XIIIth Quark Confinement and the Hadron Spectrum

On the analytic structure of QCD propagators

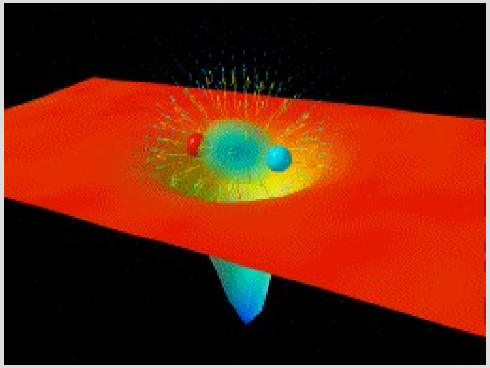
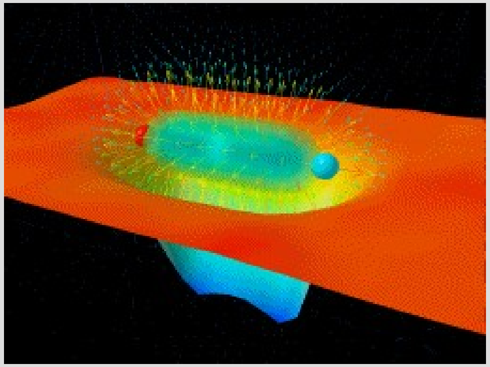
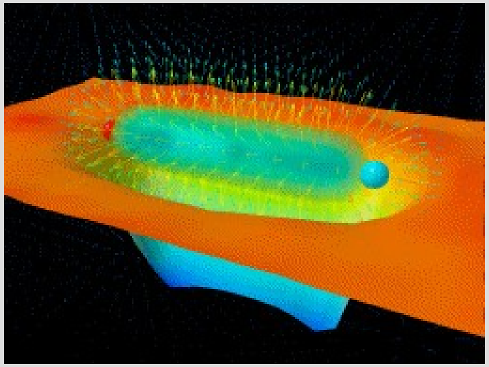
Peter Lowdon

(SLAC)

SLAC NATIONAL ACCELERATOR LABORATORY

Outline

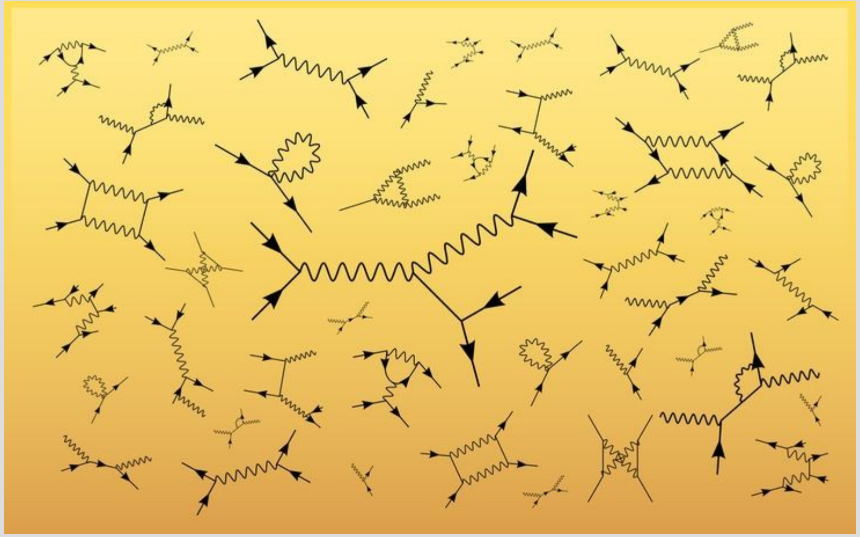
1. LQFT: an axiomatic approach to QFT
2. Correlation functions in LQFT
3. Singular terms and the CDP
4. The QCD propagators
5. Summary and outlook

[The University of Adelaide (2015)]

1. LQFT: an axiomatic approach to QFT

- Perturbation theory has proven to be an extremely successful tool for investigating problems in particle physics



But by definition this procedure is only valid in a *weakly interacting regime*

- *Form factors?*
- *Hadronic observables?*
- *Confinement mechanism?*

- This emphasises the need for a non-perturbative approach!

