Conformal Perturbation description of Deconfinement.[1]

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Quark Confinement 01/08/2018

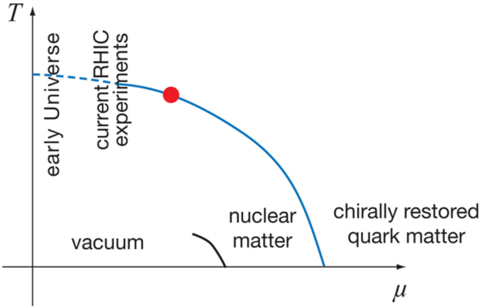
[1] M. Caselle, G. Costagliola, N. Magnoli arXiv:1605.05133 Phys. Rev. D 94, 026005 (2016)
M. Caselle, N. Magnoli, A.Nada, M.Panero, M. Scanavino in preparation.

Summary:

1. Introduction and motivations
2. Conformal Perturbation Theory
3. Comparison with Montecarlo Simulations of the 3d Ising Model
4. Polyakov Loop correlators in (3+1) SU(2) LGT near the Deconfinement Transition

Introduction and motivations

- Our long-term goal is to derive theoretical predictions for the dynamics of strong interactions in the neighbourhood of the critical end-point appearing at finite temperature T and quark chemical potential μ in the QCD phase diagram which is expected to belong to the 3d Ising Universality class.



Introduction and motivations

- From the knowledge of the universality class, general RG arguments allow to predict the behaviour of one point functions in the scaling region. However a precise understanding of the dynamics of the system requires also the knwoledge of correlators. This goal is accomplished by the so called Conformal Perturbation Theory (CPT).

- The main advantage of CPT is that it allows to understand some general features of the model: presence of bound states, estimate of universal quantitites... and at the same time it can be compared with other approximation or effective models and can be used to fine tune the parameters of these effective models.

Introduction and motivations

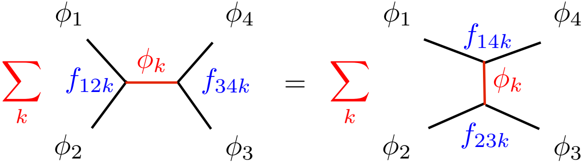
- To this end it is mandatory to compare CPT predictions with Montecarlo Simulations so as to understand the range of validity (the "scaling region") of CPT.

- This is for the moment impossible for the critical point in the μ, T plane. For this reason we decided to study, as a benchmark of the program, a different LGT realization of this same universality class: the Deconfinement transition of the (3+1) dimensional SU(2) Lattice Gauge Theory.

- Besides being an useful benchmark to test methods and ideas, the SU(2) model represents a very interesting subject in itself. The CPT analysis will allow us
  - to fine tune the parameters of the effective string theory which describes the Polyakov loop correlators,
  - to explore the structure of bound states of the model
  - to evaluate (and succesfully compare with simulations) for the first time an universal ratio of amplitudes above and below the deconfinement transition, whose value only depends on the symmetries of the system.

Introduction and motivations: CPT

- The choice of the 3d Ising universality class is not random. CPT requires a precise knwoledge of universal quantities of the Conformal Field Theory (CFT) at the critical point to predict the behaviour of off-critical correlators in the scaling region
- For this reason in the past it was mainly applied to models in $d = 2$ where conformal symmetry has an infinite number of generators leading to a host of exact results[2].
- In $d > 2$ the conformal group has a finite number of generators and it is in general much less predictive.
- However this situation drastically changed in the last few years thanks to the use of conformal invariant Operator Product Expansion and to the idea of the so called Conformal Bootstrap.



