

Cuts and coaction of Feynman integrals

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- There is a **coaction on Feynman diagrams**.
- At 1 loop, there is a basis for which the coaction is simply related to pinches and cuts of the original diagram and corresponds to Goncharov's Hopf algebra on MPLs.
- Natural interplay with discontinuities and differential equations is potentially useful for computation.
- Larger framework: coactions of the form

$$\Delta \left(\int_{\gamma} \omega \right) = \sum_i \int_{\gamma} \omega_i \otimes \int_{\gamma_i} \omega$$

Coaction operation

$$\Delta \left[\begin{array}{c} \text{---} 1 \text{---} \\ \text{---} e_2 \text{---} \diagup \quad \diagdown \text{---} 2 \text{---} \\ \text{---} e_3 \text{---} \text{---} \\ \text{---} e_1 \text{---} \diagdown \quad \diagup \text{---} 3 \text{---} \end{array} \right] = \begin{array}{c} \text{---} 1 \text{---} \text{---} e_2 \text{---} \text{---} 1 \text{---} \\ \text{---} e_1 \text{---} \end{array} \otimes \begin{array}{c} \text{---} 1 \text{---} \text{---} e_2 \text{---} \diagup \quad \diagdown \text{---} 2 \text{---} \\ \text{---} e_1 \text{---} \text{---} e_3 \text{---} \text{---} \\ \text{---} \text{---} \end{array} + \begin{array}{c} \text{---} 2 \text{---} \text{---} e_3 \text{---} \text{---} 2 \text{---} \\ \text{---} e_2 \text{---} \end{array} \otimes \begin{array}{c} \text{---} 1 \text{---} \text{---} e_2 \text{---} \diagup \quad \diagdown \text{---} 2 \text{---} \\ \text{---} e_1 \text{---} \text{---} e_3 \text{---} \text{---} \\ \text{---} \text{---} \end{array} \\ + \begin{array}{c} \text{---} 3 \text{---} \text{---} e_3 \text{---} \text{---} 3 \text{---} \\ \text{---} e_1 \text{---} \end{array} \otimes \begin{array}{c} \text{---} 1 \text{---} \text{---} e_2 \text{---} \diagup \quad \diagdown \text{---} 2 \text{---} \\ \text{---} e_1 \text{---} \text{---} e_3 \text{---} \text{---} \\ \text{---} \text{---} \end{array} + \begin{array}{c} \text{---} 1 \text{---} \text{---} e_2 \text{---} \diagup \quad \diagdown \text{---} 2 \text{---} \\ \text{---} e_1 \text{---} \text{---} e_3 \text{---} \text{---} \\ \text{---} \text{---} \end{array} \otimes \begin{array}{c} \text{---} 1 \text{---} \text{---} e_2 \text{---} \diagup \quad \diagdown \text{---} 2 \text{---} \\ \text{---} e_1 \text{---} \text{---} e_3 \text{---} \text{---} \\ \text{---} \text{---} \end{array}$$

Coaction operation

$$\Delta \left[\text{Diagram} \right] = \text{Diagram}_1 \otimes \text{Diagram}_2 + \text{Diagram}_3 \otimes \text{Diagram}_4 + \text{Diagram}_5 \otimes \text{Diagram}_6 + \text{Diagram}_7 \otimes \text{Diagram}_8$$

The diagram on the left is a triangle with a horizontal line labeled '1' on the left, a vertical line labeled '3' on the right, and a diagonal line labeled '2' on the top. Internal edges are labeled e_1 , e_2 , and e_3 .

The four terms on the right are:

- Diagram 1: A loop with a horizontal line labeled '1' on the left and a horizontal line labeled '1' on the right. Internal edges are e_1 (bottom) and e_2 (top).
- Diagram 2: A triangle with a horizontal line labeled '1' on the left, a vertical line labeled '3' on the right, and a diagonal line labeled '2' on the top. Internal edges are e_1 , e_2 , and e_3 . Dashed red lines indicate a cut along the diagonal edge e_2 .
- Diagram 3: A loop with a horizontal line labeled '2' on the left and a horizontal line labeled '2' on the right. Internal edges are e_1 (bottom) and e_3 (top).
- Diagram 4: A triangle with a horizontal line labeled '1' on the left, a vertical line labeled '3' on the right, and a diagonal line labeled '2' on the top. Internal edges are e_1 , e_2 , and e_3 . Dashed red lines indicate a cut along the diagonal edge e_2 .
- Diagram 5: A loop with a horizontal line labeled '3' on the left and a horizontal line labeled '3' on the right. Internal edges are e_1 (bottom) and e_3 (top).
- Diagram 6: A triangle with a horizontal line labeled '1' on the left, a vertical line labeled '3' on the right, and a diagonal line labeled '2' on the top. Internal edges are e_1 , e_2 , and e_3 . Dashed red lines indicate a cut along the diagonal edge e_2 .
- Diagram 7: A triangle with a horizontal line labeled '1' on the left, a vertical line labeled '3' on the right, and a diagonal line labeled '2' on the top. Internal edges are e_1 , e_2 , and e_3 .
- Diagram 8: A triangle with a horizontal line labeled '1' on the left, a vertical line labeled '3' on the right, and a diagonal line labeled '2' on the top. Internal edges are e_1 , e_2 , and e_3 . Dashed red lines indicate a cut along the diagonal edge e_2 .

$$\Delta(\log z) = 1 \otimes \log z + \log z \otimes 1$$

$$\Delta(\log^2 z) = 1 \otimes \log^2 z + 2 \log z \otimes \log z + \log^2 z \otimes 1$$

$$\Delta(\text{Li}_2(z)) = 1 \otimes \text{Li}_2(z) + \text{Li}_2(z) \otimes 1 + \text{Li}_1(z) \otimes \log z$$

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Discontinuities and cuts:

$$\Delta \text{ Disc} = (\text{Disc} \otimes 1) \Delta$$

Differential operators:

$$\Delta \partial = (1 \otimes \partial) \Delta$$

Master formula for coaction on integrals

We conjecture a framework as follows.

