Confinement criteria and the Higgs Mechanism
a distinction between color confinement, and confinement

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Quark Confinement and
the Hadron Spectrum 13

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What is "confinement"?

Suppose we have an SU(N) gauge theory with matter fields in the fundamental representation, e.g. QCD. Wilson loops have perimeter-law falloff asymptotically, Polyakov lines have a non-zero VEV, what does it mean to say such theories (QCD in particular) are confining?

Most people take it to mean “color confinement” or

> **C-confinement**
>
> There are only color neutral particles in the asymptotic spectrum.

The problem with C-confinement is that it also holds true for gauge-Higgs theories, deep in the Higgs regime, where there are

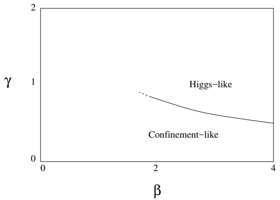
- only Yukawa forces,
- no linearly rising Regge trajectories,
- no color electric flux tubes.

If C-confinement is “confinement,” then the Higgs phase is also confining.

C-confinement in gauge-Higgs theories

How we know this:

1. **Elitzur's** Theorem: No such thing as spontaneous symmetry breaking of a local gauge symmetry.
2. The **Fradkin-Shenker-Osterwalder-Seiler** (FSOS) Theorem: There is no transition in coupling-constant space which isolates the Higgs phase from a confinement-like phase.
3. **Frölich-Morchio-Strocchi** (FMS) and also **'t Hooft** (1980): physical particles (e.g. W's) in the spectrum are created by gauge-invariant operators in the Higgs region.



FMS show how to recover the usual results of perturbation theory, starting from gauge-invariant composite operators.

Conclusion: If the confinement-like (QCD-like) region has a color neutral spectrum, then so does the Higgs-like region.

Beyond C-confinement?

In a pure SU(N) gauge theory there is a different and stronger meaning that can be assigned to the word “confinement," which goes beyond C-confinement.

Of course the spectrum consists only of color neutral objects: glueballs.

But such theories *also* have the property that the static quark potential rises linearly or, equivalently, that large planar Wilson loops have an area-law falloff.

Is there any way to generalize this property to gauge theories with matter in the fundamental representation?

Separation-of-charge (“S_c”) confinement

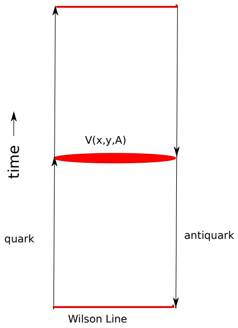
The Wilson area-law criterion for pure gauge theories is equivalent to “S_c-confinement.”

A static $q\overline{q}$ pair, connected by a Wilson line, evolves in Euclidean time to some state

$$\Psi_V \equiv \overline{q}^a(\mathbf{x}) V^{ab}(\mathbf{x},\mathbf{y};A) q^b(\mathbf{y}) \Psi_0$$

where $V(\mathbf{x},\mathbf{y};A)$ is a gauge bi-covariant operator transforming as

$$V^{ab}(\mathbf{x},\mathbf{y};A) \rightarrow g^{ac}(\mathbf{x},t) V^{cd}(\mathbf{x},\mathbf{y};A) g^{\dagger db}(\mathbf{y},t)$$



The energy above the vacuum energy $\mathcal{E}_{vac}$ is

$$E_V(R) = \langle \Psi_V | H | \Psi_V \rangle - \mathcal{E}_{vac}$$

S_c-confinement

means that there exists an asymptotically linear function $E_0(R)$, i.e.

$$\lim_{R\to\infty} \frac{dE_0}{dR} = \sigma > 0$$

such that

$$E_V(R) \geq E_0(R)$$

for ***ANY*** choice of bi-covariant $V(\mathbf{x}, \mathbf{y}; A)$.

For an SU(N) pure gauge theory, $E_0(R)$ is the ground state energy of a static quark-antiquark pair, and σ is the string tension. This is equivalent to the Wilson area-law criterion.