[1] A. M. Polyakov, *Conformal symmetry of critical fluctuations* JETP Lett. 12 (1970) 381

[2] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. B241 (1984) 333

Introduction and motivations

- Numerical implementations of conformal bootstrap equations allowed to obtain very precise estimates of critical dimensions and structure constants of several $d > 2$ theories and in particular of the 3d Ising model[3] [4]
- The 3d Ising model is used as a benchmark for CFT applications since very precise numerical and experimental results exist for this model, both at the critical point and in the whole scaling region. In the Ising case we have only two relevant operators: σ and ϵ,

$$\Delta_\sigma = 0.5181489(10), \quad \Delta_\epsilon = 1.412625(10)$$

$$C^{\epsilon}_{\sigma\sigma} = 1.0518537(41), \quad C^{\epsilon}_{\epsilon\epsilon} = 1.532435(19)$$

[3] Z. Komargodski and D. Simmons-Duffin, (2016), 1603.04444

[4] F. Kos, D. Poland, D. Simmons-Duffin, and A. Vichi, (2016), 1603.04436

Introduction and motivations

CPT works for any universality class, for any correlator and any relevant perturbation, but we shall concentrate in this talk on the thermal perturbation of the $\langle\sigma(r)\sigma(0)\rangle$ correlator of the 3D Ising universality class both above and below the critical temperature, because for symmetry reasons [1] this is the relevant one to study the behaviour of Polyakov loop correlators in the (3+1) SU(2) LGT both above and below the deconfinement point.

[1] B. Svetitsky and L. G. Yaffe, Nucl. Phys. B210, 423 (1982)

Conformal Perturbation Theory: The thermal perturbation

The perturbed action is given by the conformal action S_{cft}, plus a term proportional to the energy operator:

$$S = S_{cft} + t \int \epsilon(r)\, d\mathbf{r}$$

where t is a parameter related in the continuum limit to the deviation from the critical temperature. Using the operator product expansion, we can write the two point function of two generic operators $O_i,\ O_j$

$$\langle O_i(r) O_j(0) \rangle_t = \sum_k C_{ij}^k(t, r) \langle O_k(0) \rangle_t .$$

where $C_{ij}^k(t, r)$ are the Wilson coefficients, calculated outside of the critical point.

Conformal Perturbation Theory: The thermal perturbation

By Taylor expanding the Wilson coefficients we find:

$$\langle O_i(r)O_j(0)\rangle_t = \sum_k [C_{ij}^k(0,r) + \partial_t C_{ij}^k(0,r) + ...]\langle O_k(0)\rangle_t$$

where $\partial_t C_{ij}^k(0,r)$ denotes the derivatives of the Wilson coefficients with respect to t evaluated at the critical point.

- The derivatives $\partial_t C_{ij}^k(0,r)$ can be evaluated using conformal symmetry
- The one point functions $\langle O_k(0)\rangle_t$ must be provided as external inputs
- However if $O_k(0)$ is the identity operator this input is not needed and the corresponding term $\partial_t C_{ij}^1(0,r)$ is universal and can be used as a "test of conformality" of the underlying theory.

Thermal Perturbation Theory: the 3d Ising Model

In the Ising case we have only two relevant operators σ and ϵ, with scaling dimensions $\Delta_\sigma = 0.5181489(10)$ and $\Delta_\epsilon = 1.412625(10)$.
We find for the first three orders of perturbed two-point function of σ:

$$\langle \sigma(r)\sigma(0)\rangle_t = C^1_{\sigma\sigma}(0,r) + C^\epsilon_{\sigma\sigma}(0,r)\langle\epsilon\rangle_t + t\partial_t C^1_{\sigma\sigma}(0,r) + \dots$$

To make contact with the usual definition for the structure constants we factorize the r dependence in the Wilson coefficients:

$$C^1_{\sigma\sigma}(0,r) = \frac{1}{r^{2\Delta_\sigma}}, \qquad C^\epsilon_{\sigma\sigma}(0,r) = C^\epsilon_{\sigma\sigma} r^{\Delta_\epsilon - 2\Delta_\sigma}$$

where we have chosen the usual normalization $C^1_{\sigma\sigma} = 1$.
Defining $\Delta_t = 3 - \Delta_\epsilon$, the perturbed one-point functions is:

$$\langle\epsilon\rangle_t = A^\pm |t|^{\frac{\Delta_\epsilon}{\Delta_t}} \tag{1}$$

Where $A^\pm$ are non-universal amplitudes. Introducing the scaling variable $s = tr^{\Delta_t}$ we end up with the following expression for the perturbed two-point function:

$$r^{2\Delta_\sigma}\langle \sigma(r)\sigma(0)\rangle_t = 1 + C^\epsilon_{\sigma\sigma} A^\pm |s|^{\frac{\Delta_\epsilon}{\Delta_t}} + t\partial_t C^1_{\sigma\sigma}(0,r) + \dots$$

Thermal Perturbation Theory: Evaluation of the derivative term.