Coactions of the following form:

$$\Delta \left(\int_{\gamma} \omega \right) = \sum_i \int_{\gamma} \omega_i \otimes \int_{\gamma_i} \omega$$

with a duality condition

$$P_{ss} \int_{\gamma_i} \omega_j = \delta_{ij}.$$

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$$P_{ss} \int_{\gamma_i} \omega_j = \delta_{ij}.$$

P_{ss} is semi-simple projection (“drop logarithms but not π ”).

The master formula coaction is like inserting a complete set of states (“ ω_i are a set of master integrands for ω ”).

- 1 Diagrammatic coaction
- 2 Cut integrals
- 3 Master formula for coaction on integrals

The “incidence” bialgebra [Joni, Rota]

A simple combinatorial algebra: let $[n] = \{1, 2, \dots, n\}$.

Elements: pairs of nested subsets $S \subseteq T$, where $S \subseteq T \subseteq [n]$.

$\{1\} \subseteq \{1, 2\}$ represented by **1 2**

$\emptyset \subseteq \{1, 2\}$ represented by **1 2**

$\emptyset \subset \emptyset$ represented by *

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Multiplication is a free operation, and the coproduct is defined by

$$\Delta(S \subseteq T) = \sum_{S \subseteq X \subseteq T} (S \subseteq X) \otimes (X \subseteq T).$$

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For example:

$$\Delta(\mathbf{12}) = \mathbf{12} \otimes \mathbf{12} + \mathbf{1} \otimes \mathbf{12} + \mathbf{2} \otimes \mathbf{12} + * \otimes \mathbf{12}$$

$$\Delta(\mathbf{12}) = \mathbf{12} \otimes \mathbf{12} + \mathbf{2} \otimes \mathbf{12}$$

$$\Delta(\mathbf{2}) = \mathbf{2} \otimes \mathbf{2} + * \otimes \mathbf{2}$$

$$\Delta(\mathbf{2}) = \mathbf{2} \otimes \mathbf{2}$$

$$\Delta(S \subseteq S) = (S \subseteq S) \otimes (S \subseteq S)$$

Example of the incidence algebra: edges of graphs

$$\Delta(123) = 123 \otimes 123 + 12 \otimes 123 + 23 \otimes 123 + 13 \otimes 123 \\ + 1 \otimes 123 + 2 \otimes 123 + 3 \otimes 123 + * \otimes 123$$

Pinch and cut *complementary* subsets of edges:

$$\Delta_{\text{Inc}} \left[\begin{array}{c} e_2 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ e_1 \quad e_3 \end{array} \right] = \begin{array}{c} \begin{array}{c} e_2 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ e_1 \quad e_3 \end{array} \otimes \begin{array}{c} e_2 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ e_1 \quad e_3 \end{array} \\ + \begin{array}{c} e_1 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ e_2 \end{array} \otimes \begin{array}{c} e_2 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ e_1 \quad e_3 \end{array} \\ + \begin{array}{c} e_2 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ e_3 \end{array} \otimes \begin{array}{c} e_2 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ e_1 \quad e_3 \end{array} \\ + \begin{array}{c} e_1 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ e_3 \end{array} \otimes \begin{array}{c} e_2 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ e_1 \quad e_3 \end{array} \\ + \begin{array}{c} e_1 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ e_1 \end{array} \otimes \begin{array}{c} e_2 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ e_1 \quad e_3 \end{array} \\ + \begin{array}{c} e_2 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ e_2 \end{array} \otimes \begin{array}{c} e_2 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ e_1 \quad e_3 \end{array} \\ + \begin{array}{c} e_3 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ e_3 \end{array} \otimes \begin{array}{c} e_2 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ e_1 \quad e_3 \end{array} \end{array}$$

Example of the incidence algebra: edges of graphs

Can also start with a cut diagram.

$$\Delta(\mathbf{12}) = \mathbf{12} \otimes \mathbf{12} + \mathbf{1} \otimes \mathbf{12}$$

$$\Delta_{\text{Inc}} \left[\begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \\ \text{---} \text{---} \\ \text{---} \text{---} \end{array} \right] = \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \\ \text{---} \text{---} \\ \text{---} \text{---} \end{array} \otimes \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \\ \text{---} \text{---} \\ \text{---} \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \otimes \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \\ \text{---} \text{---} \\ \text{---} \text{---} \end{array}$$

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Multiple polylogarithms (MPL)

A large class of iterated integrals are described by multiple polylogarithms:

$$G(a_1, \dots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \dots, a_n; t)$$

Examples:

$$G(0; z) = \log z, \quad G(a; z) = \log \left(1 - \frac{z}{a} \right)$$

$$G(\vec{a}_n; z) = \frac{1}{n!} \log^n \left(1 - \frac{z}{a} \right), \quad G(\vec{0}_{n-1}, a; z) = -\text{Li}_n \left(\frac{z}{a} \right)$$

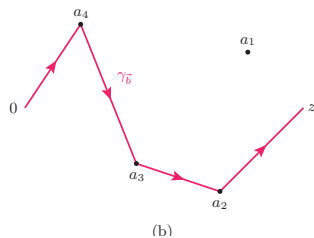
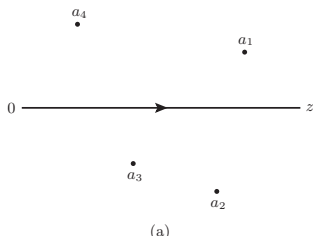
n is the *transcendental weight*.

MPLs obey shuffle product relations. There is a coaction on MPLs, graded by weight, which thus breaks MPLs into simpler functions (lower weight). [Goncharov; Duhr]

Contour integrals

The coaction is a pairing of contours and integrands. Recalls the incidence algebra.

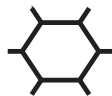
$$\Delta_{\text{MPL}}(G(\vec{a}; z)) = \sum_{\vec{b} \subseteq \vec{a}} G(\vec{b}; z) \otimes G_{\vec{b}}(\vec{a}; z)$$



Contour (b) encircles a subset of residues in a given order.

