjecture a framework as follows.

Coactions of the following form:

$$\Delta\left(\int_\gamma \omega\right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$$

with a duality condition

$$P_{ss} \int_{\gamma_i} \omega_j = \delta_{ij}\,.$$

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$$P_{ss} \int_{\gamma_i} \omega_j = \delta_{ij} \, .$$

P_{ss} is semi-simple projection ("drop logarithms but not π").

The master formula coaction is like inserting a complete set of states ("ω_i are a set of master integrands for ω").

Outline

1. Diagrammatic coaction
2. Cut integrals
3. Master formula for coaction on integrals

The "incidence" bialgebra [Joni, Rota]

A simple combinatorial algebra: let $[\mathbf{n}] = \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}\}$.
Elements: pairs of nested subsets $S \subseteq T$, where $S \subseteq T \subseteq [\mathbf{n}]$.

$\{\mathbf{1}\} \subseteq \{\mathbf{1}, \mathbf{2}\}$ represented by $\mathbf{1}\,\mathbf{2}$
$\emptyset \subseteq \{\mathbf{1}, \mathbf{2}\}$ represented by $\mathbf{1}\,\mathbf{2}$
$\emptyset \subset \emptyset$ represented by $*$

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Multiplication is a free operation, and the coproduct is defined by

$$\Delta(S \subseteq T) = \sum_{S \subseteq X \subseteq T} (S \subseteq X) \otimes (X \subseteq T).$$

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$$\Delta(S \subseteq T) = \sum_{S \subseteq X \subseteq T} (S \subseteq X) \otimes (X \subseteq T).$$

For example:

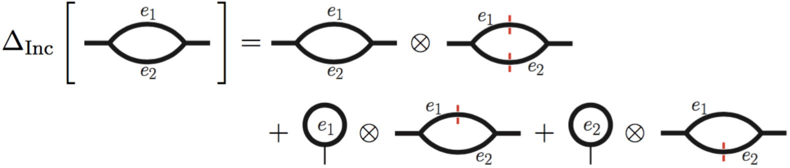
$$\begin{aligned}
\Delta(\mathbf{1}\,\mathbf{2}) &= \mathbf{1}\,\mathbf{2} \otimes \mathbf{1}\,\mathbf{2} + \mathbf{1} \otimes \mathbf{1}\,\mathbf{2} + \mathbf{2} \otimes \mathbf{1}\,\mathbf{2} + * \otimes \mathbf{1}\,\mathbf{2} \\
\Delta(\mathbf{1}\,\mathbf{2}) &= \mathbf{1}\,\mathbf{2} \otimes \mathbf{1}\,\mathbf{2} + \mathbf{2} \otimes \mathbf{1}\,\mathbf{2} \\
\Delta(\mathbf{2}) &= \mathbf{2} \otimes \mathbf{2} + * \otimes \mathbf{2} \\
\Delta(\mathbf{2}) &= \mathbf{2} \otimes \mathbf{2} \\
\Delta(S \subseteq S) &= (S \subseteq S) \otimes (S \subseteq S)
\end{aligned}$$

Example of the incidence algebra: edges of graphs

$$\Delta(\mathbf{1}\,\mathbf{2}) \quad = \quad \mathbf{1}\,\mathbf{2} \otimes \mathbf{1}\,\mathbf{2} + \mathbf{1} \otimes \mathbf{1}\,\mathbf{2} + \mathbf{2} \otimes \mathbf{1}\,\mathbf{2} + * \otimes \mathbf{1}\,\mathbf{2}$$

For graphs, set $* = (\emptyset \subseteq \emptyset) = 0$.

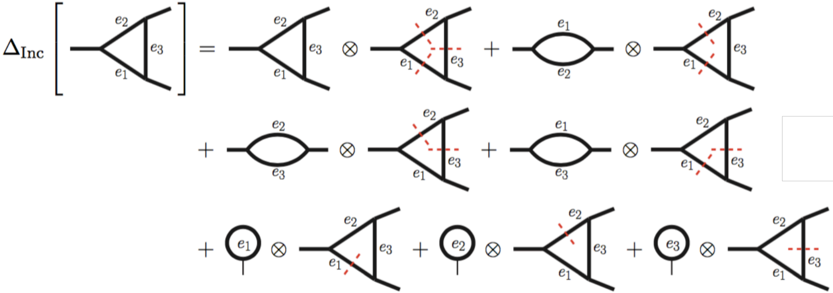
Pinch and cut *complementary* subsets of edges:



Example of the incidence algebra: edges of graphs

$$\Delta(1\,2\,3) = 1\,2\,3 \otimes 1\,2\,3 + 1\,2 \otimes 1\,2\,3 + 2\,3 \otimes 1\,2\,3 + 1\,3 \otimes 1\,2\,3$$
$$+1 \otimes 1\,2\,3 + 2 \otimes 1\,2\,3 + 3 \otimes 1\,2\,3 + * \otimes 1\,2\,3$$

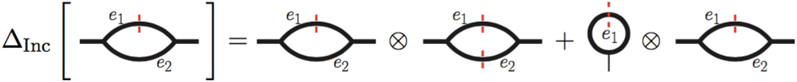
Pinch and cut *complementary* subsets of edges:



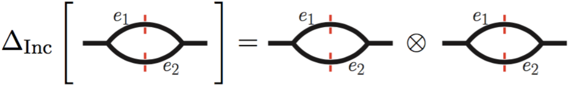
Example of the incidence algebra: edges of graphs