The derivatives of the Wilson coefficient can be written as:

$$\partial_t C^1_{\sigma\sigma}(0,r) = -\int (\langle\sigma(r)\sigma(0)\epsilon(r_1)\rangle_0 - C^\epsilon_{\sigma\sigma}\langle\epsilon(r_1)\epsilon(0)\rangle_0) d\mathbf{r_1},$$

where:

$$\langle\sigma(r)\sigma(0)\epsilon(r_1)\rangle_0 = \frac{C^\epsilon_{\sigma\sigma}}{r_1^{\Delta_\epsilon} r^{2\Delta_\sigma - \Delta_\epsilon}(r^2 + r_1^2 - 2rr_1 \ \cos\theta)^{\frac{\Delta_\epsilon}{2}}}$$

while the second term in the integral $\langle\epsilon(r)\epsilon(0)\rangle_0 = \frac{1}{r^{2\Delta_\epsilon}}$ acts as an infrared counterterm. The integral can be calculated using a Mellin transform. Setting $y = \frac{r_1}{r}$ one finds

$$\partial_t C^1_{\sigma\sigma}(0,r) \equiv r^{\Delta_t - 2\Delta_\sigma} C^\epsilon_{\sigma\sigma} X_{Ising}$$

with

$$X_{Ising} = \int \frac{1}{y^{\Delta_\epsilon}} \frac{1}{(1 + y^2 - 2ycos\theta)^{\frac{\Delta_\epsilon}{2}}} d\mathbf{y} = -62.5336$$

Thus we end up with:

$$r^{2\Delta_\sigma}\langle\sigma(r)\sigma(0)\rangle_t = 1 + C^\epsilon_{\sigma\sigma} A^\pm |s|^{\frac{\Delta_\epsilon}{\Delta_t}} - C^\epsilon_{\sigma\sigma} s X_{Ising}$$

Thermal Perturbation Theory: Results for the 3d Ising Model.

Inserting the known numerical estimates of $A^{\pm}$ for the 3d Ising model we end up with:

$$r^{2\Delta_\sigma}\langle\sigma(r)\sigma(0)\rangle_t = 1 - 51.2(3)|s|^{\frac{\Delta_\epsilon}{\Delta_t}} + 65.7762..s \quad (t>0) + ...$$

$$r^{2\Delta_\sigma}\langle\sigma(r)\sigma(0)\rangle_t = 1 + 95.6(6)|s|^{\frac{\Delta_\epsilon}{\Delta_t}} + 65.7762..s \quad (t<0) + ...$$

Notice that:

- The second term is not universal and must be independently evaluated with numerical or experimental methods, but the third term is indeed universal and represents a highly non trivial test of conformal invariance!
- The second and the third term are of the same size for $t > 0$ and almost cancel, while this is not true for $t < 0$ thus the behaviour is completely different in the $t > 0$ and $t < 0$ cases.

Comparison with Montecarlo Simulations of the 3d Ising Model

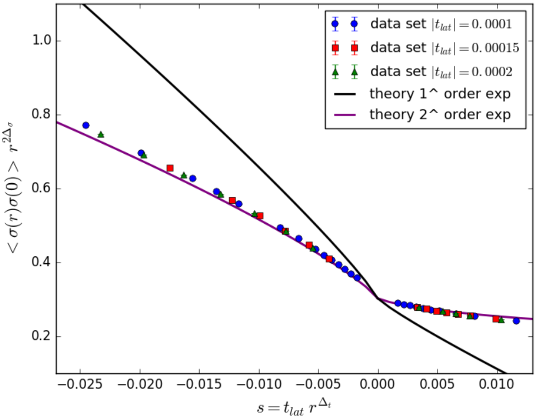
Simulations with a standard Metropolis updating with multispin coding on a cubic lattice of size $L = 300$ with periodic boundary condition.
$t_{lat} \equiv \beta_c - \beta$ with $\beta_c = 0.22165462$

