Higher-order condensate corrections to bottomonium observables

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XIIIth Quark Confinement and the Hadron Spectrum
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Ip3

Outline

- Motivation

- Review of the perturbative description of bottomonium

- Non-perturbative corrections in the local condensate approach

- Phenomenological analysis

  - Masses of S-wave bottomonium

  - Non-relativistic moments

- Conclusions

Motivation

- Bottomonium is very interesting for studying QCD

  - Effective field theories

  - Perturbation theory

  - Renormalons

  - Non-perturbative effects

  - ...

Motivation

- Bottomonium is very interesting for studying QCD

    - Effective field theories

    - Perturbation theory

    - Renormalons

    - Non-perturbative effects

    - ...

- Determinations of the bottom-quark mass

    - Fundamental parameter of nature

    - Important input for flavour physics

    - Dominant uncertainty for many Higgs branching ratios

$$\frac{[\delta \operatorname{Br}(H \to X)]_{m_b}}{\operatorname{Br}(H \to X)} = \operatorname{Br}(H \to b\bar{b}) \frac{2\delta m_b}{m_b}$$

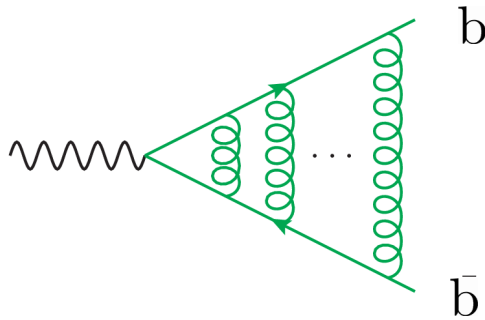   - ...

Non-relativistic description

Bottomonium is a non-relativistic system with velocity $v \sim \alpha_s$

- Multiple scales are relevant

hard scale m_b mass

soft scale $m_b\, v$ momentum

ultrasoft scale $m_b\, v^2$ energy

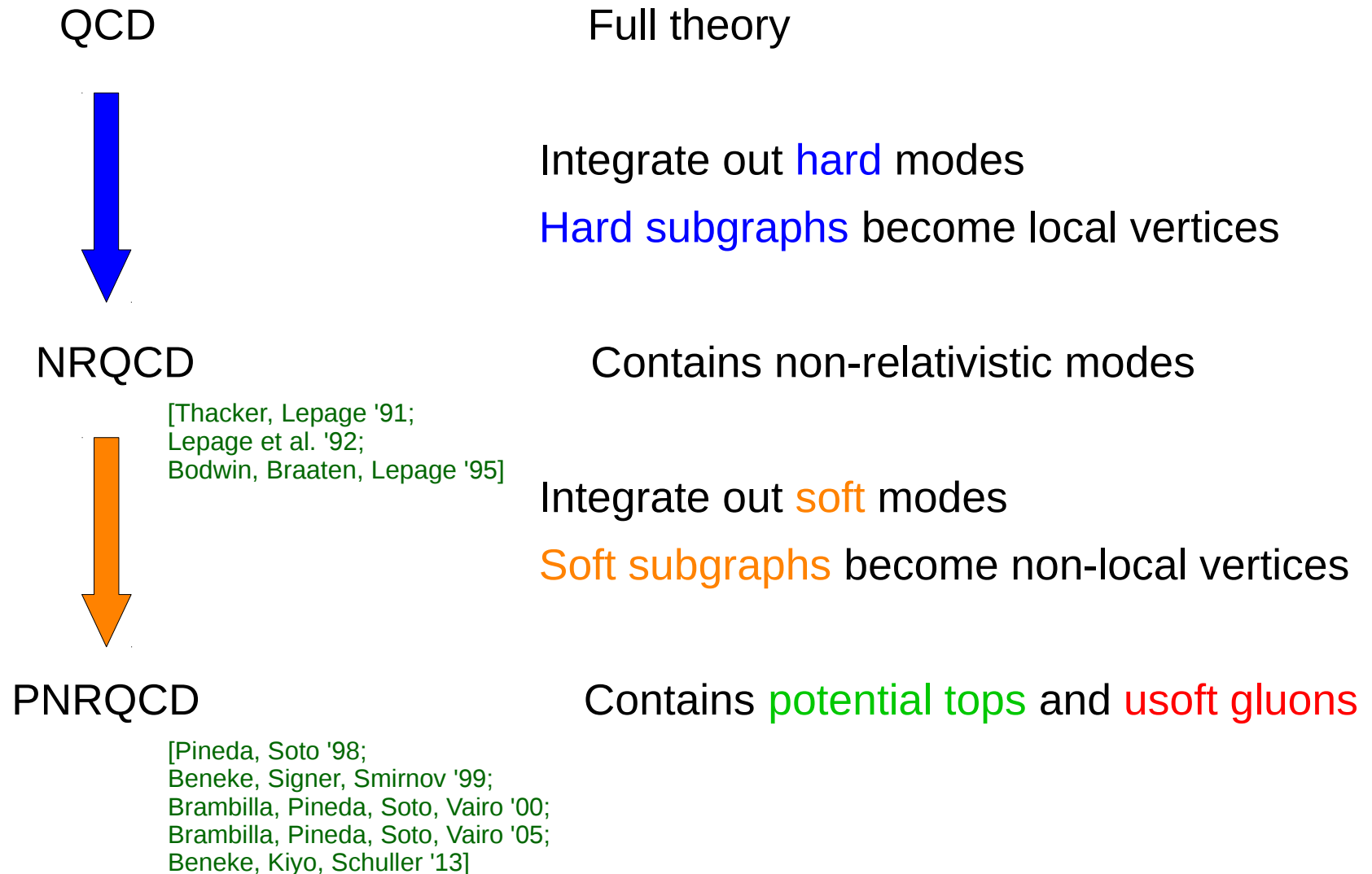- Coulomb singularities $(\alpha_s/v)^n$ from n exchanges of potential gluons

$$k^0 \sim m_b\, v^2, \quad \mathbf{k} \sim m_b\, v$$

- Conventional perturbation theory in α_s fails
- Coulomb singularities must be resummed to all orders

Effective field theory setup

Construct EFT by integrating out the hard and soft scale.