A useful **basis** for all 1-loop integrals:

$$J_n = \frac{ie^{\gamma_E \epsilon}}{\pi^{D_n/2}} \int d^{D_n} k \prod_{j=1}^n \frac{1}{(k - q_j)^2 - m_j^2}$$



- k is the loop momentum
- q_j are sums of external momenta, m_j are internal masses
- Dimensions:

$$D_n = \begin{cases} n - 2\epsilon, & \text{for } n \text{ even,} \\ n + 1 - 2\epsilon, & \text{for } n \text{ odd.} \end{cases}$$

e.g. tadpoles and bubbles in $2 - 2\epsilon$ dimensions,
triangles and boxes in $4 - 2\epsilon$ dimensions, etc.

- Each J_n has uniform transcendent weight.

2 equivalent Hopf algebras

The **combinatorial** algebra agrees with the Hopf algebra on the **MPL** of evaluated diagrams!

- The graph with n edges is interpreted as J_n , i.e. in D_n dimensions, no numerator.
- Need to insert extra terms in the diagrammatic equation:

$$\begin{aligned} \Delta \left[\text{Diagram with edges } e_1 \text{ and } e_2 \right] &= \text{Diagram with edges } e_1 \text{ and } e_2 \otimes \text{Diagram with edges } e_1 \text{ and } e_2 \text{ (cut)} \\ &+ \text{Diagram with edge } e_1 \text{ (loop)} \otimes \left(\text{Diagram with edges } e_1 \text{ and } e_2 \text{ (cut)} + \frac{1}{2} \text{Diagram with edges } e_1 \text{ and } e_2 \text{ (cut)} \right) \\ &+ \text{Diagram with edge } e_2 \text{ (loop)} \otimes \left(\text{Diagram with edges } e_1 \text{ and } e_2 \text{ (cut)} + \frac{1}{2} \text{Diagram with edges } e_1 \text{ and } e_2 \text{ (cut)} \right) \end{aligned}$$

Isomorphic to the more basic construction. (For any value of 1/2.)

- How do we evaluate the cut graphs?

[related work: Brown; Bloch and Kreimer]

- 1 Diagrammatic coaction
- 2 Cut integrals**
- 3 Master formula for coaction on integrals

Generalized cuts as residues and determinants

1-loop cuts defined as **residues**:

$$\mathcal{C}_C[I_n] = \frac{e^{\gamma_E \epsilon}}{i\pi^{\frac{D}{2}}} \int_{\Gamma_C} d^D k \prod_{j \notin C} \frac{1}{(k - q_j)^2 - m_j^2 + i0} \quad \text{mod } i\pi,$$

- C is the set of cut propagators
- Contour Γ_C encircles poles of cut propagators

Cut integrals give **discontinuities** of their uncut counterparts.

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Some results:

$$\mathcal{C}_C I_n = \frac{e^{\gamma_E \epsilon}}{\sqrt{Y_C}} \left(\frac{Y_C}{G_C} \right)^{(D-c)/2} \int \frac{d\Omega_{D-c+1}}{i\pi^{D/2}} \left[\prod_{j \notin C} \frac{1}{(k - q_j)^2 - m_j^2} \right]_C$$

where we often find Gram and modified Cayley determinants:

$$G_C = \det(q_i \cdot q_j)_{i,j \in C \setminus *}$$

$$Y_C = \det \left(\frac{1}{2}(m_i^2 + m_j^2 + (q_i - q_j)^2) \right)_{i,j \in C}$$

Maximal and next-to-maximal cuts

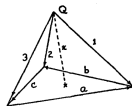
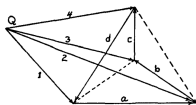
Some special cases:

$$\begin{aligned}C_{2k}[J_{2k}] &= \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{e^{\gamma_E \epsilon}}{\sqrt{Y_{2k}}} \left(\frac{Y_{2k}}{G_{2k}}\right)^{-\epsilon} \\C_{2k+1}[J_{2k+1}] &= \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon) \sqrt{G_{2k+1}}} \left(\frac{Y_{2k+1}}{G_{2k+1}}\right)^{-\epsilon}.\end{aligned}$$

$$C_{2k-1}[J_{2k}] = -\frac{\Gamma(1-2\epsilon)^2}{\Gamma(1-\epsilon)^3} \frac{e^{\gamma_E \epsilon}}{\sqrt{Y_{2k}}} \left(\frac{Y_{2k-1}}{G_{2k-1}}\right)^{-\epsilon} {}_2F_1\left(\frac{1}{2}, -\epsilon; 1-\epsilon; \frac{G_{2k} Y_{2k-1}}{Y_{2k} G_{2k-1}}\right).$$

For more complicated cuts, we set up a Feynman parametrization.

Landau conditions are expressed in [polytope geometry](#): these determinants are volumes of simplices. [Cutkosky]



$$\alpha_i \left[(k^E - q_i^E)^2 + m_i^2 \right] = 0, \quad \forall i.$$

$$\sum_{i=1}^n \alpha_i (k^E - q_i^E) = 0.$$

Therefore

$$\begin{pmatrix} (k^E - q_1^E) \cdot (k^E - q_1^E) & \dots & (k^E - q_1^E) \cdot (k^E - q_c^E) \\ \vdots & \ddots & \vdots \\ (k^E - q_c^E) \cdot (k^E - q_1^E) & \dots & (k^E - q_c^E) \cdot (k^E - q_c^E) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_c \end{pmatrix} = 0.$$

Nontrivial solution $\Leftrightarrow Y_C = 0$ and integral over Γ_C : Landau singularities of the first type.

Second-type singularities come from $G_C = 0$ and integral over $\Gamma_{C \cup \infty}$. Contour is pinched at infinity.

Homology theory for Feynman contours

Homology describes the inequivalent integration contours. Also explains why cuts are discontinuities. Use **Leray residues**. [Fotiadi, Pham; Hwa, Teplitz; Federbusch; Eden, Fairlie, Landshoff, Nuttall, Olive, Polkinghorne,...]

Residues: if

$$\omega = \frac{ds}{s} \wedge \psi + \theta$$

and $S = \{s = 0\}$, while ψ, θ are regular on S , then

$$\text{Res}_S[\omega] = \psi|_S.$$

Cut integrals: if $I_n = \int \omega_n$, then

$$\begin{aligned} C_C[I_n] &= \int_{S_C} \text{Res}_{S_C}[\omega_n] \\ &= (2\pi i)^{-k} \int_{\delta S_C} \omega_n \end{aligned}$$

where δ constructs a “tubular neighborhood” around $S_C = \bigcap_{i \in C} S_i$, the spherical locus of the cut conditions.