t_{lat}	r_{min}	r_{max}	$\partial_t C^1_{\sigma\sigma}$	$\chi^2/d.o.f$
$+10^{-4}$	6	20	61.4 (0.9)[1.2]	0.7
-10^{-4}	6	20	60.9 (0.9)[1.5]	0.8
$1.5 \cdot 10^{-4}$	7	14	61.3 (0.8)[1.0]	1.0
$-1.5 \cdot 10^{-4}$	8	20	61.1 (0.9)[1.8]	1.1
$2 \cdot 10^{-4}$	6	13	61.0 (0.8)[1.0]	0.7
$-2 \cdot 10^{-4}$	8	20	61.6 (0.7)[1.5]	1.2

Table : Results of the fits to the spin-spin correlator performed keeping $\partial_t C^1_{\sigma\sigma}$ as free parameter. The columns r_{min}, r_{max} indicate the range of distances sampled which are always much smaller than the correlation length. Statistical errors are reported in round brackets, while the systematic ones, due to the continuum to lattice conversion, are reported in square brackets.

The theoretical estimate $\partial_t C^1_{\sigma\sigma} = 65.7762...$ almost saturates the two point function!

Comparison with Montecarlo Simulations



Comparison with Experimental data

- CPT predictions can be compared with experimental estimates of the critical scattering functions
- The Scattering Function can be obtained by Fourier transforming the spin-spin correlator. Its functional form is dictated by scaling and was already known more than 40 years ago [1]

$$g(q) = \frac{C_1^{\pm}}{q^{2-\eta}} \left(1 + \frac{C_2^{\pm}}{q^{(1-\alpha)/\nu}} + \frac{C_3^{\pm}}{q^{1/\nu}} \right)$$

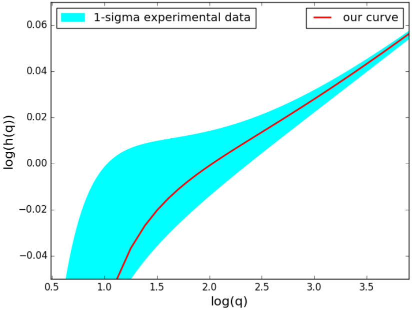
with $q = k\xi$, where k is the momentum-transfer vector and ξ the correlation length.

- CPT allows to evaluate the coefficients in terms of critical amplitudes and conformal data.

[1] M. E. Fisher and J. S. Langer, Phys. Rev. Lett. **20**, 665 (1968).

Comparison with Experimental data

Scattering Function from a small-angle neutron scattering experiment on a sample of **CO_2** at critical density [1]. As usual $g(q)$ is plotted in a log-log scale and normalized to the Ornstein-Zernicke function: $h(q) = g(q)/g_{OZ}(q)$ with $g_{OZ} = 1/(1+q^2)$.



The continuous line is our prediction while the light-blue region marks the experimental estimate

[1] P. Damay, F. Leclercq, R. Magli, F. Formisano, and P. Lindner, Phys. Rev. **B58**, 12038 (1998).

Interquark Potential and Deconfinement Transition

- For some choices of the gauge group the finite temperature Deconfinement Transition of a pure Yang-Mills Theory is critical and is described by a CFT
- The order parameter of the transition is the Polyakov loop defined as the trace of a Wilson line winding around the lattice in the temporal ("0") direction:

$$P(x,y,z) = \mathrm{Tr} \prod_{0 \le t < N_t} U_0(x,y,z,t\,a)\ .$$

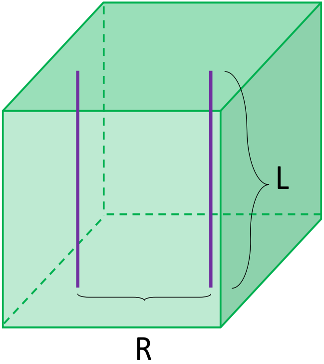
- The analogous of the $\langle \sigma(r)\sigma(0) \rangle_t$ correlator is the correlator of two Polyakov loops, which is related to the interquark potential

$$V(R) = -\lim_{L\to\infty} \frac{1}{L} \log < P(0)P(R)^\dagger >$$

- The confining phase is the analogous of the *symmetric* phase of the perturbed CFT (i.e the high T phase of the Ising realization, even if for the Yang-Mills theory it corresponds to the low T regime!!).