→ **Local quantum field theory** (LQFT) is one such approach

1. LQFT: an axiomatic approach to QFT

- LQFT is defined by a core set of physically motivated axioms

Axiom 1 (Hilbert space structure). *The states of the theory are rays in a Hilbert space $\mathcal{H}$ which possesses a continuous unitary representation $U(a,\alpha)$ of the Poincaré spinor group $\overline{\mathscr{P}_+^\uparrow}$.*

Axiom 2 (Spectral condition). *The spectrum of the energy-momentum operator P^μ is confined to the closed forward light cone $\overline{V}^+ = \{p^\mu \mid p^2 \geq 0,\, p^0 \geq 0\}$, where $U(a,1) = e^{iP^\mu a_\mu}$.*

Axiom 3 (Uniqueness of the vacuum). *There exists a unit state vector $|0\rangle$ (the vacuum state) which is a unique translationally invariant state in $\mathcal{H}$.*

Axiom 4 (Field operators). *The theory consists of fields $\varphi^{(\kappa)}(x)$ (of type (κ)) which have components $\varphi_l^{(\kappa)}(x)$ that are operator-valued tempered distributions in $\mathcal{H}$, and the vacuum state $|0\rangle$ is a cyclic vector for the fields.*

Axiom 5 (Relativistic covariance). *The fields $\varphi_l^{(\kappa)}(x)$ transform covariantly under the action of $\overline{\mathscr{P}_+^\uparrow}$:*

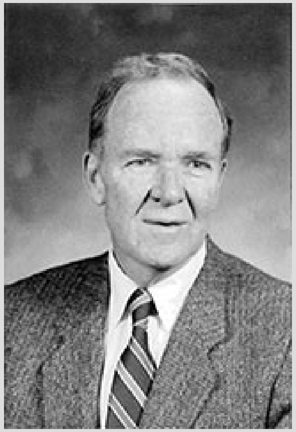
$$U(a,\alpha)\varphi_i^{(\kappa)}(x)U(a,\alpha)^{-1} = S_{ij}^{(\kappa)}(\alpha^{-1})\varphi_j^{(\kappa)}(\Lambda(\alpha)x + a)$$

where $S(\alpha)$ is a finite dimensional matrix representation of the Lorentz spinor group $\overline{\mathscr{L}_+^\uparrow}$, and $\Lambda(\alpha)$ is the Lorentz transformation corresponding to $\alpha \in \overline{\mathscr{L}_+^\uparrow}$.

Axiom 6 (Local (anti-)commutativity). *If the support of the test functions f, g of the fields $\varphi_l^{(\kappa)}, \varphi_m^{(\kappa')}$ are space-like separated, then:*

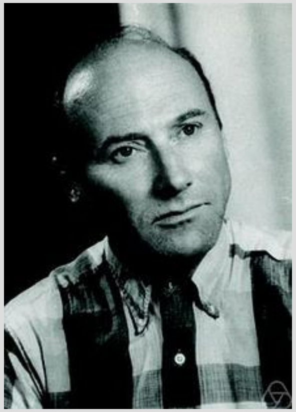
$$[\varphi_l^{(\kappa)}(f), \varphi_m^{(\kappa')}(g)]_\pm = \varphi_l^{(\kappa)}(f)\varphi_m^{(\kappa')}(g) \pm \varphi_m^{(\kappa')}(g)\varphi_l^{(\kappa)}(f) = 0$$

when applied to any state in $\mathcal{H}$, for any fields $\varphi_l^{(\kappa)}, \varphi_m^{(\kappa')}$.