Can also start with a cut diagram.

$$\Delta(1\,2) \quad = \quad 1\,2 \otimes 1\,2 + 1 \otimes 1\,2$$



$$\Delta(1\,2) \quad = \quad 1\,2 \otimes 1\,2$$



Multiple polylogarithms (MPL)

A large class of iterated integrals are described by multiple polylogarithms:

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

Examples:

$$G(0; z) = \log z, \quad G(a; z) = \log\left(1 - \frac{z}{a}\right)$$

$$G(\vec{a}_n; z) = \frac{1}{n!} \log^n\left(1 - \frac{z}{a}\right), \quad G(\vec{0}_{n-1}, a; z) = -\mathrm{Li}_n\left(\frac{z}{a}\right)$$

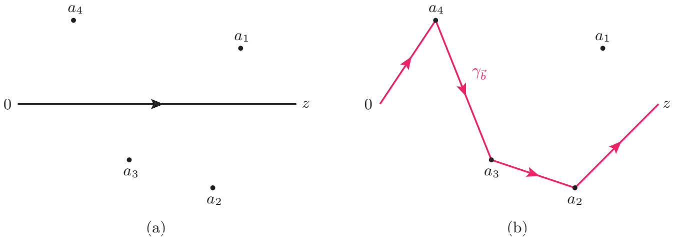
n is the *transcendental weight*.

MPLs obey shuffle product relations. There is a coaction on MPLs, graded by weight, which thus breaks MPLs into simpler functions (lower weight). [Goncharov; Duhr]

Contour integrals

The coaction is a pairing of contours and integrands. Recalls the incidence algebra.

$$\Delta_{\mathrm{MPL}}(G(\vec{a};z)) = \sum_{\vec{b}\subseteq\vec{a}} G(\vec{b};z) \otimes G_{\vec{b}}(\vec{a};z)$$



Contour (b) encircles a subset of residues in a given order.

Feynman integrals

A useful basis for all 1-loop integrals:

$$J_n = \frac{ie^{\gamma_E \epsilon}}{\pi^{D_n/2}} \int d^{D_n} k \prod_{j=1}^{n} \frac{1}{(k-q_j)^2 - m_j^2}$$

- k is the loop momentum
- q_j are sums of external momenta, m_j are internal masses
- Dimensions:

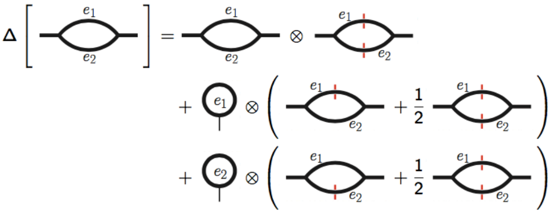
$$D_n = \begin{cases} n - 2\epsilon\,, & \text{for } n \text{ even}\,, \\ n + 1 - 2\epsilon\,, & \text{for } n \text{ odd}\,. \end{cases}$$

  e.g. tadpoles and bubbles in $2 - 2\epsilon$ dimensions,
  triangles and boxes in $4 - 2\epsilon$ dimensions, etc.
- Each J_n has uniform transcendental weight.

2 equivalent Hopf algebras

The combinatorial algebra agrees with the Hopf algebra on the MPL of evaluated diagrams!

- The graph with n edges is interpreted as J_n, i.e. in D_n dimensions, no numerator.
- Need to insert extra terms in the diagrammatic equation:



Isomorphic to the more basic construction. (For any value of 1/2.)

- How do we evaluate the cut graphs?

[related work: Brown; Bloch and Kreimer]

Outline

Generalized cuts as residues and determinants

1-loop cuts defined as residues:

$$\mathcal{C}_C[I_n] = \frac{e^{\gamma_E \epsilon}}{i\pi^{\frac{D}{2}}} \int_{\Gamma_C} d^D k \prod_{j \notin C} \frac{1}{(k - q_j)^2 - m_j^2 + i0} \quad \text{mod } i\pi \,,$$

- C is the set of cut propagators
- Contour Γ_C encircles poles of cut propagators

Cut integrals give discontinuities of their uncut counterparts.

Generalized cuts as residues and determinants

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Cut integrals give discontinuities of their uncut counterparts.
Some results:

$$\mathcal{C}_C I_n = \frac{e^{\gamma_E \epsilon}}{\sqrt{Y_C}} \left(\frac{Y_C}{G_C}\right)^{(D-c)/2} \int \frac{d\Omega_{D-c+1}}{i\pi^{D/2}} \left[\prod_{j \notin C} \frac{1}{(k-q_j)^2 - m_j^2}\right]_C$$

where we often find Gram and modified Cayley determinants:

$$\begin{aligned} G_C &= \det\left(q_i \cdot q_j\right)_{i,j \in C \setminus *} \\ Y_C &= \det\left(\frac{1}{2}(m_i^2 + m_j^2 + (q_i - q_j)^2)\right)_{i,j \in C} \end{aligned}$$

Maximal and next-to-maximal cuts

Some special cases:

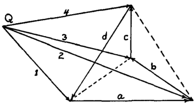
$$\mathcal{C}_{2k}[J_{2k}] = \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{e^{\gamma_E \epsilon}}{\sqrt{Y_{2k}}} \left(\frac{Y_{2k}}{G_{2k}}\right)^{-\epsilon}$$