QCD   Full theory

Integrate out hard modes

Hard subgraphs become local vertices

NRQCD   Contains non-relativistic modes

[Thacker, Lepage '91;
Lepage et al. '92;
Bodwin, Braaten, Lepage '95]

Integrate out soft modes

Soft subgraphs become non-local vertices

PNRQCD   Contains potential tops and usoft gluons

[Pineda, Soto '98;
Beneke, Signer, Smirnov '99;
Brambilla, Pineda, Soto, Vairo '00;
Brambilla, Pineda, Soto, Vairo '05;
Beneke, Kiyo, Schuller '13]

QCD cross section

Normalized cross section: $R_b(s) = \frac{\sigma(e^+e^- \to b\bar{b}X)}{\sigma_0(e^+e^- \to \mu^+\mu^-)}$

Resummed cross section at NNNLO:

$$R_b \sim v \sum_k \left(\frac{\alpha_s}{v}\right)^k \begin{cases} 1 & \text{LO} \\ \alpha_s, v & \text{NLO} \\ \alpha_s^2, \alpha_s v, v^2 & \text{NNLO} \\ \alpha_s^3, \alpha_s^2 v, \alpha_s v^2, v^3 & \text{NNNLO} \end{cases}$$
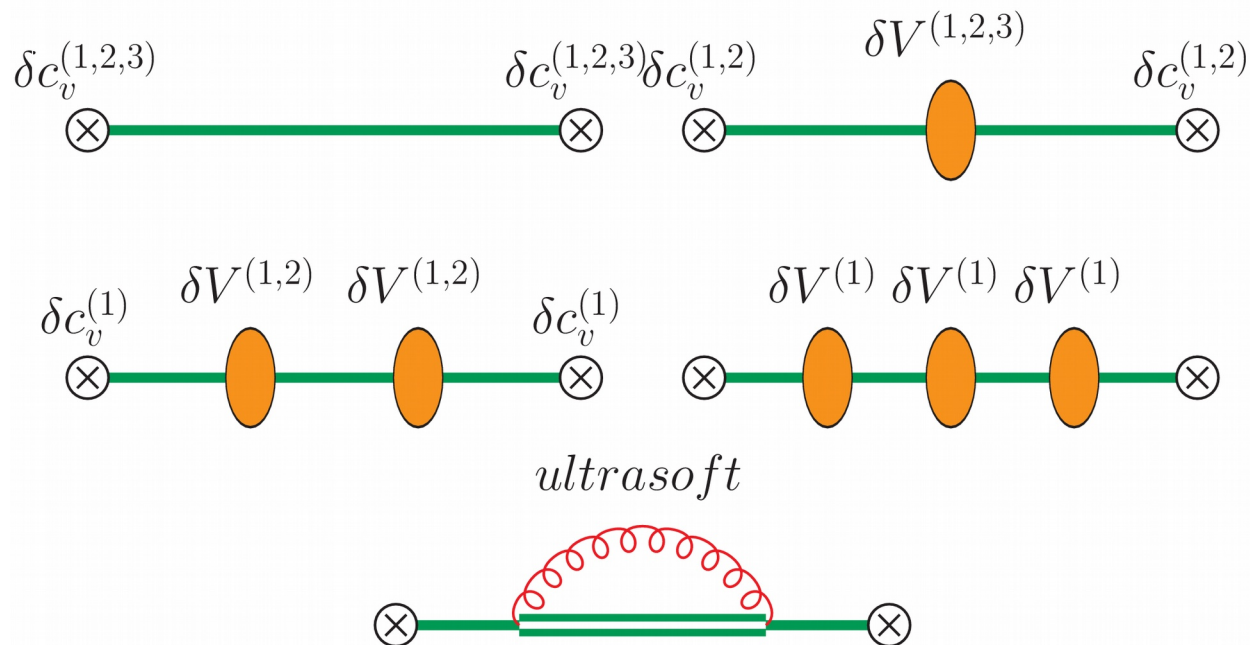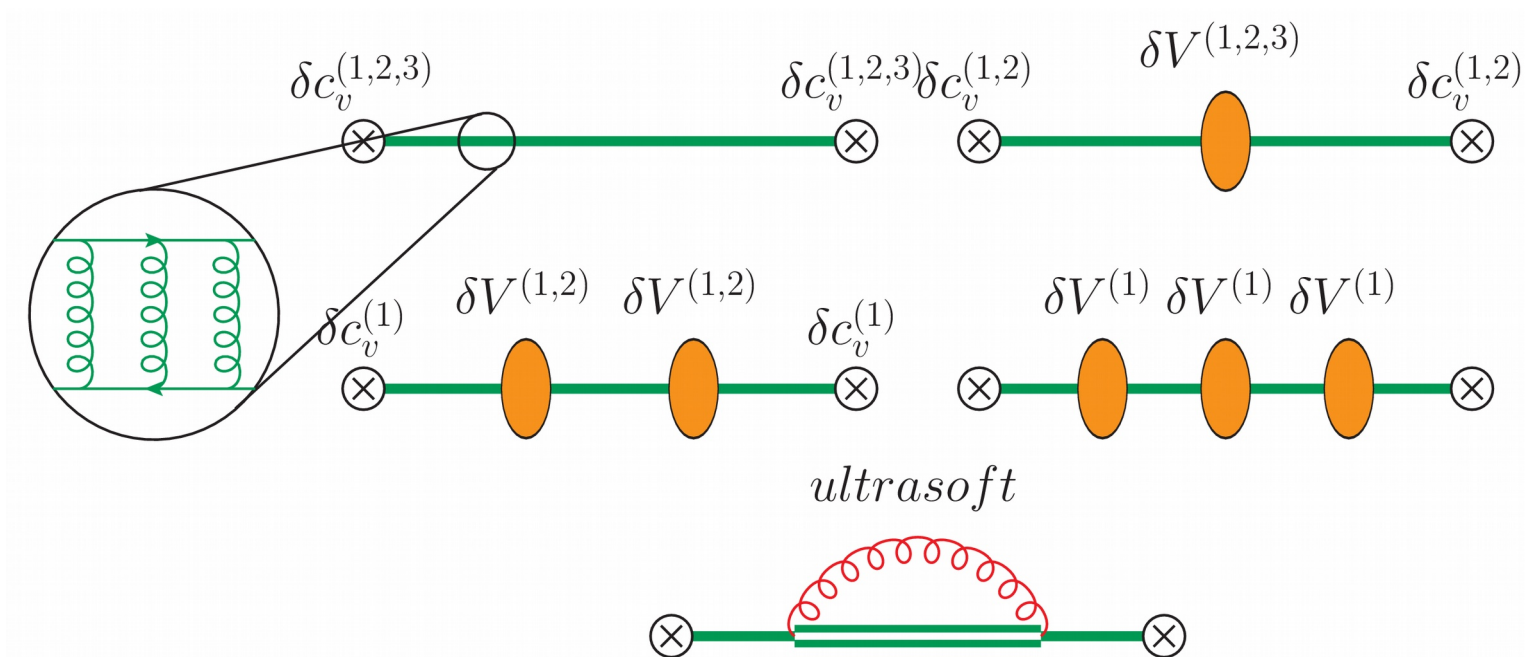
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$$R_b \sim v \sum_k \left(\frac{\alpha_s}{v}\right)^k \begin{cases} 1 & \text{LO} \\ \alpha_s, v & \text{NLO} \\ \alpha_s^2, \alpha_s v, v^2 & \text{NNLO} \\ \alpha_s^3, \alpha_s^2 v, \alpha_s v^2, v^3 & \text{NNNLO} \end{cases}$$