A. Wightman

[R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and all that* (1964).]



R. Haag

[R. Haag, *Local Quantum Physics*, Springer-Verlag (1996).]

2. Correlation functions in LQFT

- Due to the properties of the field operators it follows that correlation functions are (tempered) *distributions*

- Since the fields are Lorentz covariant (*Axiom 5*), the Fourier transform of general correlation functions can be written

$$\boxed{\widehat{T}_{(1,2)}(p) = \mathcal{F}\left[\langle 0|\phi_1(x)\phi_2(y)|0\rangle\right] = \sum_{\alpha=1}^{\mathcal{N}} Q_\alpha(p)\,\widehat{T}_{\alpha(1,2)}(p)}$$

Lorentz covariant polynomial

Lorentz invariant component

- Moreover, the spectral condition (*Axiom 2*) implies that the Lorentz invariant components *vanish* outside the (closed) forward light cone

$$\boxed{\widehat{T}_{\alpha(1,2)}(p) = P(\partial^2)\delta(p) + \int_0^\infty ds\,\theta(p^0)\delta(p^2-s)\rho_\alpha(s)}$$

Singular component

"Spectral function"

[N. N. Bogolubov, A. A. Logunov and A. I. Oksak, General Principles of Quantum Field Theory, (1990).]

3. Singular terms and the CDP

- For QFTs that satisfy the standard LQFT axioms one can prove that the correlation strength between clusters of fields always **decreases** with separation [Araki; Araki, Hepp, Ruelle]

  → this is called the **cluster decomposition property** (CDP)

- If QCD satisfied these axioms one would be permitted to 'pull apart' coloured states

- It turns out though that gauge theories violate these axioms

  → *charged fields are non-local!*

- There are two approaches for defining a quantised gauge theory:

  (1) One preserves positivity of the Hilbert space, but loses locality (e.g. Coulomb gauge)

  (2) One preserves locality, but loses positivity (e.g. BRST quantised gauge theories)

In this work we choose to preserve locality

[H. Araki, *Ann. Phys.* **11**, 260 (1960).]
[H. Araki, K. Hepp and D. Ruelle, *Helv. Phys. Acta* **35**, 164 (1962).]

3. Singular terms and the CDP

- By choosing option (2) one maintains many of the standard properties of LQFT, but now the space of states can contain negative norm states

  → referred to as the *Pseudo-Wightman* approach [Bogolubov et al.]

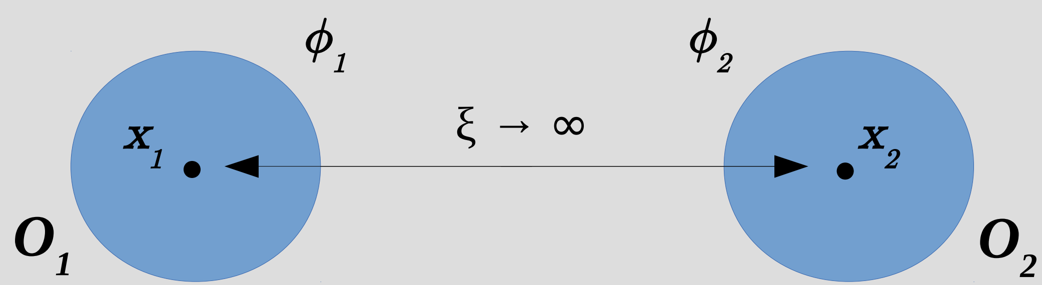
- In Pseudo-Wightman QFTs the CDP can be <u>violated</u> [Strocchi]:

Theorem (Cluster Decomposition).