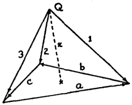
$$\mathcal{C}_{2k+1}[J_{2k+1}] = \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)\sqrt{G_{2k+1}}} \left(\frac{Y_{2k+1}}{G_{2k+1}}\right)^{-\epsilon} .$$

$$\mathcal{C}_{2k-1}[J_{2k}] = -\frac{\Gamma(1-2\epsilon)^2}{\Gamma(1-\epsilon)^3} \frac{e^{\gamma_E \epsilon}}{\sqrt{Y_{2k}}} \left(\frac{Y_{2k-1}}{G_{2k-1}}\right)^{-\epsilon} {}_2F_1\left(\frac{1}{2}, -\epsilon; 1-\epsilon; \frac{G_{2k}\, Y_{2k-1}}{Y_{2k}\, G_{2k-1}}\right) .$$

For more complicated cuts, we set up a Feynman parametrization.

Landau conditions are expressed in polytope geometry: these determinants are volumes of simplices. [Cutkosky]





Landau conditions

$$\alpha_i \left[(k^E - q_i^E)^2 + m_i^2 \right] = 0, \quad \forall i \,.$$
$$\sum_{i=1}^{n} \alpha_i (k^E - q_i^E) = 0 \,.$$

Therefore

$$\begin{pmatrix} (k^E - q_1^E)\cdot(k^E - q_1^E) & \dots & (k^E - q_1^E)\cdot(k^E - q_c^E) \\ \vdots & \ddots & \vdots \\ (k^E - q_c^E)\cdot(k^E - q_1^E) & \dots & (k^E - q_c^E)\cdot(k^E - q_c^E) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_c \end{pmatrix} = 0 \,.$$

Nontrivial solution $\Leftrightarrow Y_C = 0$ and integral over Γ_C: Landau singularities of the first type.

Second-type singularities come from $G_C = 0$ and integral over $\Gamma_{C \cup \infty}$. Contour is pinched at infinity.

Homology theory for Feynman contours

Homology describes the inequivalent integration contours. Also explains why cuts are discontinuities. Use Leray residues. [Fotiadi, Pham; Hwa, Teplitz; Federbusch; Eden, Fairlie, Landshoff, Nuttall, Olive, Polkinghorne,...]

Residues: if

$$\omega = \frac{ds}{s} \wedge \psi + \theta$$

and $S = \{s = 0\}$, while ψ, θ are regular on S, then

$$\text{Res}_S[\omega] = \psi|_S \,.$$

Cut integrals: if $I_n = \int \omega_n$, then

$$\begin{aligned} \mathcal{C}_C[I_n] &= \int_{S_C} \text{Res}_{S_C}[\omega_n] \\ &= (2\pi i)^{-k} \int_{\delta S_C} \omega_n \end{aligned}$$

where δ constructs a "tubular neighborhood" around $S_C = \cap_{i \in C} S_i$, the spherical locus of the cut conditions.

Composed residues, e.g.:

$$\omega = \frac{ds_1}{s_1} \wedge \frac{ds_2}{s_2} \wedge \psi_{12} + \frac{ds_1}{s_1} \wedge \psi_1 + \frac{ds_2}{s_2} \wedge \psi_2 + \theta$$

$$\mathrm{Res}_{S_1 S_2}[\omega] = \psi_{12}|_{S_1 \cap S_2}$$

Residue theorem:

$$\int_{\delta_{S_1 S_2} \sigma} \omega = (2\pi i)^2 \int_\sigma \mathrm{Res}_{S_2 S_1}[\omega] \,,$$

where $\delta_{S_1 S_2} \equiv \delta_{S_1} \delta_{S_2}$.

Compactification

Embedding formalism variables.

$$Z = \begin{bmatrix} z^\mu \\ Z^- \\ Z^+ \end{bmatrix} \in \mathbb{CP}^{D+1}$$

with bilinear form

$$(Z_1 Z_2) = z_1^\mu \, z_{2\mu} - \frac{1}{2} \, Z_1^+ Z_2^- - \frac{1}{2} \, Z_1^- Z_2^+$$

Compactification

Uncut integrals:

$$I_n^D = \frac{e^{\gamma_E \epsilon}}{\pi^{D/2}} \int_{(YY)=0} \frac{d^{D+2}Y}{\mathrm{Vol}(GL(1))} \frac{(X_\infty Y)^{n-D}}{(X_1 Y)\dots(X_n Y)}\,,$$

where

$$X_i = \begin{bmatrix} (q_i^E)^\mu \\ (q_i^E)^2 + m_i^2 \\ 1 \end{bmatrix} \text{ for } 1 \leq i \leq n\,, \qquad X_\infty = \begin{bmatrix} 0^\mu \\ 1 \\ 0 \end{bmatrix}\,,$$

[Dirac embedding formalism; Fotiadi et al; Simmons-Duffin; Caron-Huot, Henn]

Compactification

Landau conditions:

$$\det(X_i X_j)_{i,j \in C} = 0$$

X_∞ → second type singularities.

$$\det(X_i X_j)_{i,j \in C} = \begin{cases} (-1)^c \, Y_C \,, & \text{if } \infty \notin C \,, \\ \frac{(-1)^{c-1}}{4} \, G_{C \setminus \{\infty\}} \,, & \text{if } \infty \in C \,. \end{cases}$$