Homology theory for Feynman contours

Composed residues, e.g.:

$$\omega = \frac{ds_1}{s_1} \wedge \frac{ds_2}{s_2} \wedge \psi_{12} + \frac{ds_1}{s_1} \wedge \psi_1 + \frac{ds_2}{s_2} \wedge \psi_2 + \theta$$

$$\text{Res}_{S_1 S_2}[\omega] = \psi_{12}|_{S_1 \cap S_2}$$

Residue theorem:

$$\int_{\delta_{S_1 S_2} \sigma} \omega = (2\pi i)^2 \int_{\sigma} \text{Res}_{S_2 S_1}[\omega],$$

where $\delta_{S_1 S_2} \equiv \delta_{S_1} \delta_{S_2}$.

Embedding formalism variables.

$$Z = \begin{bmatrix} z^\mu \\ Z^- \\ Z^+ \end{bmatrix} \in \mathbb{CP}^{D+1}$$

with bilinear form

$$(Z_1 Z_2) = z_1^\mu z_{2\mu} - \frac{1}{2} Z_1^+ Z_2^- - \frac{1}{2} Z_1^- Z_2^+$$

Uncut integrals:

$$I_n^D = \frac{e^{\gamma_E \epsilon}}{\pi^{D/2}} \int_{(\gamma\gamma)=0} \frac{d^{D+2}Y}{\text{Vol}(GL(1))} \frac{(X_\infty Y)^{n-D}}{(X_1 Y) \dots (X_n Y)},$$

where

$$X_i = \begin{bmatrix} (q_i^E)^\mu \\ (q_i^E)^2 + m_i^2 \\ 1 \end{bmatrix} \text{ for } 1 \leq i \leq n, \quad X_\infty = \begin{bmatrix} 0^\mu \\ 1 \\ 0 \end{bmatrix},$$

[Dirac embedding formalism; Fotiadi et al; Simmons-Duffin; Caron-Huot, Henn]

Landau conditions:

$$\det(X_i X_j)_{i,j \in C} = 0$$

$X_\infty \rightarrow$ second type singularities.

$$\det(X_i X_j)_{i,j \in C} = \begin{cases} (-1)^c Y_C, & \text{if } \infty \notin C, \\ \frac{(-1)^{c-1}}{4} G_{C \setminus \{\infty\}}, & \text{if } \infty \in C. \end{cases}$$

Decomposition Theorem

For 1-loop Feynman integrals, the **Decomposition Theorem** shows that the contours $\Gamma_C = \delta S_C$ form a basis. [Fotiadi, Pham]

$$\Gamma_{\infty C} = -2x_C \Gamma_C - \sum_{C \subset X \subseteq [n]} (-1)^{\lceil \frac{|C|}{2} \rceil + \lceil \frac{|X|}{2} \rceil} \Gamma_X, \quad x_C = \begin{cases} 1, & |C| \text{ odd,} \\ 0, & |C| \text{ even.} \end{cases}$$

It follows that

- for $|C|$ even,

$$\mathcal{C}_{\infty C} I_n = \sum_{i \in [n] \setminus C} \mathcal{C}_{Ci} I_n + \sum_{\substack{i, j \in [n] \setminus C \\ i < j}} \mathcal{C}_{Cij} I_n \pmod{i\pi}.$$

- for $|C|$ odd,

$$\mathcal{C}_{\infty C} I_n = -2\mathcal{C}_C I_n - \sum_{i \in [n] \setminus C} \mathcal{C}_{Ci} I_n \pmod{i\pi}.$$

Relations among cut integrals

- Any linear relation among cut/uncut integrals can be cut as a whole (cf. “reverse unitarity”)
- For $D = n$, i.e. $\epsilon = 0$, there is no singularity at $(X_\infty Y) = 0$, so $C_\infty C I_n^n = 0$. If n is even and c is odd, then

$$2C_C I_n^n + \sum_i C_{Ci} I_n^n = 0 \pmod{i\pi}$$

Special case:

$$C_{n-1} I_n^n = -\frac{1}{2} C_n I_n^n + \mathcal{O}(\epsilon)$$

- By examining Disc_∞ , we see that

$$\begin{aligned} \sum_i C_i I_n^D + \sum_{i < j} C_{ij} I_n^D &= C_\infty I_n^D \\ &= -\epsilon I_n^D \pmod{i\pi} \end{aligned}$$

Parametrization and evaluation

Both cut and uncut integrals are **unified** in the following class of functions:

$$Q_n^D(X_1, \dots, X_n, X_0) = \frac{e^{\gamma_E \epsilon}}{\pi^{D/2}} \int_{(Y\bar{Y})=0} \frac{d^{D+2}Y}{\text{Vol}(GL(1))} \frac{(X_0 Y)^{n-D}}{(X_1 Y) \dots (X_n Y)},$$

Uncut: $I_n^D(X_1, \dots, X_n) = Q_n^D(X_1, \dots, X_n, X_\infty)$

Cut: $C_c I_n^D = \frac{1}{\sqrt{Y_c}} Q_{n-c}^{D-c}(X'_{C,c+1}, \dots, X'_{C,n}, X'_{C,\infty})$

where $X'_{C,i}$ are projections of X_i onto the cut locus,

$$X'_{C,i} = \frac{1}{Y_c} \det \begin{pmatrix} (X_1 X_1) & \dots & (X_1 X_c) & X_1 \\ \vdots & & \vdots & \vdots \\ (X_c X_1) & \dots & (X_c X_c) & X_c \\ (X_i X_1) & \dots & (X_i X_c) & X_i \end{pmatrix}.$$

Parametrization and evaluation

Feynman parameters for cut integrals:

$$Q_n^D(X_1, \dots, X_n, X_0) = \frac{e^{\gamma_E \epsilon}}{\pi^{D/2}} \frac{\Gamma(D)}{\Gamma(D-n)} \int [d\mathbf{a}] \mathbf{a}_0^{D-n-1} (\xi\xi)^{-D/2}, \quad (1)$$

where

$$\xi \equiv \mathbf{a}_0 X_0 + \dots + \mathbf{a}_n X_n, \quad \int [d\mathbf{a}] \equiv \int_0^\infty d\mathbf{a}_0 \cdots \int_0^\infty d\mathbf{a}_n \delta(1 - h(\mathbf{a})), \quad (2)$$

and $h(\mathbf{a}) = \sum_{i=0}^n h_i \mathbf{a}_i$ such that $h_i \geq 0$.