$$\left|\langle 0|\mathcal{B}_1(x_1)\mathcal{B}_2(x_2)|0\rangle^T\right| \leq \begin{cases} C_{1,2}[\xi]^{2N-\frac{3}{2}}e^{-M[\xi]}\left(1+\frac{|\xi_0|}{[\xi]}\right), & \textit{with a mass gap } (0,M) \textit{ in } \mathcal{V} \\ \widetilde{C}_{1,2}[\xi]^{2N-2}\left(1+\frac{|\xi_0|}{[\xi]^2}\right), & \textit{without a mass gap in } \mathcal{V} \end{cases}$$

where: $\langle 0|\mathcal{B}_1(x_1)\mathcal{B}_2(x_2)|0\rangle^T = \langle 0|\mathcal{B}_1(x_1)\mathcal{B}_2(x_2)|0\rangle - \langle 0|\mathcal{B}_1(x_1)|0\rangle\langle 0|\mathcal{B}_2(x_2)|0\rangle$, $N \in \mathbb{Z}_{\geq 0}$, $\xi = x_1 - x_2$ *is large and space-like, and* $C_{1,2}, \widetilde{C}_{1,2}$ *are constants independent of* ξ *and* N.

$$\boxed{\mathcal{B}_i(x_i) := \int_{\mathcal{O}_i} d^4y\, \phi_i(y) f(y-x_i), \quad f \in \mathcal{D}(\mathbb{R}^{1,3})}$$



→ This violation is related to the presence of singular terms [PL, 1511.02780]

[N. N. Bogolubov, A. A. Logunov and A. I. Oksak, General Principles of Quantum Field Theory, (1990).]
[F. Strocchi, *Phys. Lett.* B **62**, 60 (1976).]

- Using the previous structural results the QCD propagators take the following general forms [PL, 1702.02954; PL, 1711.07569]:

Gluon propagator

$$\boxed{\widehat{D}^{F}_{\mu\nu}(p) = i\int_0^\infty \frac{ds}{2\pi}\,\frac{\left[g_{\mu\nu}\rho_1^{ab}(s) + p_\mu p_\nu \rho_2^{ab}(s)\right]}{p^2 - s + i\epsilon} + \left[g_{\mu\nu}P_1^{ab}(\partial^2) + p_\mu p_\nu P_2^{ab}(\partial^2)\right]\delta(p)}$$

Quark propagator

$$\boxed{\widehat{S}^{ij}_{F}(p) = i\int_0^\infty \frac{ds}{2\pi}\,\frac{\left[\rho_1^{ij}(s) + \not{p}\rho_2^{ij}(s)\right]}{p^2 - s + i\epsilon} + \left[P_1^{ij}(\partial^2) + \not{p}P_2^{ij}(\partial^2)\right]\delta(p)}$$

Ghost propagator

$$\boxed{\widehat{G}^{ab}_{F}(p) = i\int_0^\infty \frac{ds}{2\pi}\,\frac{\rho_C^{ab}(s)}{p^2 - s + i\epsilon} + P_C^{ab}(\partial^2)\delta(p)}$$

4. The QCD propagators

- What constraints do the dynamical properties of the fields (i.e. equations of motion, ETCRs) impose on these propagators?

<u>Gluon propagator</u> [PL, 1801.09337]

→ The spectral functions are no longer independent, and both satisfy sum rules

$$\boxed{\rho_1^{ab}(s) + s\rho_2^{ab}(s) = -2\pi\xi\delta^{ab}\delta(s), \quad \int_0^\infty ds\, \rho_1^{ab}(s) = -2\pi\delta^{ab} Z_3^{-1}, \quad \int_0^\infty ds\, \rho_2^{ab}(s) = 0}$$

→ The coefficients of the singular components are linearly related

→ One can write:

$$\begin{aligned}\widehat{D}_{\mu\nu}^{ab\,F}(p) = i\int_0^\infty \frac{ds}{2\pi}\left(-sg_{\mu\nu} + p_\mu p_\nu\right)\frac{\rho_2^{ab}(s)}{p^2 - s + i\epsilon} - \frac{ig_{\mu\nu}\xi\delta^{ab}}{p^2 + i\epsilon} \\ + \sum_{n=0}^{N+1}\left[c_n^{ab}\, g_{\mu\nu}(\partial^2)^n + d_n^{ab}\partial_\mu\partial_\nu(\partial^2)^{n-1}\right]\delta(p).\end{aligned}$$