For 1-loop Feynman integrals, the Decomposition Theorem shows that the contours $\Gamma_C = \delta S_C$ form a basis. [Fotiadi, Pham]

$$\Gamma_{\infty C} = -2x_C\,\Gamma_C - \sum_{C\subset X\subseteq[n]} (-1)^{\lceil \frac{|C|}{2} \rceil + \lceil \frac{|X|}{2} \rceil}\,\Gamma_X\,, \qquad x_c = \begin{cases} 1\,, & |C| \text{ odd}\,,\\ 0\,, & |C| \text{ even}\,. \end{cases}$$

It follows that

- for $|C|$ even,

$$\mathcal{C}_{\infty C} I_n = \sum_{i\in[n]\setminus C} \mathcal{C}_{Ci} I_n + \sum_{\substack{i,j\in[n]\setminus C\\ i<j}} \mathcal{C}_{Cij} I_n \mod i\pi\,.$$

- for $|C|$ odd,

$$\mathcal{C}_{\infty C} I_n = -2\mathcal{C}_C I_n - \sum_{i\in[n]\setminus C} \mathcal{C}_{Ci} I_n \mod i\pi\,.$$

Relations among cut integrals

- Any linear relation among cut/uncut integrals can be cut as a whole (cf. "reverse unitarity")

- For $D = n$, i.e. $\epsilon = 0$, there is no singularity at $(X_\infty Y) = 0$, so $\mathcal{C}_{\infty C} I_n^n = 0$. If n is even and c is odd, then

$$2\mathcal{C}_C I_n^n + \sum_i \mathcal{C}_{Ci} I_n^n = 0 \quad \text{mod } i\pi$$

Special case:

$$\mathcal{C}_{n-1} I_n^n = -\frac{1}{2}\mathcal{C}_n I_n^n + \mathcal{O}(\epsilon)$$

- By examining Disc_∞, we see that

$$\begin{aligned}\sum_i \mathcal{C}_i I_n^D + \sum_{i<j} \mathcal{C}_{ij} I_n^D &= \mathcal{C}_\infty I_n^D \\ &= -\epsilon\, I_n^D \qquad \text{mod } i\pi\end{aligned}$$

Both cut and uncut integrals are unified in the following class of functions:

$$Q_n^D(X_1, \ldots, X_n, X_0) = \frac{e^{\gamma_E \epsilon}}{\pi^{D/2}} \int_{(YY)=0} \frac{d^{D+2}Y}{\mathrm{Vol}(GL(1))} \frac{(X_0 Y)^{n-D}}{(X_1 Y) \ldots (X_n Y)},$$

Uncut: $I_n^D(X_1, \ldots, X_n) = Q_n^D(X_1, \ldots, X_n, X_\infty)$

Cut: $\mathcal{C}_C I_n^D = \frac{1}{\sqrt{Y_C}} Q_{n-c}^{D-c}(X'_{C,c+1}, \ldots, X'_{C,n}, X'_{C,\infty})$

where $X'_{C,i}$ are projections of X_i onto the cut locus,

$$X'_{C,i} = \frac{1}{Y_C} \det \begin{pmatrix} (X_1 X_1) & \ldots & (X_1 X_c) & X_1 \\ \vdots & & \vdots & \vdots \\ (X_c X_1) & \ldots & (X_c X_c) & X_c \\ (X_i X_1) & \ldots & (X_i X_c) & X_i \end{pmatrix}.$$

Parametrization and evaluation

Feynman parameters for cut integrals:

$$Q_n^D(X_1,\ldots,X_n,X_0) = \frac{e^{\gamma_E \epsilon}}{\pi^{D/2}} \frac{\Gamma(D)}{\Gamma(D-n)} \int [da]\, a_0{}^{D-n-1} \, (\xi\xi)^{-D/2}, \qquad (1)$$

where

$$\xi \equiv a_0 X_0 + \cdots + a_n X_n\,, \qquad \int [da] \equiv \int_0^\infty da_0 \cdots \int_0^\infty da_n\, \delta(1-h(a))\,, \quad (2)$$

and $h(a) = \sum_{i=0}^n h_i a_i$ such that $h_i \geq 0$.

[see also: Arkani-Hamed and Yuan]

Statement of the graphical conjecture

The coaction on 1-loop graphs defined by pinching and cutting subsets of propagators,

when evaluated by Feynman rules,
if expanded order by order in ϵ,

agrees with the coaction on MPLs.

Examples of the graphical conjecture

$$\Delta_{\mathrm{MPL}}\left[\right.$$ 

$$\left.\right] = $$ 

$$\otimes$$ 

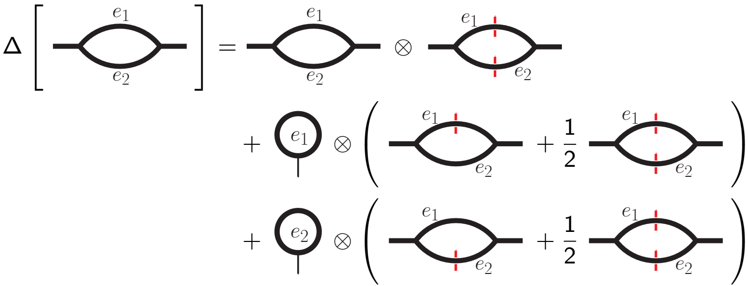
$$.$$

$$= \; -\frac{e^{\gamma_E \epsilon}\Gamma(1+\epsilon)\left(m^2\right)^{-\epsilon}}{\epsilon}$$