[see also: Arkani-Hamed and Yuan]

Statement of the graphical conjecture

The coaction on 1-loop graphs defined by pinching and cutting subsets of propagators,

when evaluated by Feynman rules,
if expanded order by order in ϵ ,

agrees with the coaction on MPLs.

Examples of the graphical conjecture

$$\Delta_{\text{MPL}} \left[\text{circle}(e) \right] = \text{circle}(e) \otimes \text{circle}(e) \text{ with red dashed line} .$$

$$\text{circle}(e) = - \frac{e^{\gamma E \epsilon} \Gamma(1 + \epsilon) (m^2)^{-\epsilon}}{\epsilon}$$

$$\text{circle}(e) \text{ with red dashed line} = \frac{e^{\gamma E \epsilon} (-m^2)^{-\epsilon}}{\Gamma(1 - \epsilon)} .$$

$$\Delta_{\text{MPL}} \left[(m^2)^{-\epsilon} \right] = (m^2)^{-\epsilon} \otimes (m^2)^{-\epsilon} ,$$

$$\Delta_{\text{MPL}} \left[e^{\gamma E \epsilon} \Gamma(1 + \epsilon) \right] = (e^{\gamma E \epsilon} \Gamma(1 + \epsilon)) \otimes \frac{e^{\gamma E \epsilon}}{\Gamma(1 - \epsilon)} ,$$

Examples of the graphical conjecture

$$\Delta \left[\text{Diagram} \right] = \text{Diagram} \otimes \text{Diagram} + \text{Diagram} \otimes \left(\text{Diagram} + \frac{1}{2} \text{Diagram} \right) + \text{Diagram} \otimes \left(\text{Diagram} + \frac{1}{2} \text{Diagram} \right)$$

The diagram on the left is a lens-shaped graph with two vertices and two edges, labeled e_1 and e_2 . The first term on the right is the tensor product of two such lens-shaped graphs. The second term is the tensor product of a circle with edge e_1 and a lens-shaped graph with red dashed lines on edges e_1 and e_2 . The third term is the tensor product of a circle with edge e_2 and a lens-shaped graph with red dashed lines on edges e_1 and e_2 .

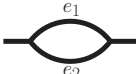


$$\begin{aligned} \Delta \left(\int_{\Gamma_\emptyset} \omega_{12} \right) &= \int_{\Gamma_\emptyset} \omega_{12} \otimes \int_{\Gamma_{12}} \omega_{12} + \int_{\Gamma_\emptyset} \omega_1 \otimes \left(\int_{\Gamma_1} \omega_{12} + \frac{1}{2} \int_{\Gamma_{12}} \omega_{12} \right) + \dots \\ &= \int_{\Gamma_\emptyset} \omega_{12} \otimes \int_{\Gamma_{12}} \omega_{12} + \int_{\Gamma_\emptyset} \omega_1 \otimes \int_{-\frac{1}{2}\Gamma_{1\infty}} \omega_{12} + \int_{\Gamma_\emptyset} \omega_2 \otimes \int_{-\frac{1}{2}\Gamma_{2\infty}} \omega_{12} \end{aligned}$$

The odd-shaped integrals have poles at infinity.

Examples of the graphical conjecture

$$\begin{aligned} \Delta \left(\int_{\Gamma_\emptyset} \omega_{12} \right) &= \int_{\Gamma_\emptyset} \omega_{12} \otimes \int_{\Gamma_{12}} \omega_{12} + \int \omega_1 \otimes \left(\int_{\Gamma_1} \omega_{12} + \frac{1}{2} \int_{\Gamma_{12}} \omega_{12} \right) + \dots \\ &= \int_{\Gamma_\emptyset} \omega_{12} \otimes \int_{\Gamma_{12}} \omega_{12} + \int_{\Gamma_\emptyset} \omega_1 \otimes \int_{-\frac{1}{2}\Gamma_{1\infty}} \omega_{12} + \int_{\Gamma_\emptyset} \omega_2 \otimes \int_{-\frac{1}{2}\Gamma_{2\infty}} \omega_{12} \end{aligned}$$

Basis of integrands and corresponding contours:

			
$\omega_j :$	ω_{12}	ω_1	ω_2
$\gamma_j :$	Γ_{12} \mathcal{C}_{12}	$-\frac{1}{2}\Gamma_{1\infty}$ $\mathcal{C}_1 + \frac{1}{2}\mathcal{C}_{12}$	$-\frac{1}{2}\Gamma_{2\infty}$ $\mathcal{C}_2 + \frac{1}{2}\mathcal{C}_{12}$

They satisfy

$$\int_{\gamma_i} \omega_j \sim \delta_{ij} \quad \text{after dropping logs}$$

- 1 Diagrammatic coaction
- 2 Cut integrals
- 3 Master formula for coaction on integrals**

The master formula for the ${}_2F_1$ family

Consider the diagrammatic coaction

$$\Delta \left[\text{Diagram} \right] = \text{Diagram} \otimes \left(\text{Diagram} + \frac{1}{2} \text{Diagram} \right) + \text{Diagram} \otimes \text{Diagram}$$

The diagrammatic coaction is represented as follows:

- The left side is the coaction Δ applied to a diagram with a horizontal line labeled '1' on the left, which splits into two lines labeled 'e₂' (top) and 'e₁' (bottom). These lines meet a vertical line labeled 'e₃' on the right.
- The right side is the tensor product of a diagram with a circle labeled 'e₁' and a vertical line, and a sum of two diagrams. The first diagram in the sum is the same as the left diagram but with dashed red lines on the 'e₁' line. The second diagram is the same as the first but with a factor of $\frac{1}{2}$.
- The bottom part of the equation shows a diagram with a horizontal line labeled '1' on the left, a curved line labeled 'e₁' above it, and another horizontal line labeled '1' on the right, with a vertical line labeled 'e₂' below it. This is tensor product with a diagram similar to the one in the top part but with dashed red lines on the 'e₁' line.