Our proposal: S_c-confinement should also be regarded as the confinement criterion in gauge+matter theories. The crucial element is that the bi-covariant operators $V^{ab}(\mathbf{x}, \mathbf{y}; A)$ *must depend only on the gauge field A* at a fixed time, and not on the matter fields.

The idea is to study the energy $E_V(R)$ of physical states with large separations R of static color charges, *unscreened by matter fields*.

If $V^{ab}(\mathbf{x},\mathbf{y};A)$ would also depend on the matter field(s), then it is easy to violate the S_c-confinement criterion, e.g. let ϕ be a matter field in the fundamental representation, and

$$V^{ab}(\mathbf{x},\mathbf{y},\phi) = \phi^a(\mathbf{x})\phi^{\dagger b}(\mathbf{y})$$

Then

$$\Psi_V = \{\overline{q}^a(\mathbf{x})\phi^a(\mathbf{x})\} \times \{\phi^{\dagger b}(\mathbf{y})q^b(\mathbf{y})\}\Psi_0$$

corresponds to two color singlet (static quark + Higgs) states, only weakly interacting at large separations. Operators V of this kind, which depend on the matter fields, are excluded.

This also means that the lower bound $E_0(R)$, unlike in pure gauge theories, is *not* the lowest energy of a state containing a static quark-antiquark pair.

It is the lowest energy of such states when color screening by matter is excluded.

SU(2) doublet to group element

We consider a unimodular $|\phi| = 1$ Higgs field. In SU(2) the doublet can be mapped to an SU(2) group element

$$\vec{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \Longrightarrow \phi = \begin{bmatrix} \phi_2^* & \phi_1 \\ -\phi_1^* & \phi_2 \end{bmatrix}$$

and the corresponding action is

$$S = \beta \sum_{plaq} \frac{1}{2} \mathrm{Tr}[UUU^\dagger U^\dagger] + \gamma \sum_{x,\mu} \frac{1}{2} \mathrm{Tr}[\phi^\dagger(x) U_\mu(x) \phi(x+\widehat{\mu})]$$

Existence of S_c-confinement

1. Does S_c-confinement exist *anywhere* in the $\beta - \gamma$ phase diagram, apart from pure gauge theory ($\gamma = 0$)?

   Yes. We can show that gauge-Higgs theory is S-confining at least in the region

   $$\gamma \ll \beta \ll 1 \quad \text{and} \quad \gamma \ll \frac{1}{10}$$

   This is based on strong-coupling expansions and a theorem (Gershgorim) in linear algebra.

2. Then does S_c-confinement hold *everywhere* in the $\beta - \gamma$ phase diagram?

   No. We can construct V operators which violate the S_c-confinement criterion when γ is large enough.

So there must exist a transition between S and C confinement.

Strong Coupling: string-breaking takes time

Simple example: V=Wilson line, $\gamma \ll \beta \ll 1$.

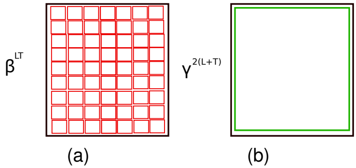
Leading contributions in β, γ are confining and screening:

$$W(L,T) = 2\left(\frac{\beta}{4}\right)^{LT} + 2\left(\frac{\gamma}{4}\right)^{2(L+T)}$$

For times

$$T < T_{break} = 2\frac{\log\gamma}{\log\beta}$$

confinement dominates. Beyond this limit, the string breaks, and we have screening.



For small $T < T_{break}$, $E_V \approx -\log(\beta/4)R$. This is S_c-confinement.

But we have to *prove it for any V*, not just a straight Wilson line.