$$= \; \frac{e^{\gamma_E \epsilon}\left(-m^2\right)^{-\epsilon}}{\Gamma(1-\epsilon)}\,.$$

$$\Delta_{\mathrm{MPL}}\left[(m^2)^{-\epsilon}\right] = (m^2)^{-\epsilon} \otimes (m^2)^{-\epsilon}\,,$$

$$\Delta_{\mathrm{MPL}}\left[e^{\gamma_E \epsilon}\Gamma(1+\epsilon)\right] = \left(e^{\gamma_E \epsilon}\Gamma(1+\epsilon)\right) \otimes \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)}\,,$$



$$\Delta\left(\int_{\Gamma_\emptyset}\omega_{12}\right) = \int_{\Gamma_\emptyset}\omega_{12}\otimes\int_{\Gamma_{12}}\omega_{12} + \int\omega_1\otimes\left(\int_{\Gamma_1}\omega_{12}+\frac{1}{2}\int_{\Gamma_{12}}\omega_{12}\right)+\cdots$$

$$= \int_{\Gamma_\emptyset}\omega_{12}\otimes\int_{\Gamma_{12}}\omega_{12}+\int_{\Gamma_\emptyset}\omega_1\otimes\int_{-\frac{1}{2}\Gamma_{1\infty}}\omega_{12}+\int_{\Gamma_\emptyset}\omega_2\otimes\int_{-\frac{1}{2}\Gamma_{2\infty}}\omega_{12}$$

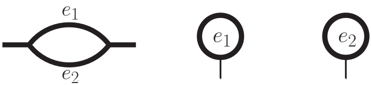
The odd-shaped integrals have poles at infinity.

Examples of the graphical conjecture

$$\Delta\left(\int_{\Gamma_\emptyset}\omega_{12}\right) = \int_{\Gamma_\emptyset}\omega_{12}\otimes\int_{\Gamma_{12}}\omega_{12} + \int\omega_1\otimes\left(\int_{\Gamma_1}\omega_{12}+\frac{1}{2}\int_{\Gamma_{12}}\omega_{12}\right)+\cdots$$

$$= \int_{\Gamma_\emptyset}\omega_{12}\otimes\int_{\Gamma_{12}}\omega_{12} + \int_{\Gamma_\emptyset}\omega_1\otimes\int_{-\frac{1}{2}\Gamma_{1\infty}}\omega_{12} + \int_{\Gamma_\emptyset}\omega_2\otimes\int_{-\frac{1}{2}\Gamma_{2\infty}}\omega_{12}$$

Basis of integrands and corresponding contours:



ω_j :	ω_{12}	ω_1	ω_2
γ_j :	Γ_{12}	$-\frac{1}{2}\Gamma_{1\infty}$	$-\frac{1}{2}\Gamma_{2\infty}$
	$\mathcal{C}_{12}$	$\mathcal{C}_1+\frac{1}{2}\mathcal{C}_{12}$	$\mathcal{C}_2+\frac{1}{2}\mathcal{C}_{12}$

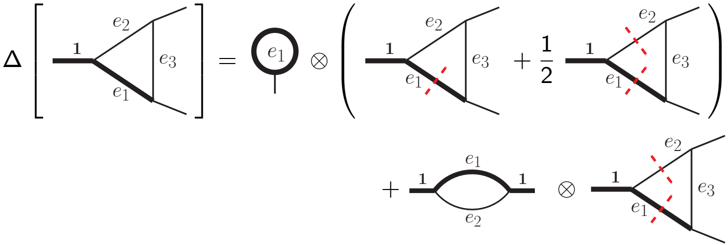
They satisfy

$$\int_{\gamma_i}\omega_j \sim \delta_{ij} \qquad \text{after dropping logs}$$

Outline

The master formula for the $_2F_1$ family

Consider the diagrammatic coaction



There is a coaction on $_2F_1$ that gives

$$
\begin{aligned}
\Delta\, {}_2F_1\left(1, 1+\epsilon, 2-\epsilon, x\right) \quad = \quad & {}_2F_1\left(1, \epsilon, 1-\epsilon, x\right) \otimes {}_2F_1\left(1, 1+\epsilon, 2-\epsilon, x\right) \\
& + {}_2F_1\left(1, 1+\epsilon, 2-\epsilon, x\right) \otimes {}_2F_1\left(1, \epsilon, 1-\epsilon, \frac{1}{x}\right)
\end{aligned}
$$

without expanding in ϵ!

Master formula for Hopf algebra on integrals

Coaction of the form

$$\Delta\left(\int_\gamma \omega\right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$$

with a duality condition

$$P_{ss} \int_{\gamma_i} \omega_j = \delta_{ij} \,.$$

P_{ss} is semi-simple projection ("drop logarithms but not π").

To be precise, P_{ss} projects onto the space of semi-simple numbers x satisfying $\Delta(x) = x \otimes 1$.