There is a coaction on ${}_2F_1$ that gives

$$\begin{aligned} \Delta {}_2F_1(1, 1 + \epsilon, 2 - \epsilon, x) &= {}_2F_1(1, \epsilon, 1 - \epsilon, x) \otimes {}_2F_1(1, 1 + \epsilon, 2 - \epsilon, x) \\ &\quad + {}_2F_1(1, 1 + \epsilon, 2 - \epsilon, x) \otimes {}_2F_1\left(1, \epsilon, 1 - \epsilon, \frac{1}{x}\right) \end{aligned}$$

without expanding in ϵ !

Master formula for Hopf algebra on integrals

Coaction of the form

$$\Delta\left(\int_{\gamma} \omega\right) = \sum_i \int_{\gamma} \omega_i \otimes \int_{\gamma_i} \omega$$

with a duality condition

$$P_{ss} \int_{\gamma_i} \omega_j = \delta_{ij}.$$

P_{ss} is semi-simple projection (“drop logarithms but not π ”).

To be precise, P_{ss} projects onto the space of semi-simple numbers x satisfying $\Delta(x) = x \otimes 1$.

The master formula for the ${}_2F_1$ family

Consider the family of integrands

$$\omega(\alpha_1, \alpha_2, \alpha_3) = x^{\alpha_1}(1-x)^{\alpha_2}(1-zx)^{\alpha_3} dx$$

where $\alpha_i = n_i + \epsilon_i$ and $n_i \in \mathbb{Z}$.

$$\int_0^1 \omega(\alpha_1, \alpha_2, \alpha_3) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2 - \alpha_1)}{\Gamma(\alpha_2)} {}_2F_1(-\alpha_3, \alpha_1 + 1; \alpha_2 + \alpha_1 + 2; z)$$

Basis of master integrands:

$$\int_0^1 \omega = c_0 \int_0^1 \omega_0 + c_1 \int_0^1 \omega_1$$

where

$$\omega_0 = x^{\epsilon_1}(1-x)^{-1+\epsilon_2}(1-zx)^{\epsilon_3} dx$$

$$\omega_1 = x^{\epsilon_1}(1-x)^{\epsilon_2}(1-zx)^{-1+\epsilon_3} dx$$

With the two contours $\gamma_0 = [0, 1]$ and $\gamma_1 = [0, 1/z]$, we have $P_{ss} \int_{\gamma_i} \omega_j \sim \delta_{ij}$.

Similar for half-integer arguments.

Master formula for Appell F_1

Family of integrands for F_1 .

$$\omega(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = x^{\alpha_1}(1-x)^{\alpha_2}(1-z_1x)^{\alpha_3}(1-z_2x)^{\alpha_4} dx$$

where $\alpha_i = n_i + \epsilon_i$ and $n_i \in \mathbb{Z}$.

$$\int_0^1 \omega(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2 - \alpha_1)}{\Gamma(\alpha_2)} F_1(\alpha_1, \alpha_3, \alpha_4, \alpha_2; z_1, z_2)$$

Master integrands:

$$\omega_0 = x^{\epsilon_1}(1-x)^{-1+\epsilon_2}(1-z_1x)^{\epsilon_3}(1-z_2x)^{\epsilon_4} dx$$

$$\omega_1 = x^{\epsilon_1}(1-x)^{\epsilon_2}(1-z_1x)^{-1+\epsilon_3}(1-z_2x)^{\epsilon_4} dx$$

$$\omega_2 = x^{\epsilon_1}(1-x)^{\epsilon_2}(1-z_1x)^{\epsilon_3}(1-z_2x)^{-1+\epsilon_4} dx$$

Master contours: $\gamma_0 = [0, 1]$, $\gamma_1 = [0, z_1^{-1}]$, $\gamma_2 = [0, z_2^{-1}]$.

Diagrammatic example with F_1

$$\Delta \left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} e_2 \\ e_3 \\ e_1 \end{array} \right] = \begin{array}{l} \textcircled{e_1} \otimes \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} e_2 \\ e_3 \\ e_1 \end{array} + \frac{1}{2} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} e_2 \\ e_3 \\ e_1 \end{array} \right) \\ + \textcircled{e_2} \otimes \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} e_2 \\ e_3 \\ e_1 \end{array} + \frac{1}{2} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} e_2 \\ e_3 \\ e_1 \end{array} \right) \\ + \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} e_1 \\ e_2 \end{array} \otimes \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} e_2 \\ e_3 \\ e_1 \end{array} \\ + \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} e_2 \\ e_3 \\ e_1 \end{array} \otimes \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} e_2 \\ e_3 \\ e_1 \end{array} \end{array}$$

The diagrammatic equation shows the expansion of a triangle with external legs e_1, e_2, e_3 into a sum of four terms. The first two terms involve a loop on the left side, with the loop labeled e_1 and e_2 respectively. The third and fourth terms involve a loop on the left side, with the loop labeled e_1 and e_2 respectively. The diagrams are connected by tensor products (\otimes) and sums (+). The first two terms include a factor of $\frac{1}{2}$ for the second diagram in each pair.

Master formula for ${}_{p+1}F_p$

Family of integrands for ${}_3F_2$.

$$\omega(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = x^{\alpha_1}(1-x)^{\alpha_2}y^{\alpha_3}(1-y)^{\alpha_4}(1-zxy)^{\alpha_5} dx \wedge dy$$

where $\alpha_i = n_i + \epsilon_i$ and $n_i \in \mathbb{Z}$.