⇒ *Slavnov-Taylor* identity is explicitly satisfied $\boxed{p^\mu p^\nu \widehat{D}_{\mu\nu}^{ab\,F}(p) = -i\xi\delta^{ab}}$

4. The QCD propagators

- Using the Dyson-Schwinger equation one can further constrain the spectral functions and coefficients of the singular terms

$$\boxed{\left[\partial^2 g_{\mu}^{\ \alpha} - \left(1 - \frac{1}{\xi_0}\right)\partial_\mu \partial^\alpha\right]\langle 0|T\{A^a_\alpha(x)A^b_\nu(y)\}|0\rangle = i\delta^{ab}g_{\mu\nu}Z_3^{-1}\delta(x-y) + \langle 0|T\{\mathcal{J}^a_\mu(x)A^b_\nu(y)\}|0\rangle}$$

$$\Rightarrow \quad \boxed{-\left[p^2 g_{\mu}^{\ \alpha} - \left(1 - \frac{1}{\xi_0}\right)p_\mu p^\alpha\right]\widehat{D}^{ab\,F}_{\alpha\nu}(p) = i\delta^{ab}g_{\mu\nu}Z_3^{-1} + \widehat{J}^{ab}_{\mu\nu}(p)}$$

- Inserting the spectral representations of the propagators, and separately equating the different Lorentz components (as distributions) implies:

  → the coefficients of the singular terms in the gluon and current propagators are linearly related

  → the gluon spectral functions satisfy the constraints:

$$\boxed{\begin{aligned} \rho_1^{ab}(s) &= -2\pi\delta^{ab}Z_3^{-1}\delta(s) + \widetilde{\rho}_2^{ab}(s) \\ s\rho_2^{ab}(s) &= 2\pi\delta^{ab}\left(Z_3^{-1} - \xi\right)\delta(s) - \widetilde{\rho}_2^{ab}(s) \end{aligned}} \qquad \boxed{\int ds\, \widetilde{\rho}_2^{ab}(s) = 0}$$

4. The QCD propagators

$$\boxed{\begin{aligned} \rho_1^{ab}(s) &= -2\pi\delta^{ab}Z_3^{-1}\delta(s) + \widetilde{\rho}_2^{ab}(s) \\ s\rho_2^{ab}(s) &= 2\pi\delta^{ab}\left(Z_3^{-1} - \xi\right)\delta(s) - \widetilde{\rho}_2^{ab}(s) \end{aligned}}$$

$$\boxed{\int ds\, \widetilde{\rho}_2^{ab}(s) = 0}$$

- These spectral properties have important consequences

  (*i*) The *Oehme-Zimmerman superconvergence relation:* $\boxed{\int_0^\infty ds\rho_1^{ab}(s) = -2\pi\delta^{ab}Z_3^{-1}}$ holds due to the explicit massless component

  (*ii*) Z_3^{-1} vanishes in Landau gauge – this implies the absence of massless gluon states

  (*iii*) Non-negativity violations can arise due to the sum rule satisfied by the second spectral component

  → This behaviour is *not* related to whether the gluon is absent or not from the spectrum – it also holds for the corresponding spectral function of the photon propagator

[R. Oehme, W. Zimmermann, *Phys. Rev. D* **21**, 471 (1980).]