The master formula for the ${}_2F_1$ family

Consider the family of integrands

$$\omega(\alpha_1, \alpha_2, \alpha_3) \quad = \quad x^{\alpha_1}(1-x)^{\alpha_2}(1-zx)^{\alpha_3} \; dx$$

where $\alpha_i = n_i + \epsilon_i$ and $n_i \in \mathbb{Z}$.

$$\int_0^1 \omega(\alpha_1, \alpha_2, \alpha_3) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2 - \alpha_1)}{\Gamma(\alpha_2)} \, {}_2F_1(-\alpha_3, \alpha_1 + 1; \alpha_2 + \alpha_1 + 2; z)$$

Basis of master integrands:

$$\int_0^1 \omega \quad = \quad c_0 \int_0^1 \omega_0 + c_1 \int_0^1 \omega_1$$

where

$$\begin{aligned} \omega_0 &= x^{\epsilon_1}(1-x)^{-1+\epsilon_2}(1-zx)^{\epsilon_3} \, dx \\ \omega_1 &= x^{\epsilon_1}(1-x)^{\epsilon_2}(1-zx)^{-1+\epsilon_3} \, dx \end{aligned}$$

With the two contours $\gamma_0 = [0, 1]$ and $\gamma_1 = [0, 1/z]$, we have $P_{ss} \int_{\gamma_i} \omega_j \sim \delta_{ij}$.

Similar for half-integer arguments.

Master formula for Appell F_1

Family of integrands for F_1.

$$\omega(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = x^{\alpha_1}(1-x)^{\alpha_2}(1-z_1x)^{\alpha_3}(1-z_2x)^{\alpha_4}\, dx$$

where $\alpha_i = n_i + \epsilon_i$ and $n_i \in \mathbb{Z}$.

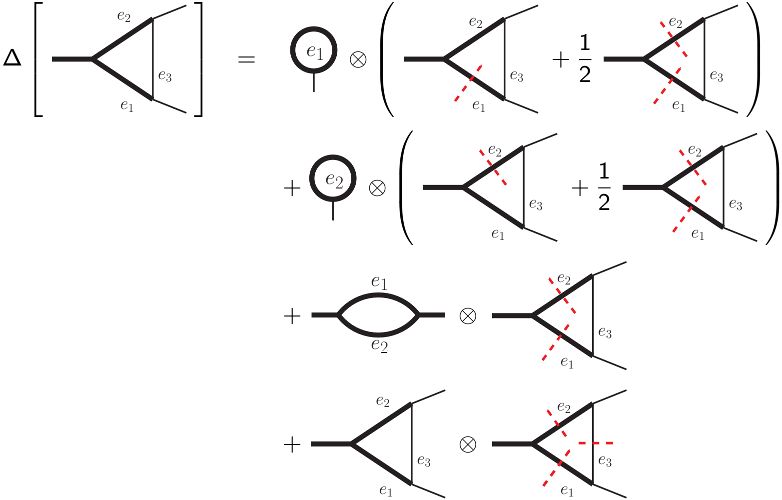
$$\int_0^1 \omega(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2 - \alpha_1)}{\Gamma(\alpha_2)} F_1(\alpha_1, \alpha_3, \alpha_4, \alpha_2; z_1, z_2)$$

Master integrands:

$$\begin{aligned}
\omega_0 &= x^{\epsilon_1}(1-x)^{-1+\epsilon_2}(1-z_1x)^{\epsilon_3}(1-z_2x)^{\epsilon_4}\, dx \\
\omega_1 &= x^{\epsilon_1}(1-x)^{\epsilon_2}(1-z_1x)^{-1+\epsilon_3}(1-z_2x)^{\epsilon_4}\, dx \\
\omega_2 &= x^{\epsilon_1}(1-x)^{\epsilon_2}(1-z_1x)^{\epsilon_3}(1-z_2x)^{-1+\epsilon_4}\, dx
\end{aligned}$$

Master contours: $\gamma_0 = [0, 1]$, $\gamma_1 = [0, z_1^{-1}]$, $\gamma_2 = [0, z_2^{-1}]$.

Diagrammatic example with F_1



Master formula for ${}_{p+1}F_p$

Family of integrands for ${}_3F_2$.

$$\omega(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = x^{\alpha_1}(1-x)^{\alpha_2}y^{\alpha_3}(1-y)^{\alpha_4}(1-zxy)^{\alpha_5}\,dx\wedge dy$$

where $\alpha_i = n_i + \epsilon_i$ and $n_i \in \mathbb{Z}$.
Then

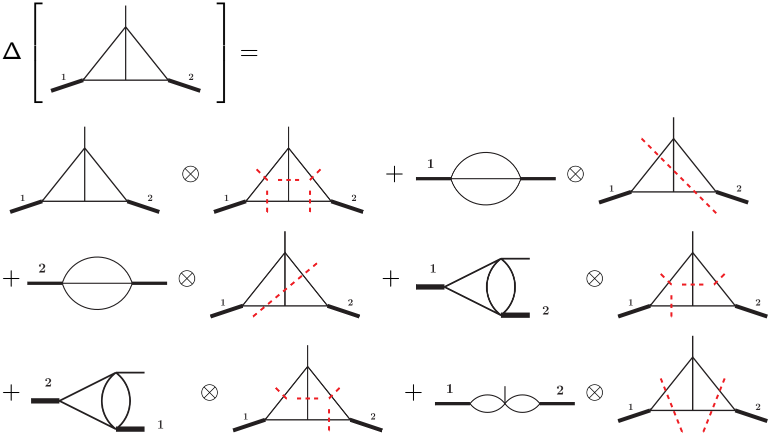
$$\int_0^1\int_0^1 \omega(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = \frac{\Gamma()\Gamma()\Gamma()\Gamma()}{\Gamma()\Gamma()}\,{}_3F_2(\alpha_1+1,\alpha_3+1,-\alpha_5;2+\alpha_1+\alpha_2,2+\alpha_3+\alpha_4;z)$$