Polyakov loop correlator.

Expectation value of two Polyakov loops at distance R and Temperature $T = 1/L$



$$V(R) = -\lim_{L\to\infty} \frac{1}{L} \log < P(0)P(R)^{\dagger} >$$

SU(2) LGT in (3+1) dimensions.

The most intersting example is the LGT with gauge group $SU(2)$ in (3+1) dimensions whose deconfinement transition belongs to the universality class of the 3d Ising model.

β	$N_t \times N_s^3$	T/T_c	N_{conf}
2.55	10×80^3	0.90	10^5
2.569	10×80^3	0.96	10^5
2.572	10×80^3	0.97	10^5
2.58101	10×80^3	1	8×10^4
2.58984	10×80^3	1.02	1.6×10^5
2.59271	10×80^3	1.05	1.6×10^5
2.61	10×80^3	1.10	1.6×10^5

Table : Setup of our lattice simulations for $N_t = 10$.

SU(2) LGT in (3+1) dimensions.

β	$N_t \times N_s^3$	T/T_c	N_{conf}
2.48479	8×80^3	0.90	8×10^4
2.50311	8×80^3	0.96	8×10^4
2.50598	8×80^3	0.97	8×10^4
2.51165	8×80^3	1	8×10^4
2.52295	8×80^3	1.02	8×10^4
2.52567	8×80^3	1.05	8×10^4
2.54189	8×80^3	1.10	8×10^4

Table : Setup of our lattice simulations.

SU(2) LGT in (3+1) dimensions for $N_t = 8$.

β	$N_t \times N_s^3$	T/T_c	N_{conf}
2.60573	12×96^3	0.90	8×10^4
2.626	12×96^3	0.96	8×10^4
2.62923	12×96^3	0.97	8×10^4
2.63896	12×96^3	1	8×10^4
2.64558	12×96^3	1.02	1.6×10^5
2.65541	12×96^3	1.05	1.6×10^5
2.67085	12×96^3	1.10	1.6×10^5

Table : Setup of our lattice simulations for $N_t = 12$.

SU(2) LGT in (3+1) dimensions: effective string description

The large distance behaviour of the interquark potential is well described by the Nambu-Goto effective string action:

$$\langle P(0)^* P(R) \rangle = \sum_{n=0}^{\infty} \frac{N_t}{2R} w_n e^{-E_n r}$$

where R is the interquark distance, N_t the size of the lattice in the compactified time direction, $D = 4$ and w_n denotes the multiplicity of the state and can be evaluated using the expansion of the Dedekind function:

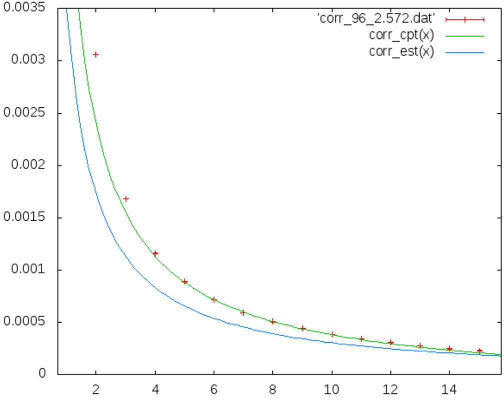
$$\left(\prod_{r=1}^{\infty} \frac{1}{1 - q^r} \right)^{D-2} = \sum_{k=0}^{\infty} w_k q^k .$$

and E_n the closed-string energies:

$$E_n = \sigma N_t \left\{ 1 + \frac{8\pi}{\sigma N_t^2} \left[-\frac{1}{24} (D-2) + n \right] \right\}^{1/2} .$$

SU(2) LGT in (3+1) dimensions.

However with the Nambu-Goto action there are clear deviations at short distance, where we may expect CPT to hold.