Outline of the general argument

1. Introduce a *cluster basis* $\{\mathcal{C}\}$ for operators V.
2. Define (from the lattice path integral) a transition matrix $M_T(\mathcal{C}_2, \mathcal{C}_1)$ between initial and final clusters in time T.
3. At small T, diagonal terms $M_T(\mathcal{C}, \mathcal{C})$ are dominated by pure gauge theory, off-diagonal by Higgs (screening). But there are many more off-diagonal elements. Does this imply screening at all T?
4. Compute a bound on the sum of off-diagonal elements, and from that bound show that

   $$\sum_{\mathcal{C}_1 \neq C} |M_T(C, \mathcal{C}_2)| \ll M_T(C, C)$$

   providing

   $$\gamma \ll \beta \ll \frac{1}{10}$$

   Use the Gershgorin Circle Theorem of linear algebra to show in consequence that the largest eigenvalue of M , and hence the lowest possible E_V, is approximately that of pure gauge theory.
5. This last fact implies $\mathrm{S_c}$-confinement.

Loss of S_c-confinement

Away from strong coupling, there is no guarantee of S_c-confinement.

If we can find *even one* V at some β, γ such that E_V does not grow linearly with R, then S_c-confinement is lost at that β, γ.

For $V =$ Wilson line, $E_V(R) \propto R$ even for non-confining theories. Not useful! Instead we consider

1. **The Dirac state**
   generalization of the lowest energy state with static charges in an abelian theory.

2. **Pseudomatter**
   Introduce fields built from the gauge field which transform like matter fields. See if these induce string-breaking.

3. **"Fat link" states**
   Wilson lines built from smoothed links.

The Calculation

In general

$$E_V(R) = -\lim_{t\to 0}\frac{d}{dt}\log\left[\frac{\langle\Psi_V|e^{-Ht}|\Psi_V\rangle}{\langle\Psi_V|\Psi_V\rangle}\right] - \mathcal{E}_{vac}$$

on the lattice

$$E_V(R) = -\log\left[\frac{\left\langle \mathrm{Tr}\left[U_0(x,t)V(x,y,t+1)U_0^\dagger(y,t)V(y,x,t)\right]\right\rangle}{\langle \mathrm{Tr}\left[V(x,y,t)V(y,x,t)\right]\rangle}\right]$$

and we will focus on the SU(2) gauge-Higgs action

$$S = \beta\sum_{plaq}\frac{1}{2}\mathrm{Tr}[UUU^\dagger U^\dagger] + \gamma\sum_{x,\mu}\frac{1}{2}\mathrm{Tr}[\phi^\dagger(x)U_\mu(x)\phi(x+\widehat{\mu})]$$

where ϕ is SU(2) group-valued.

The Dirac state

In an abelian theory, the gauge-invariant ground state with static $\pm$ electric charges is

$$\Psi_{\overline{q}q} = \{\overline{q}(\mathbf{x})G_C^\dagger(\mathbf{x};A)\} \times \{G_C(\mathbf{y};A)q(\mathbf{y})\}\Psi_0$$

where

$$G_C(\mathbf{x};A) = \exp\left[-i\int d^3z\, A_i(z)\partial_i \frac{1}{4\pi|\mathbf{x}-\vec{z}|}\right]$$

$G_C(\mathbf{x},A)$ is the gauge transformation $A \to$ Coulomb gauge. Non-abelian theory: define $V^{ab}(x,y;A) = G_C^{\dagger ac}(\mathbf{x};A)G_C^{cb}(\mathbf{y};A)$ and

$$\begin{aligned}\Psi_V &= \overline{q}^a(\mathbf{x})G_C^{\dagger ac}(\mathbf{x};A)G_C^{cb}(\mathbf{y};A)q^b(\mathbf{y})\Psi_0 \\ &= \overline{q}^c(\mathbf{x})q^c(\mathbf{y})\Psi_0 \quad \text{in Coulomb gauge}\end{aligned}$$