Basis of master integrands:

$$\begin{aligned}
\omega_0 &= x^{\epsilon_1}(1-x)^{-1+\epsilon_2}y^{\epsilon_3}(1-y)^{-1+\epsilon_4}(1-zxy)^{\epsilon_5}\,dx\wedge dy\\
\omega_1 &= x^{\epsilon_1}(1-x)^{-1+\epsilon_2}y^{\epsilon_3}(1-y)^{\epsilon_4}(1-zxy)^{-1+\epsilon_5}\,dx\wedge dy\\
\omega_2 &= x^{\epsilon_1}(1-x)^{\epsilon_2}y^{\epsilon_3}(1-y)^{-1+\epsilon_4}(1-zxy)^{-1+\epsilon_5}\,dx\wedge dy
\end{aligned}$$

With the master contours $\gamma_0 = (0,1)^2$, $\gamma_1 = (0,1)\times(0,\frac{1}{zx})$,
$\gamma_2 = (0,\frac{1}{zy})\times(0,1)$, we find that $P_{ss}\int_{\gamma_i}\omega_j \sim \delta_{ij}$

Diagrammatic example with ${}_3F_2$

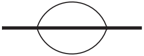


(with various prefactors inserted; terms are of uniform weight)

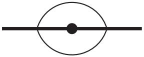
Features of diagrammatic coaction at two loops

Matrix of integrands and contours for *each topology*.

Example: sunrise with one internal mass. 2 master integrands in top topology.



$$= \int_{\Gamma_\emptyset} \omega_{111} \sim {}_2F_1\left(1+2\epsilon, 1+\epsilon, 1-\epsilon, p^2/m^2\right)$$



$$= \int_{\Gamma_\emptyset} \omega_{121} \sim {}_2F_1\left(2+2\epsilon, 1+\epsilon, 1-\epsilon, p^2/m^2\right)$$

For each, only two of the generalized cuts are linearly independent!

Thus 2 independent integration contours, e.g. $\Gamma_\emptyset$ and Γ_{123}.

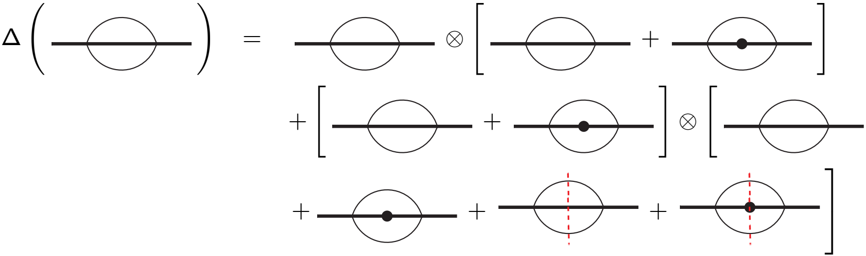
Diagonalize the matrix: $\int_{\gamma_i} \omega_j \sim \delta_{ij}$ with

$$\omega_1 = a\epsilon^2 \omega_{111}, \qquad \omega_2 = b\epsilon\omega_{111} + c\epsilon\omega_{121}$$

$$\gamma_1 = \Gamma_\emptyset, \qquad \gamma_2 = -\frac{1}{6\epsilon}\Gamma_{123} + \frac{2}{3}\Gamma_\emptyset$$

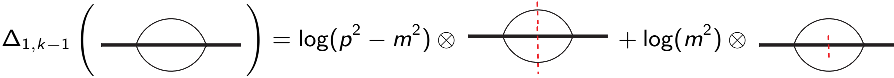
Coaction $\Delta\left(\int_\gamma \omega\right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$ is expressible in terms of diagrams.

For example:



(with prefactors as seen on previous slide)

In particular, we can recover weight 1 discontinuities:



Master formula for Appell hypergeometric functions

Choose dlog forms for a basis of master integrals.

Another view of Appell F_1:
Master integrands:

$$
\begin{aligned}
\varphi \quad &\equiv x^{a\epsilon}(1-x)^{b\epsilon}(1-z_1x)^{c\epsilon}(1-z_2x)^{d\epsilon} \\
\omega_0 \quad &= x^{a\epsilon}(1-x)^{-1+b\epsilon}(1-z_1x)^{c\epsilon}(1-z_2x)^{d\epsilon} &&= -d\log(1-x)\,\varphi \\
\omega_1 \quad &= x^{a\epsilon}(1-x)^{b\epsilon}(1-z_1x)^{-1+c\epsilon}(1-z_2x)^{d\epsilon} &&= -\frac{1}{z_1}d\log(1-z_1x)\,\varphi \\
\omega_2 \quad &= x^{a\epsilon}(1-x)^{b\epsilon}(1-z_1x)^{c\epsilon}(1-z_2x)^{-1+d\epsilon} &&= -\frac{1}{z_2}d\log(1-z_2x)\,\varphi
\end{aligned}
$$

Master contours: $\gamma_0 = (0,1)$, $\gamma_1 = (0, z_1^{-1})$, $\gamma_2 = (0, z_2^{-1})$.