Then

$$\int_0^1 \int_0^1 \omega(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \frac{\Gamma(\epsilon_1)\Gamma(\epsilon_2)\Gamma(\epsilon_3)\Gamma(\epsilon_4)}{\Gamma(\epsilon_5)\Gamma(\epsilon_6)} {}_3F_2(\alpha_1 + 1, \alpha_3 + 1, -\alpha_5; 2 + \alpha_1 + \alpha_2, 2 + \alpha_3 + \alpha_4; z)$$

Basis of master integrands:

$$\omega_0 = x^{\epsilon_1}(1-x)^{-1+\epsilon_2}y^{\epsilon_3}(1-y)^{-1+\epsilon_4}(1-zxy)^{\epsilon_5} dx \wedge dy$$

$$\omega_1 = x^{\epsilon_1}(1-x)^{-1+\epsilon_2}y^{\epsilon_3}(1-y)^{\epsilon_4}(1-zxy)^{-1+\epsilon_5} dx \wedge dy$$

$$\omega_2 = x^{\epsilon_1}(1-x)^{\epsilon_2}y^{\epsilon_3}(1-y)^{-1+\epsilon_4}(1-zxy)^{-1+\epsilon_5} dx \wedge dy$$

With the master contours $\gamma_0 = (0, 1)^2$, $\gamma_1 = (0, 1) \times (0, \frac{1}{zx})$,

$\gamma_2 = (0, \frac{1}{zy}) \times (0, 1)$, we find that $P_{ss} \int_{\gamma_i} \omega_j \sim \delta_{ij}$

Diagrammatic example with ${}_3F_2$

$$\Delta \left[\begin{array}{c} \text{---} \\ | \\ \text{---} \\ / \quad \backslash \\ \text{---} \\ \backslash \quad / \\ \text{---} \\ | \\ \text{---} \end{array} \right] =$$

The first row contains three terms:

- A triangle with a vertical line and a thick bottom line, multiplied by a triangle with a vertical line and a thick bottom line, with dashed red lines forming a square in the center.
- A circle with a horizontal line, multiplied by a triangle with a vertical line and a thick bottom line, with a dashed red diagonal line from the top-left to the bottom-right.

The second row contains three terms:

- A circle with a horizontal line, multiplied by a triangle with a vertical line and a thick bottom line, with a dashed red diagonal line from the top-right to the bottom-left.
- A lens shape with a horizontal line, multiplied by a triangle with a vertical line and a thick bottom line, with a dashed red diagonal line from the top-left to the bottom-right.

The third row contains three terms:

- A lens shape with a horizontal line, multiplied by a triangle with a vertical line and a thick bottom line, with dashed red lines forming a square in the center.
- A lens shape with a horizontal line, multiplied by a triangle with a vertical line and a thick bottom line, with dashed red lines forming a square in the center.

(with various prefactors inserted; terms are of uniform weight)

Features of diagrammatic coaction at two loops

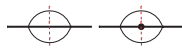
Matrix of integrands and contours for *each topology*.

Example: sunrise with one internal mass. 2 master integrands in top topology.

$$\begin{aligned} \text{---} \bigcirc \text{---} &= \int_{\Gamma_\emptyset} \omega_{111} \sim {}_2F_1 \left(1 + 2\epsilon, 1 + \epsilon, 1 - \epsilon, p^2/m^2 \right) \\ \text{---} \bullet \bigcirc \text{---} &= \int_{\Gamma_\emptyset} \omega_{121} \sim {}_2F_1 \left(2 + 2\epsilon, 1 + \epsilon, 1 - \epsilon, p^2/m^2 \right) \end{aligned}$$

For each, only two of the generalized cuts are linearly independent!

Thus 2 independent integration contours, e.g. Γ_\emptyset and Γ_{123} .



Diagonalize the matrix: $\int_{\gamma_i} \omega_j \sim \delta_{ij}$ with

$$\begin{aligned} \omega_1 &= a\epsilon^2 \omega_{111}, & \omega_2 &= b\epsilon \omega_{111} + c\epsilon \omega_{121} \\ \gamma_1 &= \Gamma_\emptyset, & \gamma_2 &= -\frac{1}{6\epsilon} \Gamma_{123} + \frac{2}{3} \Gamma_\emptyset \end{aligned}$$

Coaction $\Delta \left(\int_\gamma \omega \right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$ is expressible in terms of **diagrams**.

Features of diagrammatic coaction at two loops

For example:

$$\Delta \left(\text{Diagram} \right) = \text{Diagram} \otimes \left[\text{Diagram} + \text{Diagram} \right] + \left[\text{Diagram} + \text{Diagram} \right] \otimes \left[\text{Diagram} \right] + \left[\text{Diagram} + \text{Diagram} + \text{Diagram} \right]$$

The diagrammatic equation shows the coaction Δ on a two-loop diagram (a circle with a horizontal line through its center). The result is a sum of three terms:

- The first term is the original diagram tensorially multiplied by a sum of two diagrams: the original diagram and a diagram with a black dot on the horizontal line.
- The second term is a sum of two diagrams (the original and the one with a dot) tensorially multiplied by the original diagram.
- The third term is a sum of three diagrams: the original with a dot, the original with a vertical dashed red line, and the original with both a dot and a dashed red line.

(with prefactors as seen on previous slide)

In particular, we can recover weight 1 discontinuities:

$$\Delta_{1,k-1} \left(\text{Diagram} \right) = \log(p^2 - m^2) \otimes \text{Diagram} + \log(m^2) \otimes \text{Diagram}$$

The diagrammatic equation shows the weight 1 discontinuity $\Delta_{1,k-1}$ on the two-loop diagram. The result is a sum of two terms:

- The first term is $\log(p^2 - m^2)$ tensorially multiplied by the original diagram with a vertical dashed red line.
- The second term is $\log(m^2)$ tensorially multiplied by the original diagram with a vertical dashed red line.

Master formula for Appell hypergeometric functions

Choose dlog forms for a basis of master integrals.

Another view of Appell F_1 :

Master integrands:

$$\begin{aligned}\varphi &\equiv x^{a\epsilon}(1-x)^{b\epsilon}(1-z_1x)^{c\epsilon}(1-z_2x)^{d\epsilon} \\ \omega_0 &= x^{a\epsilon}(1-x)^{-1+b\epsilon}(1-z_1x)^{c\epsilon}(1-z_2x)^{d\epsilon} = -d \log(1-x) \varphi \\ \omega_1 &= x^{a\epsilon}(1-x)^{b\epsilon}(1-z_1x)^{-1+c\epsilon}(1-z_2x)^{d\epsilon} = -\frac{1}{z_1} d \log(1-z_1x) \varphi \\ \omega_2 &= x^{a\epsilon}(1-x)^{b\epsilon}(1-z_1x)^{c\epsilon}(1-z_2x)^{-1+d\epsilon} = -\frac{1}{z_2} d \log(1-z_2x) \varphi\end{aligned}$$

Master contours: $\gamma_0 = (0, 1)$, $\gamma_1 = (0, z_1^{-1})$, $\gamma_2 = (0, z_2^{-1})$.