Public implementation QQbar_threshold

← → C 🔒 Secure | https://qqbarthreshold.hepforge.org

- Home
- Download
- Documentation
  - Version 2
  - Version 1
- Changelog

QQbar_threshold

QQbar_threshold computes the top-quark pair production cross section near threshold in electron-positron annihilation at NNNLO in resummed non-relativistic perturbation theory [1, 2]. It includes Higgs, QED, electroweak and non-resonant corrections at various accuracies and a consistent implementation of initial-state radiation. Details can be found in

- M. Beneke, Y. Kiyo, A. Maier, and J. Piclum
  Near-threshold production of heavy quarks with QQbar_threshold
  Comput. Phys. Commun. 209 (2016) 96-115, arXiv:1605.03010 [hep-ph]
- M. Beneke, A. Maier, T. Rauh, and P. Ruiz-Femenía
  Non-resonant and electroweak NNLO correction to the $e^+ e^-$ top anti-top threshold
  arXiv:1711.10429 [hep-ph]

Please cite these (and possibly other articles, where the theoretical input was first computed) when QQbar_threshold is used for published work.

The functionality of the package can also be used to compute the bound state energies and residues of bottomonium S-wave states and high moments of the bottom production cross section at NNNLO, including the continuum (see M. Beneke, A. Maier, J. Piclum, T. Rauh, Nucl.Phys. B891 (2015) 42-72, arXiv:1411.3132 [hep-ph]).

QQbar_threshold is written in C++ and Wolfram Language. It can be used as a C++ library or through a Mathematica interface.

For questions, comments, and bug reports write to qqbarthreshold@projects.hepforge.org.

Expansion in local condensates

- Let us assume: $\Lambda_{\mathrm{QCD}} \ll m_b v^2 \ll m_b v \ll m_b$

- Then we can split the gluon field in the effective Lagrangian

$$A_\mu(t, \mathbf{x}) = A_\mu^{\mathrm{us}}(t, \mathbf{x}) + A_\mu^{\mathrm{np}}(t, \mathbf{x}).$$

- In Fock-Schwinger gauge $\mathbf{x} \cdot \mathbf{A}^{\mathrm{np}}(t, \mathbf{x}) = 0, \ A_0^{\mathrm{np}}(t, \mathbf{0}) = 0,$

$$\mathcal{L}_{\text{non-perturbative}} = \psi^\dagger \left(-g_s \mathbf{x} \cdot \mathbf{E}^{\mathrm{np}}(0, \mathbf{0}) + \dots \right) \psi + \chi^\dagger \left(-g_s \mathbf{x} \cdot \mathbf{E}^{\mathrm{np}}(0, \mathbf{0}) + \dots \right) \chi,$$

- Corrections to the Green function

$$G(E) \equiv \langle \mathbf{0} | \left(\hat{H} - E - i0 \right)^{-1} | \mathbf{0} \rangle,$$

$$\hat{H} = \hat{H}_{b\bar{b}} + \hat{H}_{\mathrm{np}} + \hat{H}_D + \dots .$$

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Perturbative
Hamiltonian
$\sim m_b v^2$

Non-perturbative
Hamiltonian
$\sim \Lambda_{\mathrm{QCD}}$

Chromo-electric
dipole term
$$\hat{H}_D = -\frac{g_s}{2} \xi^A \, \mathbf{x} \cdot \mathbf{E}^{\mathrm{np},A}(0, \mathbf{0})$$
$$\sim \Lambda_{\mathrm{QCD}}^2 / (m_b v)$$
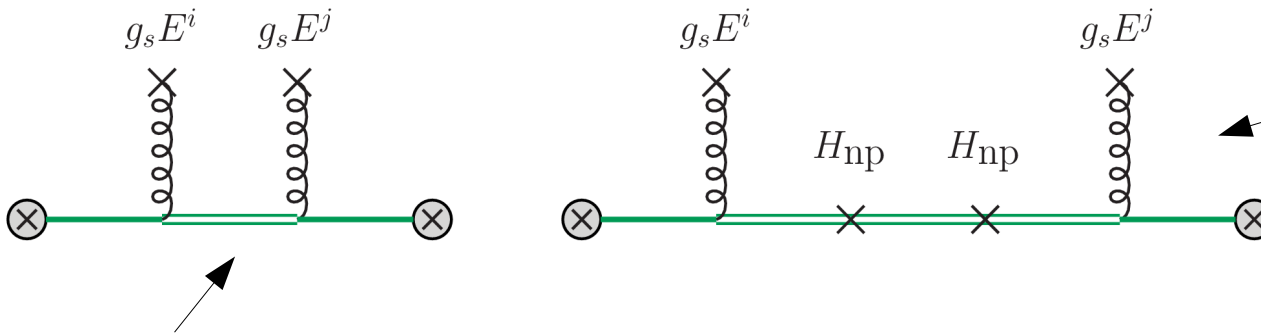
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$$\sim \Lambda_{\mathrm{QCD}}^2 / (m_b v)$$