SU(2) LGT in (3+1) dimensions: the CPT prediction

$$\langle P(x)^* P(y) \rangle \sim \frac{C^{\mathbf{1}}_{\sigma\sigma}}{r^{2\Delta_\sigma}} \left(1 + A(t) r^{\Delta_\epsilon} + B(t) r^{\Delta_t}\right)$$

where

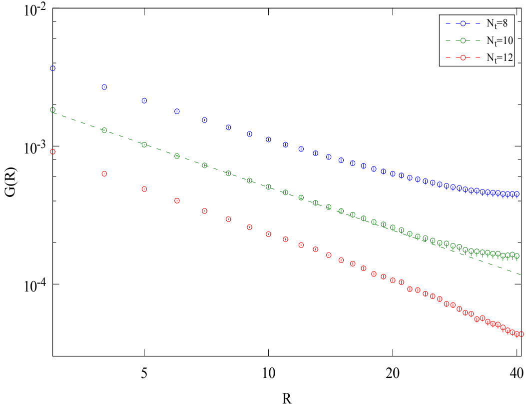
$$A(t) = A^{\pm} C^{\epsilon}_{\sigma\sigma} t^{\frac{\Delta_\epsilon}{\Delta_t}} \; .$$

$$B(t) = X_{Ising} \, C^{\epsilon}_{\sigma\sigma} \, t$$

and we know that: $\Delta_\sigma = 0.51815...$, $\Delta_\epsilon = 1.4126..$ and $\Delta_t = 3 - \Delta_\epsilon = 1.5874..$
$C^{\epsilon}_{\sigma\sigma} = 1.0518537(41)$ and $X_{Ising} = -62.5336.$
We end up with two free parameters the normalization constant for the Polyakov loops: $C^{\mathbf{1}}_{\sigma\sigma}$ and the specific heat amplitudes $A^{\pm}$

Polyakov loop correlators at the critical point

$C^{\mathbf{1}}_{\sigma\sigma}$ can be fixed looking at correlators at the critical point



Off-critical correlators

$A^{\pm}$ can be extracted fitting the off-critical correlators

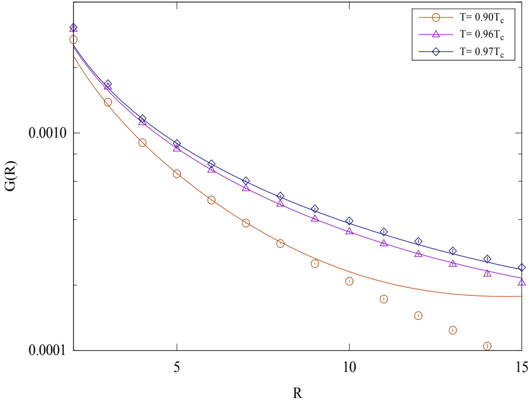
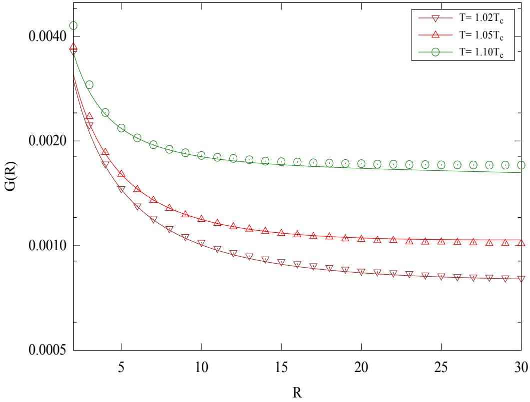


Figure : off-critical correlators below T_c

Notice the different range of validity of the CPT approximation as T/T_c changes

Off-critical correlators

$A^{\pm}$ can be extracted fitting the off-critical correlators



The range of validity of the CPT approximation is larger for $T > T_c$ than for $T < T_c$.

SU(2) LGT in (3+1) dimensions: Specific Heat Amplitudes.

β	T/T_c	$A^{\pm}$
2.55	0.90	-52.6(6)
2.569	0.96	-50(2)
2.572	0.97	-49(5)
2.58984	1.02	91(6)
2.59271	1.05	94(5)
2.61	1.10	82(3)

Table : Results for the amplitude A^{+} in the confining phase (first three rows) and for A^{-} in the deconfining phase (last three rows) for $N_t = 10$.