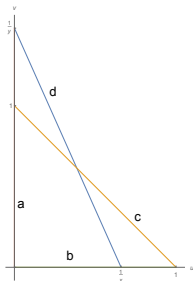
[cf. talks by He and Mizera]

Master formula for Appell hypergeometric functions

Double integral representation of Appell F_1 :

$$F_1(\alpha, \beta, \beta', \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')} \int_0^1 dv \int_0^{1-v} du u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} (1-ux-vy)^{-\alpha}$$

$$\varphi = u^{a\epsilon} v^{b\epsilon} (1-u-v)^{c\epsilon} (1-ux-vy)^{d\epsilon}$$



Combinatorial treatment.

Master formula for Appell hypergeometric functions

Basis of integration regions:

γ_{abc} bounded by lines a, b, c

γ_{abd} bounded by lines a, b, d

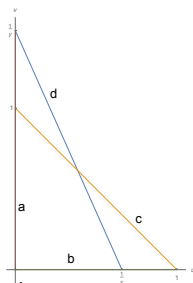
γ_{bcd} bounded by lines b, c, d

Basis of integrands:

$\omega_{ac} = d \log(1 - u - v) \wedge d \log u$

$\omega_{ad} = d \log(1 - ux - vy) \wedge d \log u$

$\omega_{cd} = d \log(1 - u - v) \wedge d \log(1 - ux - vy)$



Read off the duality matrix by intersections (residues).

$P_{ss} \int_{\gamma} \omega$	ac	ad	cd
abc	$\frac{1}{ac\epsilon^2}$	0	0
abd	0	$\frac{1}{ad\epsilon^2}$	0
bcd	0	0	$\frac{1}{cd\epsilon^2}$

Find a choice of $d \log$ forms that naturally diagonalize the matrix, or else diagonalize a posteriori.

Master formula for Appell hypergeometric functions

Appell F_2 : $\varphi = u^{a\epsilon} v^{b\epsilon} (1-u)^{c\epsilon} (1-v)^{d\epsilon} (1-ux-vy)^{e\epsilon}$

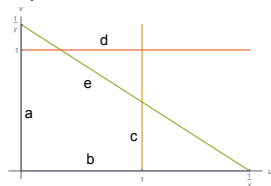
Basis of integrands: [cf. Goto]

$$\omega_{ab} = d \log u \wedge d \log v$$

$$\omega_{bc} = d \log(1-u) \wedge d \log v$$

$$\omega_{cd} = d \log(1-u) \wedge d \log(1-v)$$

$$\omega_{da} = d \log u \wedge d \log(1-v)$$



Basis of integration regions: $\gamma_{abe}, \gamma_{bce}, \gamma_{cde}, \gamma_{ade}$

Read off the duality matrix by intersections (residues).

$P_{ss} \int_{\gamma} \omega$	ab	bc	cd	da
abe	$\frac{1}{abc\epsilon^2}$	0	0	0
bce	0	$\frac{1}{bc\epsilon^2}$	0	0
cde	0	0	$\frac{1}{cd\epsilon^2}$	0
ade	0	0	0	$\frac{1}{ad\epsilon^2}$

More generally, degenerate arrangements require blowups.

Summary & Outlook

- We observe a **Hopf algebra structure on Feynman diagrams**. At 1 loop, there is a basis for which the coaction on MPLs is simply related to **pinches and cuts** of the original diagram. Beyond 1-loop: encounter matrix equations (cf. higher-order differential equations).
- Dimensional regularization works well. Deep mechanisms exist for cancelling divergences.
- Cuts should be understood through homology and Leray residues. Found 1-loop parametrization for computing cut integrals. Cuts at infinity carry physical meaning.
- Abstracted master formula: a Hopf algebra based on matched **pairs of integrands and contours**. Applications to generalized hypergeometric functions.
- To explore further: systematic description beyond 1 loop, full range of hypergeometric functions, applications to integral and amplitude computations.

Application: cuts and discontinuities

$$\Delta \text{ Disc} = (\text{Disc} \otimes 1) \Delta$$

$$\Delta (\text{Disc } I_n) = (\text{Disc} \otimes 1) (\Delta I_n)$$

Since $\Delta (\text{Disc } I_n) = 1 \otimes (\text{Disc } I_n) + \dots$, it is enough to look at the terms $\Delta_{1,w-1} I_n$.

The basis integrals of weight 1 are precisely the tadpoles and bubbles. The corresponding cut diagrams have 1 or 2 propagators cut.

Therefore: the discontinuities are precisely the unitarity cut diagrams (momentum invariant discontinuities) and the single-cut diagrams (mass discontinuities).

Generalized cuts can be interpreted as well. Steinmann relations follow.

Application: differential equations

$$\Delta \partial = (1 \otimes \partial) \Delta$$

Likewise, we get differential equations by focusing on nearly-maximal cuts in the second factor:

$$\begin{aligned}
 d \left[\text{pentagon} \right] &= \sum_{(ijk)} \left[\text{triangle } \begin{matrix} j \\ i \end{matrix} \right] d \left[\text{hexagon } \begin{matrix} i \\ j \end{matrix} \right] + \frac{1}{2} \sum_l \left[\text{hexagon } \begin{matrix} i \\ j \end{matrix} \right]_{\epsilon^0} \\
 &+ \sum_{(ijkl)} \left[\text{rectangle } \begin{matrix} j \\ i \end{matrix} \right] d \left[\text{hexagon } \begin{matrix} i \\ j \end{matrix} \right]_{\epsilon^0} + \epsilon \left[\text{pentagon} \right] d \left[\text{hexagon } \begin{matrix} i \\ j \end{matrix} \right]_{\epsilon^1}
 \end{aligned}$$

This also shows a way to identify the symbol alphabet.