- $\hat{H}_{\mathrm{np}}$ and $\hat{H}_D$ can both be treated as perturbations

- The states factorize

$$|\mathbf{0}\rangle \equiv |\mathbf{0}\rangle_{b\bar{b}} \otimes |0\rangle_{\mathrm{np}}$$

Bottom-antibottom pair at
zero spatial separation

Non-perturbative
vacuum state

Expansion in local condensates

- Let us assume: $\Lambda_{\mathrm{QCD}} \ll m_b v^2 \ll m_b v \ll m_b$

- Corrections to the Green function

$$G(E) = \langle \mathbf{0}|\hat{G}_{b\bar{b}}^{(1)}(E)|\mathbf{0}\rangle_{b\bar{b}} +$$

$$\sum_{n=0}^{\infty} \langle \mathbf{0}|\hat{G}_{b\bar{b}}^{(1)}(E)\hat{x}^i \left[\hat{G}_{b\bar{b}}^{(8)}(E)\right]^{1+2n} \hat{x}^i \, \hat{G}_{b\bar{b}}^{(1)}(E)|\mathbf{0}\rangle_{b\bar{b}} \, O_n + \dots,$$

where $O_n = \langle 0| \dfrac{g_s^2}{18} (E^{\mathrm{np}})_i^A \left[\hat{H}_{\mathrm{np}}\right]^{2n} (E^{\mathrm{np}})_i^A |0\rangle_{\mathrm{np}}$.

Dimension six computed in [Pineda '96],

Dimension six and eight computed in [TR '18]

Dimension four gives the results obtained by [Voloshin '79,'82, Leutwyler '81],

NLO corrections at dimension four

- Potential corrections determined in [TR '18]
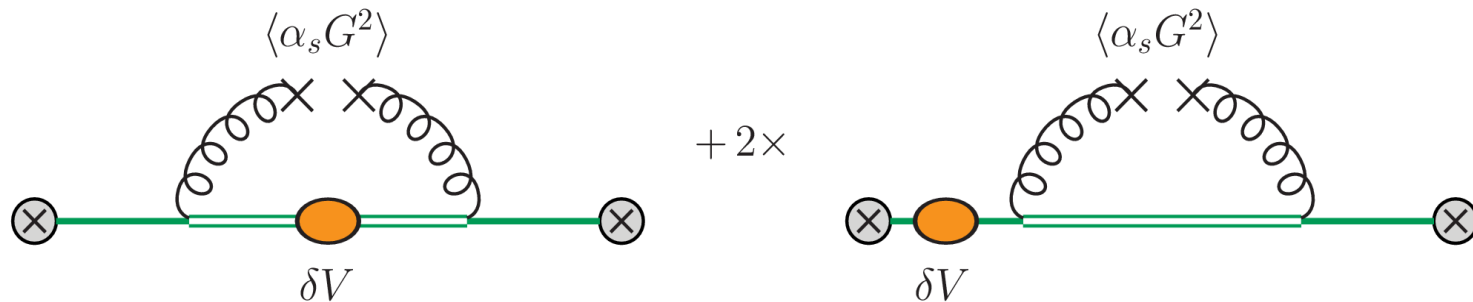
- Ultrasoft effects are missing, but less scale dependent
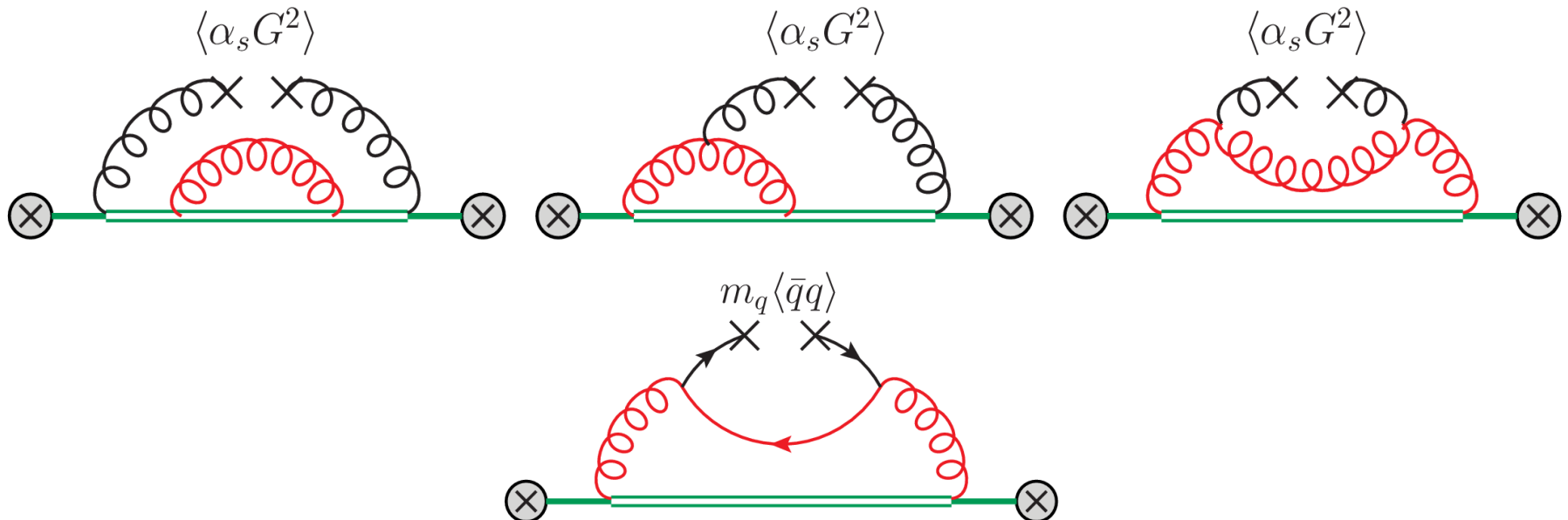
Phenomenology