We see that two major consistency checks of the whole procedure are fulfilled

- The values of $A^{\pm}$ are stable within the errors as T/T_c changes
- The ratio $-A^{+}/A^{-} = 0.56(4)$, is in remarkable agreement with the universal value $-A^{+}/A^{-} = 0.536(2)$

CPT versus Nambu-Goto.

We may use CPT to improve the effective string description. The first suggestion is that, in order to be compatible with the Ising universality class $\sigma(T)$, which for Nambu-Goto behaves as $\sigma(T) \sim (T_c - T)^{1/2}$ must behave as

$$\sigma(T) = \sigma_0(T_c - T)^{1/\Delta_t} \sim \sigma_0(T_c - T)^{0.63}$$

This corresponds to a well defined prescription for the choice of the additional terms which must be included in the effective action beyond the Nambu-Goto one.

Assuming this behaviour we find an impressively good agreement of the effective string prediction with the correlators for all the values of T/T_c

SU(2) LGT in (3+1) dimensions.

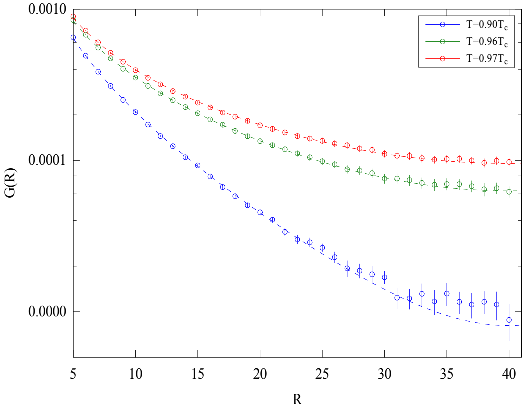
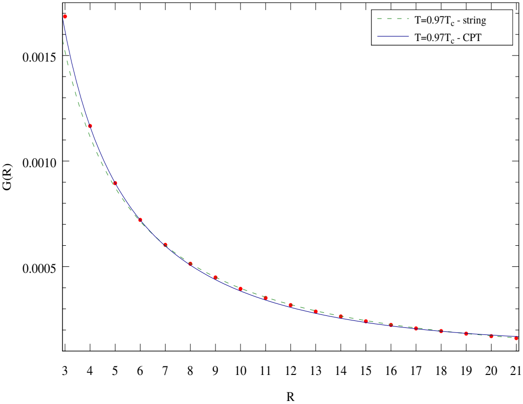
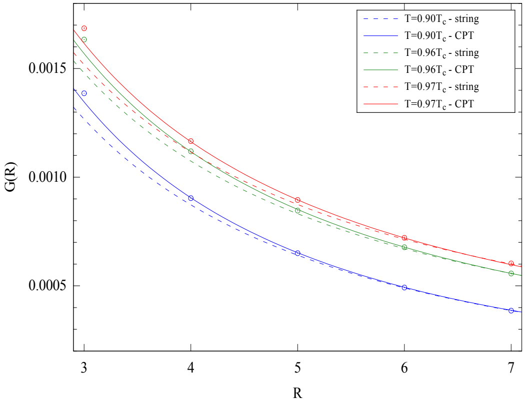


Figure : Fit to the $N_t = 10$ correlators with the effective string action in which $\sigma(T)$ is left unconstrained.

Comparison between CPT and Effective String



Zoom.



T=0.90T$_c$ - string
T=0.90T$_c$ - CPT
T=0.96T$_c$ - string
T=0.96T$_c$ - CPT
T=0.97T$_c$ - string
T=0.97T$_c$ - CPT
G(R)
0.0015
0.0010
0.0005
3
4
5
6
7
R

Conclusions

- Thanks to the very precise results of the bootstrap approach we may now use CPT also in three dimensions.
- In the Ising case there is a remarkable agreement both with Montecarlo simulations and with experimental results.
- CPT may also be used to describe the short distance behaviour of the interquark potential in the vicinity of a second order deconfinement transition.
- CPT may give indications to improve the effective string description of the interquark potential and suggests the presence of non trivial bound states in the deconfined phase of the SU(2) theory similar to those that we find in the low T phase of the 3d Ising model.

Acknowledgements

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