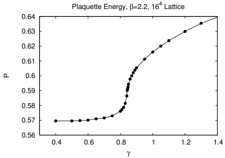
then compute in Coulomb gauge

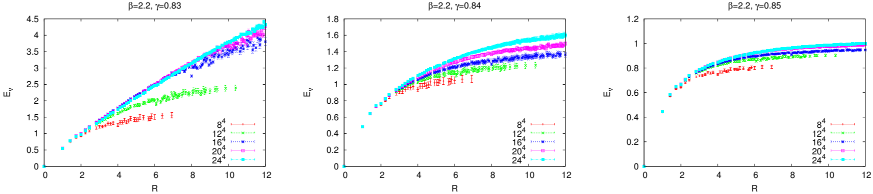
$$E_V(R) = -\log\left\langle \frac{1}{N}\mathrm{Tr}[U_0(\mathbf{0},0)U_0^\dagger(\mathbf{R},0)]\right\rangle$$

by lattice Monte Carlo.

$E_V(R)$ in the Dirac state

There is a sharp thermodynamic crossover in the SU(2) gauge model at $\beta = 2.2, \gamma \approx 0.84$.



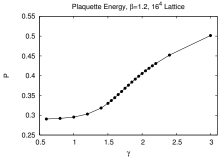


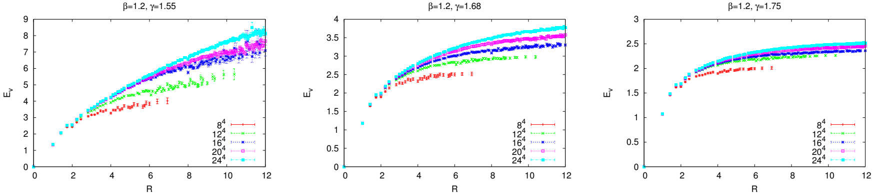
(a) (b) (c)

$E_V(R)$ rises linearly below the crossover, consistent with (but not a proof of) S_c-confinement in this region.

The theory appears to be in the C-confinement phase above the transition.

There is no thermodynamic transition or crossover in the SU(2) gauge model at $\beta = 1.2$.





(a) (b) (c)

$E_V(R)$ would appear to rise linearly below roughly $\gamma = 1.68$, at least in the large volume limit. This is consistent with the conjectured S_c-confinement at small γ.

The theory appears to be in the C-confinement phase at higher γ.

Remnant symmetry breaking

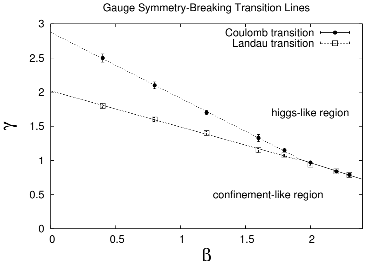
The transition in $E_V(R)$ coincides with the breaking of a remnant gauge symmetry $g(x,t) = g(t)$ that exists in Coulomb gauge. The appropriate order parameter for the symmetry breaking on a time slice is

$$u(t) = \frac{1}{\sqrt{2}V_3} \sum_{\mathbf{x}} U_0(\mathbf{x}, t)$$

and on the lattice we compute the susceptibility

$$\chi = V_3(\langle |u|^2 \rangle - \langle |u| \rangle^2) \quad \text{where} \quad |u| = \sqrt{\frac{1}{N_t} \sum_{t=1}^{N_t} \mathrm{Tr}[u^\dagger(t) u(t)]}$$

Other gauges have other remnant symmetries. However, the transition lines for remnant-symmetry breaking are gauge-dependent.



Pseudomatter

A pseudomatter field is a field constructed from the gauge field which transforms like matter in the fundamental representation. An example is any eigenstate

$$(-D_iD_i)^{ab}_{\mathbf{xy}}\varphi^b_n(\mathbf{y}) = \lambda_n\varphi^a_n(\mathbf{x})$$

of the covariant spatial Laplacian

$$(-D_iD_i)^{ab}_{\mathbf{xy}} = \sum_{k=1}^{3}\left[2\delta^{ab}\delta_{\mathbf{xy}} - U^{ab}_k(\mathbf{x})\delta_{\mathbf{y},\mathbf{x}+\hat{k}} - U^{\dagger ab}_k(\mathbf{x}-\hat{k})\delta_{\mathbf{y},\mathbf{x}-\hat{k}}\right]$$

We construct

$$\boxed{V^{ab}(\mathbf{x},\mathbf{y};A) = \varphi^a_1(\mathbf{x})\varphi^{\dagger b}_1(\mathbf{y})}$$

from the lowest-lying eigenstate, and compute $E_V(R)$ by lattice Monte Carlo.