- Values of the condensates are very uncertain
  - Take $\langle \frac{\alpha_s}{\pi} G^2 \rangle^{\mathrm{SVZ}} = 0.012 \,\mathrm{GeV}^4$ from [Shifman, Vainshtein, Zakharov '79]
  $$\Rightarrow \qquad O_0^{\mathrm{SVZ}} = -(285 \,\mathrm{MeV})^4$$

  In good agreement with [Pineda '96]

  - Then use naive rescaling
  $$O_1^{\mathrm{naive}} = (285 \,\mathrm{MeV})^6, \qquad O_2^{\mathrm{naive}} = -(285 \,\mathrm{MeV})^8.$$

  - Assign generous uncertainties:
  $$0 \geq O_0 \geq 3 O_0^{\mathrm{SVZ}}, \quad 0 \leq O_1 \leq 3^{3/2} O_1^{\mathrm{naive}}, \quad 0 \geq O_2 \geq 3^2 O_2^{\mathrm{naive}}$$

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- Determine scale choice from convergence of partial NLO corrections

$$\delta_{\Lambda_{\mathrm{QCD}}^4} E_n(\mu_c) = -\frac{m_b \alpha_s^2(\mu_c) C_F^2}{4n^2} \left(\tilde{e}_n^{(4,0)} + \frac{\alpha_s(\mu_c)}{4\pi} \tilde{e}_n^{(4,1)} + \dots \right) \frac{O_0}{m_b^4 (\alpha_s(\mu_c) C_F)^6}$$

  - The logarithm $\ln(\mu_c)$ needed to cancel the scale dependence at NLO is contained in the potential corrections

  - The missing ultrasoft correction is less scale dependent $(\propto \alpha_s^{-3}(\mu_c))$

Upsilon(1S) mass
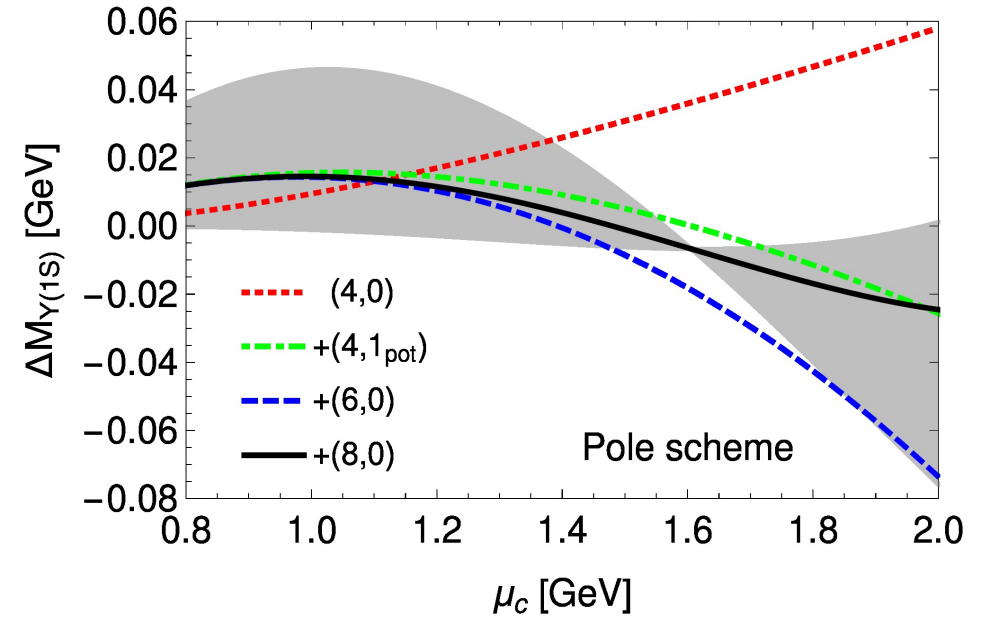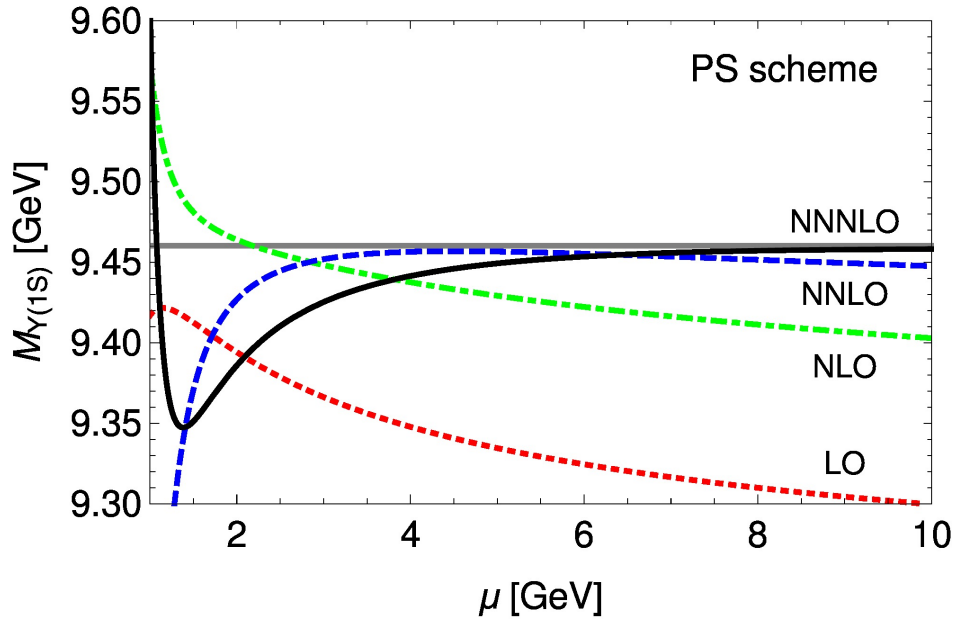
$$M^{\mathrm{exp}}_{\Upsilon(1S)} = 9\,460.30 \pm 0.26\,\mathrm{MeV},$$