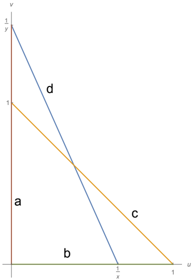
[cf. talks by He and Mizera]

Master formula for Appell hypergeometric functions

Double integral representation of Appell F_1:

$$F_1(\alpha, \beta, \beta', \gamma; x, y) =$$
$$\frac{\Gamma()}{\Gamma()\Gamma()\Gamma()} \int_0^1 dv \int_0^{1-v} du\, u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} (1-ux-vy)^{-\alpha}$$

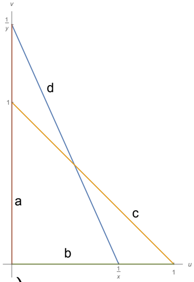
$$\varphi = u^{a\epsilon} v^{b\epsilon} (1-u-v)^{c\epsilon} (1-ux-vy)^{d\epsilon}$$



Combinatorial treatment.

Master formula for Appell hypergeometric functions

Basis of integration regions:
γ_{abc} bounded by lines a, b, c
γ_{abd} bounded by lines a, b, d
γ_{bcd} bounded by lines b, c, d



Basis of integrands:
$\omega_{ac} = d\log(1-u-v) \wedge d\log u$
$\omega_{ad} = d\log(1-ux-vy) \wedge d\log u$
$\omega_{cd} = d\log(1-u-v) \wedge d\log(1-ux-vy)$

Read off the duality matrix by intersections (residues).

$P_{ss}\int_\gamma \omega$	ac	ad	cd
abc	$\frac{1}{ac\epsilon^2}$	0	0
abd	0	$\frac{1}{ad\epsilon^2}$	0
bcd	0	0	$\frac{1}{cd\epsilon^2}$

Find a choice of dlog forms that naturally diagonalize the matrix, or else diagonalize a posteriori.

Master formula for Appell hypergeometric functions

Appell F_2: $\varphi = u^{a\epsilon} v^{b\epsilon} (1-u)^{c\epsilon} (1-v)^{d\epsilon} (1-ux-vy)^{e\epsilon}$

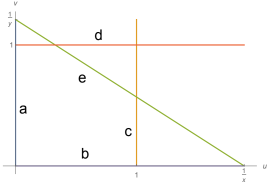
Basis of integrands: [cf. Goto]

$\omega_{ab} = d\log u \wedge d\log v$

$\omega_{bc} = d\log(1-u) \wedge d\log v$

$\omega_{cd} = d\log(1-u) \wedge d\log(1-v)$

$\omega_{da} = d\log u \wedge d\log(1-v)$



Basis of integration regions: $\gamma_{abe}, \gamma_{bce}, \gamma_{cde}, \gamma_{ade}$

Read off the duality matrix by intersections (residues).

$P_{ss} \int_\gamma \omega$	ab	bc	cd	da
abe	$\frac{1}{ab\epsilon^2}$	0	0	0
bce	0	$\frac{1}{bc\epsilon^2}$	0	0
cde	0	0	$\frac{1}{cd\epsilon^2}$	0
ade	0	0	0	$\frac{1}{ad\epsilon^2}$

More generally, degenerate arrangements require blowups.

Summary & Outlook

- We observe a Hopf algebra structure on Feynman diagrams. At 1 loop, there is a basis for which the coaction on MPLs is simply related to pinches and cuts of the original diagram. Beyond 1-loop: encounter matrix equations (cf. higher-order differential equations).
- Dimensional regularization works well. Deep mechanisms exist for cancelling divergences.
- Cuts should be understood through homology and Leray residues. Found 1-loop parametrization for computing cut integrals. Cuts at infinity carry physical meaning.
- Abstracted master formula: a Hopf algebra based on matched pairs of integrands and contours. Applications to generalized hypergeometric functions.
- To explore further: systematic description beyond 1 loop, full range of hypergeometric functions, applications to integral and amplitude computations.

Application: cuts and discontinuities

$$\Delta\ \text{Disc} = (\text{Disc} \otimes 1)\, \Delta$$

$$\Delta\, (\text{Disc}\, I_n) = (\text{Disc} \otimes 1)\ (\Delta\, I_n)$$

Since $\Delta\,(\text{Disc}\, I_n) = 1 \otimes (\text{Disc}\, I_n) + \cdots$, it is enough to look at the terms $\Delta_{1,w-1} I_n$.

The basis integrals of weight 1 are precisely the tadpoles and bubbles. The corresponding cut diagrams have 1 or 2 propagators cut.

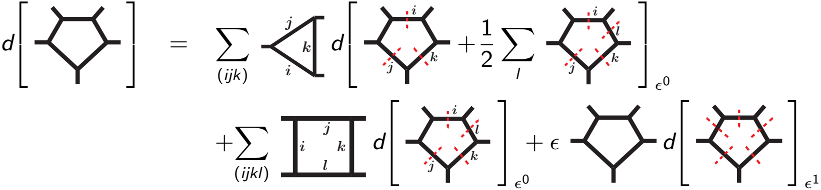
Therefore: the discontinuities are precisely the unitarity cut diagrams (momentum invariant discontinuities) and the single-cut diagrams (mass discontinuities).

Generalized cuts can be interpreted as well. Steinmann relations follow.

Application: differential equations

$$\Delta\,\partial = (1 \otimes \partial)\,\Delta$$

Likewise, we get differential equations by focusing on nearly-maximal cuts in the second factor:



This also shows a way to identify the symbol alphabet.