4. The QCD propagators

Quark propagator [PL, 1711.07569]

$$\boxed{(i\gamma^\mu \partial_\mu - m)\langle 0|T\{\psi^i(x)\overline{\psi}^j(y)\}|0\rangle = i\delta^{ij} Z_2^{-1}\delta(x-y) + \langle 0|T\{\mathcal{K}^i(x)\overline{\psi}^j(y)\}|0\rangle}$$

$$\Rightarrow \quad \boxed{(\not{p} - m)\widehat{S}_F^{ij}(p) = i\delta^{ij} Z_2^{-1} + \widehat{K}^{ij}(p)}$$

- Again, by inserting the spectral representation of the propagators, and matching the different Lorentz components, one obtains constraints

- The coefficients of the singular terms in the quark propagator are linearly related to the singular terms in the current propagator *and* the $\delta(p)$ coefficient in the quark propagator

  → In contrast to the gluon case, the appearance of $\delta(p)$ terms is sufficient to guarantee that $\delta(p)$-derivative terms must exist

4. The QCD propagators

- The quark spectral functions satisfy the constraints:

$$\boxed{\begin{aligned}\rho_1^{ij}(s) &= \left[2\pi m\,\delta^{ij} Z_2^{-1} - \int d\tilde{s}\,\kappa_1^{ij}(\tilde{s})\right]\delta(s-m^2) + \kappa_1^{ij}(s) \\ \rho_2^{ij}(s) &= \left[2\pi\delta^{ij} Z_2^{-1} - \int d\tilde{s}\,\kappa_2^{ij}(\tilde{s})\right]\delta(s-m^2) + \kappa_2^{ij}(s)\end{aligned}}$$

Coefficients of massive components are not completely fixed

Other components have no sum rule constraints, unlike gluon case

→ The presence of a massive quark pole (in specific gauges) is not so clear-cut

→ The structure of the other spectral components depend on the properties of the spectral functions of the current propagator

$$\boxed{\begin{aligned}\left(s-m^2\right)\kappa_1^{ij}(s) &= m\widetilde{\rho}_1^{ij}(s) + s\widetilde{\rho}_2^{ij}(s) \\ \left(s-m^2\right)\kappa_2^{ij}(s) &= \widetilde{\rho}_1^{ij}(s) + m\widetilde{\rho}_2^{ij}(s)\end{aligned}}$$

4. The QCD propagators

Ghost propagator [PL, 1711.07569]

$$\partial^2 \langle 0|T\{C^a(x)\overline{C}^b(y)\}|0\rangle = \delta^{ab}\widetilde{Z}_3^{-1}\delta(x-y) + \langle 0|T\{\mathcal{L}^a(x)\overline{C}^b(y)\}|0\rangle$$

$$-p^2 G_F^{ab}(p) = \delta^{ab}\widetilde{Z}_3^{-1} + L^{ab}(p)$$

- Dyson-Schwinger equation constraints:

→ The coefficients of the singular terms in the ghost propagator are linearly related to the corresponding terms in the current propagator

→ The ghost spectral function satisfies the relation:

$$\rho_C^{ab}(s) = \left[2\pi i\delta^{ab}\widetilde{Z}_3^{-1} - \int_0^\infty d\tilde{s}\,\kappa_C^{ab}(\tilde{s})\right]\delta(s) + \kappa_C^{ab}(s)$$

Coefficient vanishes in Landau gauge

Unlike gluon case, coefficient of massless component is not completely constrained

5. Summary and outlook

- LQFT can be used to determine the general spectral properties of propagators in QCD

- The momentum space gluon, quark and ghost propagators can all potentially contain purely singular terms involving derivatives of $\delta(p)$

  → these terms imply a violation of the CDP, which is relevant for understanding confinement

- The Dyson-Schwinger equations impose non-trivial constraints on the structure of the QCD spectral functions

- This LQFT approach has several potential applications:

  → improve propagator parametrisations

  → study gauge invariant correlation functions

  → understand the non-perturbative effects of finite temperature and density

Backup

- The currents appearing in the QCD Dyson-Schwinger equations have the following structure:

(*i*) **Gluon**:

$$\boxed{\left[\partial^2 g_{\mu}{}^{\alpha} - \left(1 - \frac{1}{\xi_0}\right)\partial_\mu \partial^\alpha\right] A^a_\alpha = \mathcal{J}^a_\mu}$$

$$\boxed{\mathcal{J}^a_\mu = g j^a_\mu - igf^{abc}\partial_\mu \overline{C}^b C^c + (Z_3^{-1} - 1)\partial_\mu \Lambda^a - igf^{abc}A^{b\nu}F^c_{\nu\mu} - gf^{abc}\partial^\nu (A^b_\nu A^c_\mu)}$$

(*ii*) **Quark**:

$$\boxed{\mathcal{K}^i(x) := -g\gamma^\mu A^a_\mu(x)[t^a \psi(x)]^i}$$

(*iii*) **Ghost**:

$$\boxed{\partial^2 C^a = -igf^{abc}\partial^\nu (A^b_\nu C^c) = \mathcal{L}^a}$$

Backup

- The coefficients of the singular components satisfy the following constraints:

(*i*) **Gluon**:

$$c_n^{ab} = \begin{cases} -2(n+1)(2n+3)b_{n+1}^{ab}, & 1 \leq n \leq N+1 \\ \tilde{a}_0^{ab}, & n = 0 \end{cases}$$

$$d_n^{ab} = \begin{cases} 4n(n+1)b_{n+1}^{ab}, & 1 \leq n \leq N+1 \\ 0, & n = 0 \end{cases}$$

$$c_{n+1}^{ab} = -\frac{(2n+5)}{4(2n+3)(n+1)(n+3)} C_n^{ab}, \qquad n \geq 0$$

(*ii*) **Quark**:

$$a_n^{ij} = \frac{m^{2n}}{4^n(n+1)!n!}\left[a_0^{ij} + \sum_{k=0}^{n-1} \frac{4^k(k+1)!k!\left(m\tilde{a}_k^{ij} + 4(k+1)(k+2)\tilde{b}_{k+1}^{ij}\right)}{m^{2(k+1)}}\right], \qquad n \geq 1$$

$$b_n^{ij} = \frac{m^{2n-1}}{4^n(n+1)!n!}\left[a_0^{ij} + \sum_{k=0}^{n-1} \frac{4^k(k+1)!k!\left(m\tilde{a}_k^{ij} + 4(k+1)(k+2)\tilde{b}_{k+1}^{ij}\right)}{m^{2(k+1)}}\right] - \frac{1}{m}\tilde{b}_n^{ij}, \qquad n \geq 1$$

(*iii*) **Ghost**:

$$g_{n+1}^{ab} = -\frac{\tilde{g}_n^{ab}}{4(n+1)(n+2)}, \qquad n \geq 0.$$

Backup

- How can one tell whether $N>0$ in QCD?

  → The spectral functions $\rho(s)$ are key to determining the value of N in the modified CDP (in the absence of purely singular terms)

- In general this relationship is non-trivial, but in particular one has the following result [PL, 1511.02780]:

$$\boxed{\begin{array}{l} \rightarrow\ \ \rho(s) \sim \delta(s-s_0), \quad \textit{then } N=0 \\ \rightarrow\ \ \rho(s) \sim \delta'(s-s_1), \quad \textit{then } N>0 \end{array}}$$

- An important quantity which is sensitive to the behaviour of the spectral functions $\rho_\alpha(s)$ are the Schwinger functions $\Delta_\alpha(t)$, defined by

$$\boxed{\Delta_\alpha(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp_0\, e^{ip_0 t} D_\alpha(p^2)\big|_{\mathbf{p}=0} = \int_0^{\infty} ds\, \rho_\alpha(s)\, \frac{e^{-\sqrt{s}\,t}}{2\sqrt{s}}}$$

This can be computed using non-perturbative numerical techniques like lattice QCD