Fat links

Let $V_{thin}(\mathbf{x}, \mathbf{y}; A)$ be a Wilson line running between $\mathbf{x}, \mathbf{y}$, and

$$\Psi_{thin}(R) = \overline{q}(x) V_{thin}(x, y; A) q(y)$$

Likewise, let $U_k^{(0)}(\mathbf{x}) = U_k(\mathbf{x}, t)$ and construct fat links by an iterative procedure

$$\begin{aligned} U_i^{(n+1)}(x) &= \mathcal{N}\Big\{ \alpha U_i^{(n)}(x) + \sum_{j \neq i} \Big(U_j^{(n)}(x) U_i^{(n)}(x + \hat{j}) U_j^{\dagger}(x + \hat{i}) \\ &\qquad + U_j^{(n)\dagger}(x - \hat{j}) U_i^{(n)}(x - \hat{j}) U_j^{(n)}(x - \hat{j} + \hat{i}) \Big) \Big\} \end{aligned}$$

Denote the link variables after the last iteration as $U_i^{fat}(\mathbf{x})$ and define

$$\begin{aligned} V_{fat}(x, y; A) &= U_k^{fat}(x) U_k^{fat}(x + \hat{k}) \ldots U_k^{fat}(x + (R-1)\hat{k}) \\ \Psi_{fat}(R) &= \overline{q}(x) V_{fat}(x, y; A) q(y) \end{aligned}$$

We then compute $E_V(R)$ for $V = V_{thin}, V_{fat}$.

Results

- We find an S to C-confinement transition for the V operator constructed from pseudomatter fields. The transition line is close to (but a little below) the transition line for the Dirac state.

- The fat link state seems to be everywhere S-confining. This doesn't mean the gauge-Higgs theory is everywhere S-confining. It means instead that not every operator can detect the transition to C-confinement.

Other criteria

Other criteria for distinguishing the confinement from the Higgs phase have been proposed in the past, in particular:

- the Kugo-Ojima criterion
- Non-positivity/unphysical poles in quark/gluon propagators

These criteria **assume** the existence of BRST symmetry, which is problematic non-perturbatively.

1. the Neuberger 0/0 problem: BRST symmetry $\longrightarrow$ **vanishing** of the functional integral in covariant gauges. *(Neuberger, 1986)*.
2. BRST symmetry is **broken** by gauge fixing in lattice Monte Carlo *(Cucchieri and Mendes, 2014)*.
3. BRST perturbative analysis yields the **wrong spectrum** of the SU(3) gauge-Higgs model, even deep in the Higgs region. *(Maas and Torek, 2018)*.

Brout-Englert-Higgs and symmetry breaking

Does the transition from S to C-confinement correspond to the spontaneous breaking of some symmetry in the gauge-Higgs theory?

Local Symmetry $\Longrightarrow$ Elitzur's Theorem.
Global Symmetry $\Longrightarrow$ Goldstone's Theorem.

Looks like no go. But let's look anyway at the global symmetries.