$$M_{\Upsilon(1S)} = 9\,437\,^{+61}_{-114}\,\mathrm{MeV}$$

$$= 9\,437\,^{+28}_{-74}\,(\mu)\,^{+25}_{-75}\,(m_b)\,^{+0}_{-1}\,(\alpha_s)\pm 9\,(m_c)$$

$$\pm 36\,(\mu_c)\,^{+29}_{-14}\,(O_0)\,^{+4}_{-18}\,(O_1)\,^{+10}_{-1}\,(O_2)\,\mathrm{MeV},$$

$$1.5\,\mathrm{GeV} \leq \mu \leq 6\,\mathrm{GeV},$$

$$0.8\,\mathrm{GeV} \leq \mu_c \leq 2\,\mathrm{GeV}.$$

Bottom-quark mass from spectroscopy

- Non-perturbative uncertainty "forgotten" in step from NNLO to NNNLO
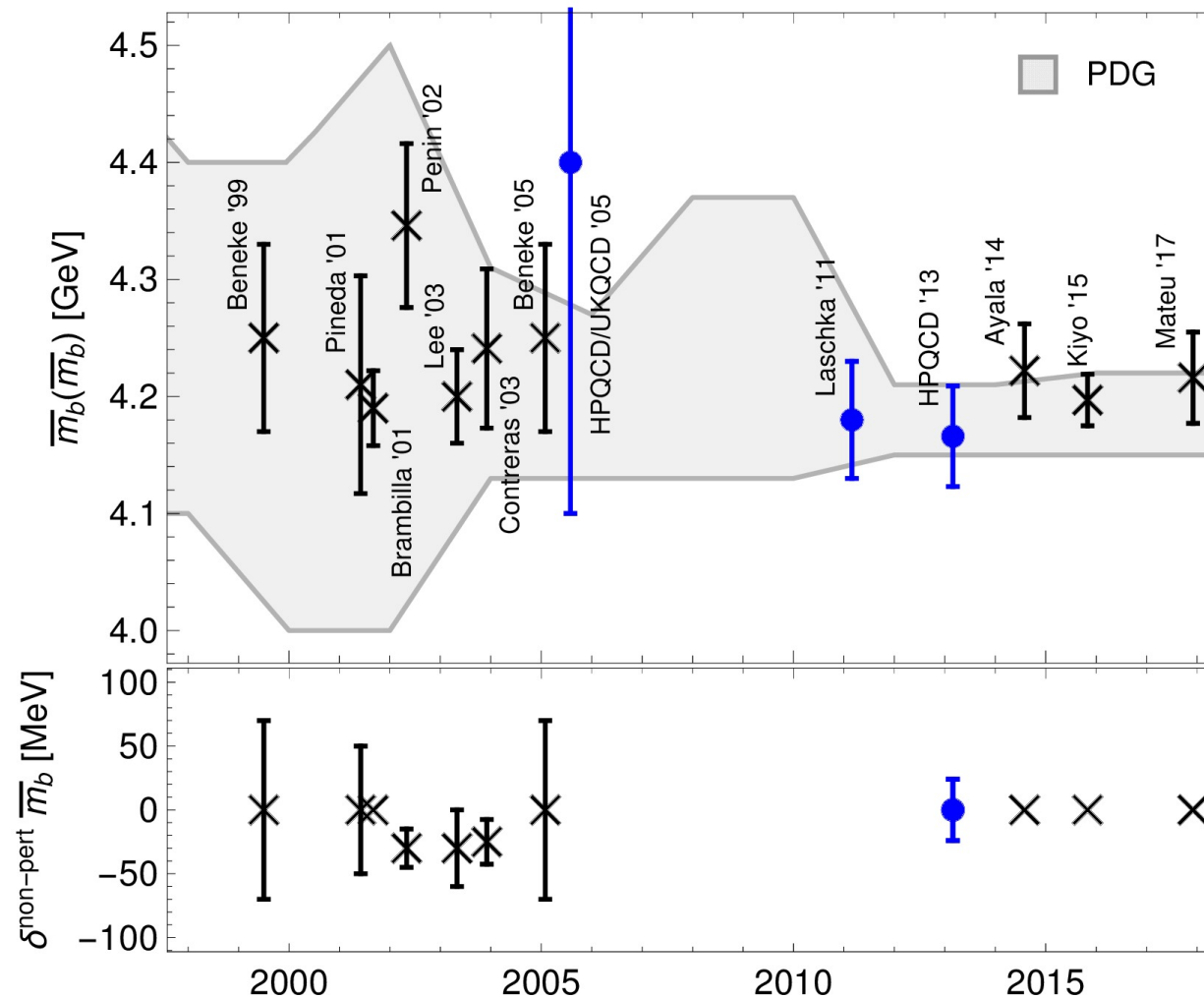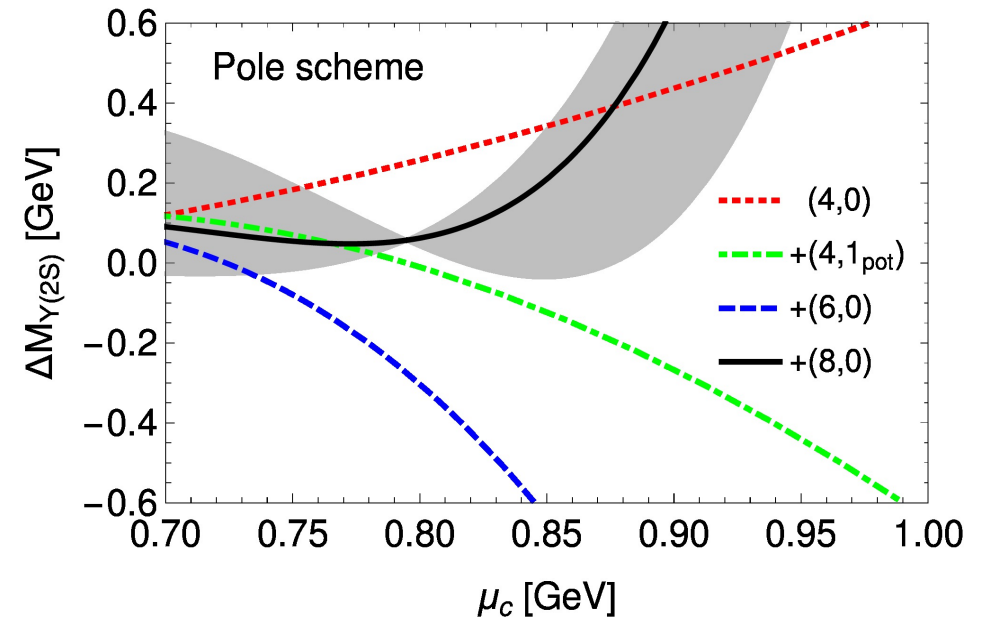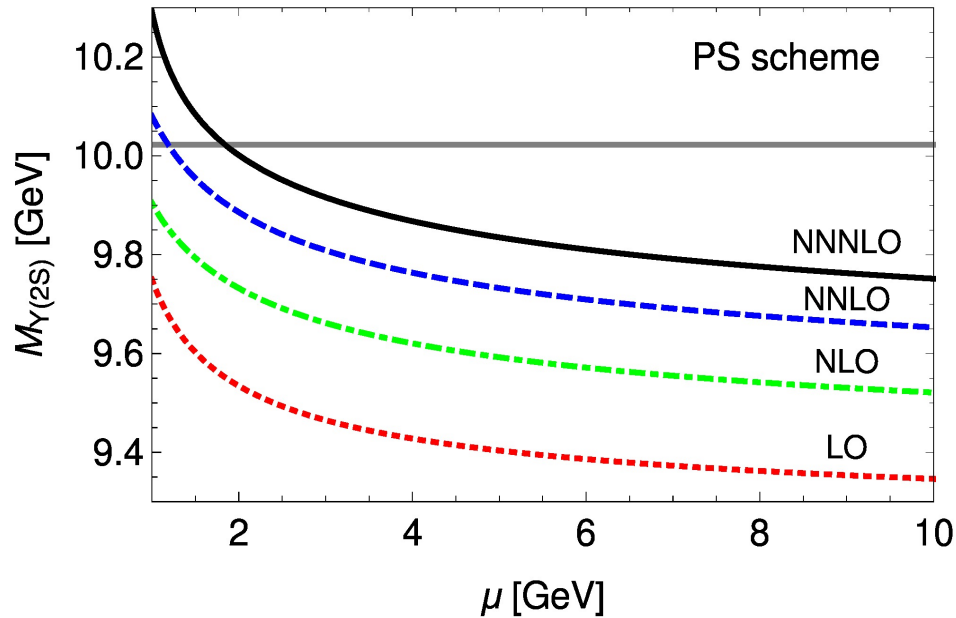
Bottom-quark mass
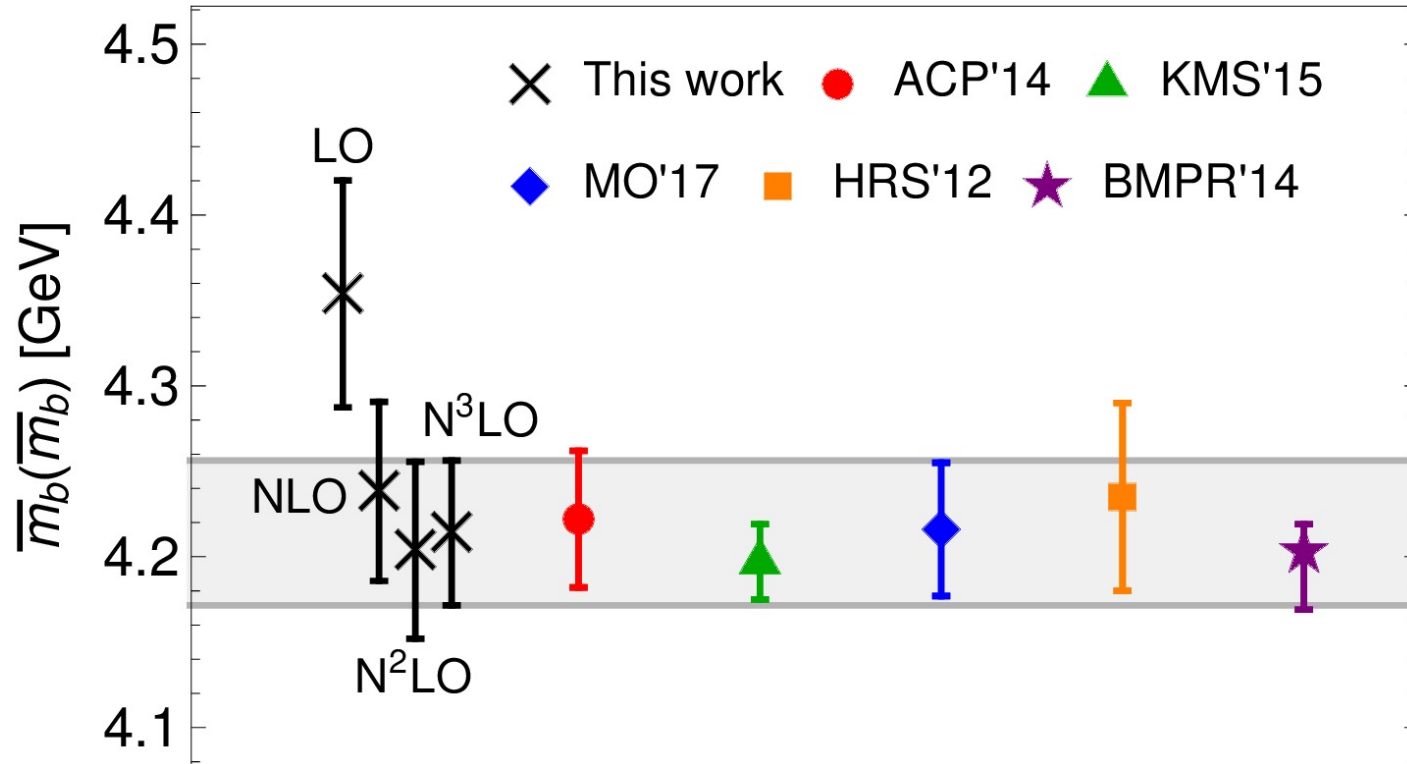
$$m_b^{\mathrm{PS}}(2\,\mathrm{GeV}) = 4544 \pm 39\,(\mathrm{pert.})\,{}^{+22}_{-25}\,(\mathrm{non\text{-}pert.})\,\mathrm{MeV} = 4544\,{}^{+44}_{-46}\,\mathrm{MeV},$$