It is well known, in the SU(2) gauge-Higgs model, that the full symmetry of the Higgs action

$$S_H = \gamma \sum_{x,\mu} \frac{1}{2} \text{Tr}[\phi^\dagger(x) U_\mu(x) \phi(x+\widehat{\mu})]$$

is SU(2)$_{gauge}\times$ SU(2)$_{global}$:

$$\begin{aligned} U_\mu(x) &\rightarrow L(x) U_\mu(x) L^\dagger(x+\hat{\mu}) \\ \phi(x) &\rightarrow L(x)\phi(x) R \end{aligned}$$

Global SU(2) symmetry

SU(2)$_{gauge}$ can't break spontaneously, but what about SU(2)$_{global}$? Note that Z is a sum of "spin systems"

$$Z(\beta,\gamma) = \int DU \, Z_{spin}(\gamma, U) e^{-S_W}$$

where

$$\begin{aligned} Z_{spin}(\gamma, U) &= \int D\phi \; e^{-S_H[\phi, U]} \\ &= e^{-\mathcal{F}_H[\gamma, U]} \end{aligned}$$

The only symmetry of the spin system, since $U_\mu(x)$ is fixed, is the SU(2)$_{global}$ symmetry $\phi(x) \rightarrow \phi(x)R$.

It is possible that the SU(2)$_{global}$ (R-transformation) symmetry breaks in *each* $Z_{spin}(\gamma, U)$ without breaking in the *sum* over spin systems.

This might be a gauge-invariant version of the gauge-dependent statement that $\langle\phi\rangle \neq 0$...and a way to evade Goldstone.

Gauge-invariant order parameter for spontaneous symmetry breaking

Consider $\phi(x)$ fluctuating in a background gauge field U, which is held fixed. Denote its average value in this background as $\overline{\phi}(x; U)$.

In general, $\int dx\phi = 0$, because if no gauge is fixed, so $U_\mu(x)$ varies wildly in space, then $\phi(x)$ also varies wildly.

On the other hand, it could be that

$$\overline{\phi}(x; U) \equiv \langle\phi(x)\rangle_U \neq 0$$

at any given point x, even if the spatial average vanishes.

Since the action at fixed U_μ is invariant under $\phi(x) \to \phi(x)R$, *this would imply SSB of an $SU(2)_{global}$ symmetry* in $Z_{spin}(\gamma, U)$, while $\langle\phi(x, U)\rangle = 0$, as it must, in the full theory.

order parameter, continued

So we introduce the following gauge-covariant operator:

$$\overline{\phi}(x;U) = \frac{1}{Z[U]}\int D\phi\ \phi(x)\exp\left[\gamma\sum_{x,\mu}\frac{1}{2}\mathrm{Tr}[\phi^\dagger(x)U_\mu(x)\phi(x+\hat{\mu})]\right]$$

$$Z[U] = \int D\phi\ \exp\left[\gamma\sum_{x,\mu}\frac{1}{2}\mathrm{Tr}[\phi^\dagger(x)U_\mu(x)\phi(x+\hat{\mu})]\right]$$

and compute the following gauge-invariant order parameter:

$$\begin{aligned} Q &= \left\langle\sqrt{\frac{1}{2}\mathrm{Tr}[\overline{\phi}^\dagger(x;U)\overline{\phi}(x;U)]}\right\rangle \\ &= \frac{1}{Z}\int DUD\phi\sqrt{\frac{1}{2}\mathrm{Tr}[\overline{\phi}^\dagger(x;U)\overline{\phi}(x;U)]}\,e^{S[U,\phi]} \end{aligned}$$

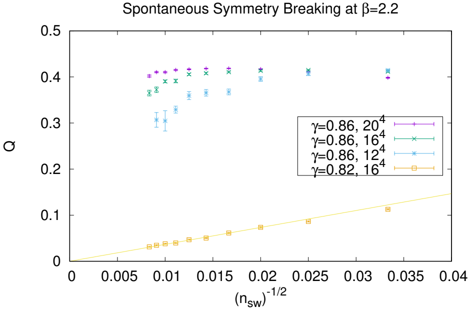
by a Monte Carlo-within-a-Monte Carlo. Of course, there is no spontaneous symmetry breaking on a finite lattice; any “broken” state is only metastable in time (just like a real magnet). “Time” in our case is the number of Monte Carlo sweeps n_{sw} used to compute $\overline{\phi}(x;U)$.

In the unbroken phase we expect $Q \propto \frac{1}{\sqrt{n_{sw}}}$.

For the broken phase, we expect Q is roughly constant with n_{sw}. Eventually $Q \to 0$ in the broken phase, but only after a Monte Carlo time which increases with lattice volume.

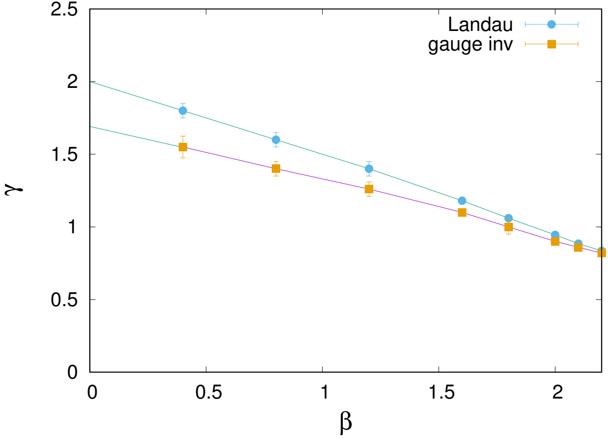
And that's what we see.

Here are the results at $\beta = 2.2$, above ($\gamma = 0.86$) and below ($\gamma = 0.82$) the thermodynamic crossover.



In this way we can map out the SSB transition line throughout the phase diagram.

Both the gauge-invariant transition line and the Landau gauge transition are shown; they are clearly not identical.



Global U(1) symmetry

The global “R” symmetry in the SU(2) gauge-Higgs model is accidental. A Higgs field in SU(N) gauge-Higgs theory at $N > 2$ cannot be expressed as an SU(N) group element.

However, the SU(N>2) Higgs action

$$S_H[U,\phi] = \gamma \sum_{x,\mu} \mathrm{Re}[\phi^\dagger(x) U_\mu(x) \phi(x+\widehat{\mu})]$$

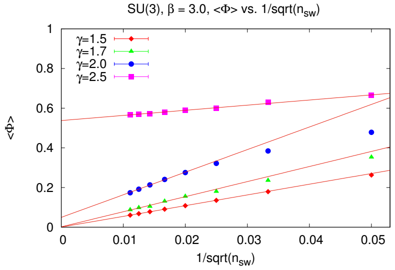
does have a global U(1) symmetry, distinct from the gauge symmetry *(Maas et al., 2017)*:

$$\phi(x) \rightarrow e^{i\theta} \phi(x)$$

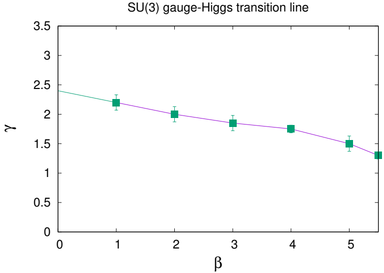
and this global symmetry can be spontaneously broken. The order parameter is the same as before

$$|\overline{\phi}(x;U)| = \sqrt{\overline{\phi}^\dagger(x;U)\overline{\phi}(x;U)}$$

except that a dot product of color indices, rather than a trace, is implied



order parameter



transition line

Conclusions

We have

1. defined a generalization of the Wilson area law criterion, "S_c-confinement," which is applicable to gauge theories with matter fields in the fundamental representation,
2. shown that in gauge-Higgs theories there must exist a transition between two physically distinct (S_c and C) types of confinement,
3. suggested an alternative distinction based on custodial symmetry in the Higgs sector, and
4. shown that this symmetry breaks spontaneously (in the sense described), as detected by a gauge-invariant order parameter.

Our conjecture is that the S_c-to-C confinement transition and the gauge-invariant symmetry-breaking transition coincide.

A last word...

The Clay Mathematics Institute offers a US $1,000,000 prize for a proof that Yang-Mills theory has a mass gap.

For that kind of money, the Clay Institute ought to get a proof of S_c-confinement in QCD.

(Not just a lousy little mass gap in Yang-Mills.)