$$\overline{m}_b(\overline{m}_b) = 4214 \pm 37\,(\mathrm{pert.})\,{}^{+20}_{-22}\,(\mathrm{non\text{-}pert.})\,\mathrm{MeV} = 4214\,{}^{+42}_{-43}\,\mathrm{MeV}.$$

[TR '18]

Upsilon(2S) mass

$$M_{\Upsilon(2S)}^{\text{pert}}\left(2\,\text{GeV}\right) = \left(9\,534 + 198 + 154 + 116\right)\,\text{MeV}.$$

$$\Delta M_{\Upsilon(2S)}^{\text{cond}}\left(0.8\,\text{GeV}\right) = \left[\left(258 - 267\right)\frac{O_0}{O_0^{\text{SVZ}}} - 293\,\frac{O_1}{O_1^{\text{naive}}} + 365\,\frac{O_2}{O_2^{\text{naive}}}\right]\,\text{MeV}.$$
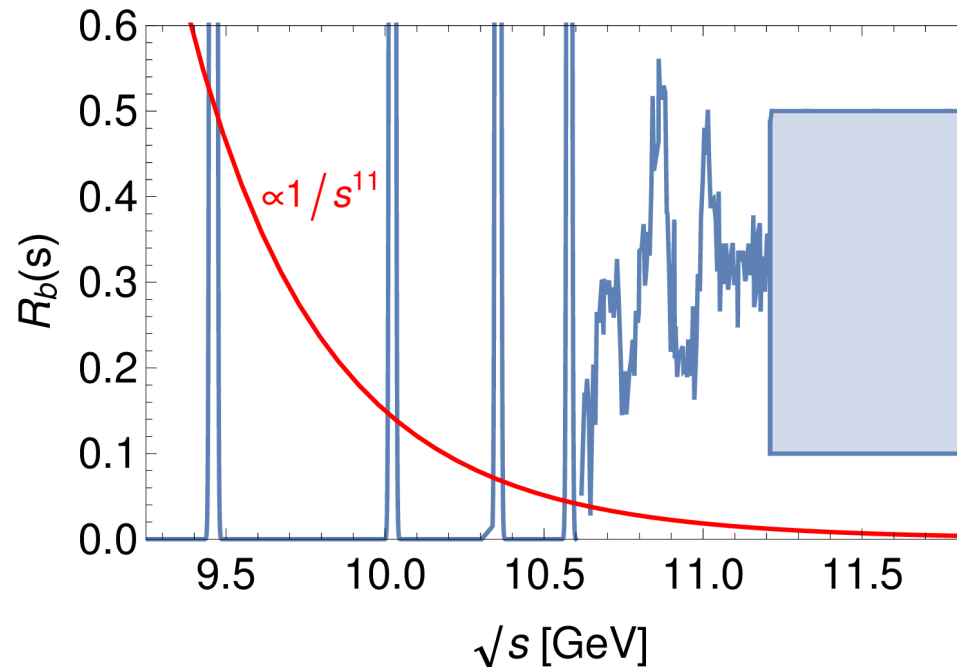
Does not even converge for scales as low as 0.8 GeV.
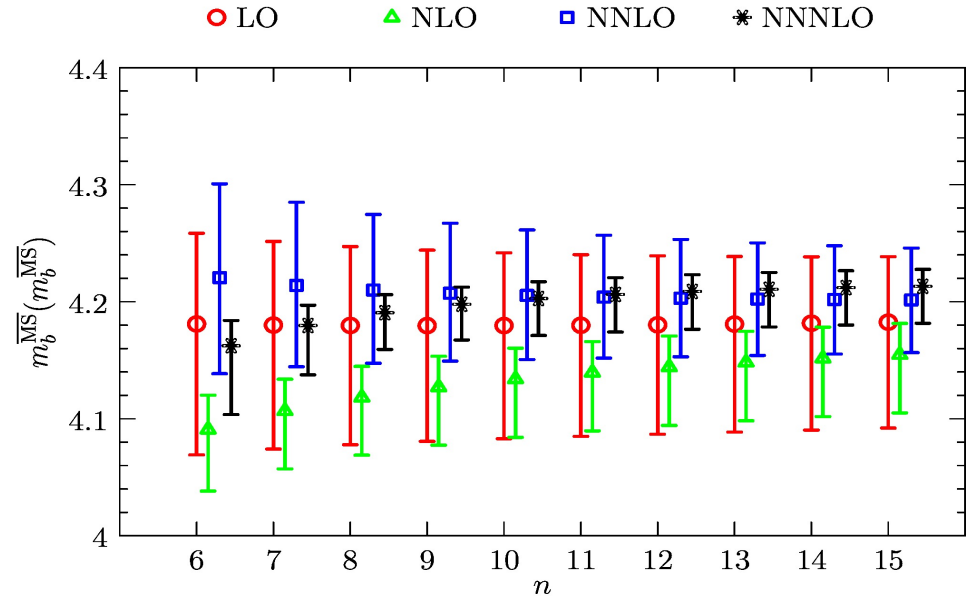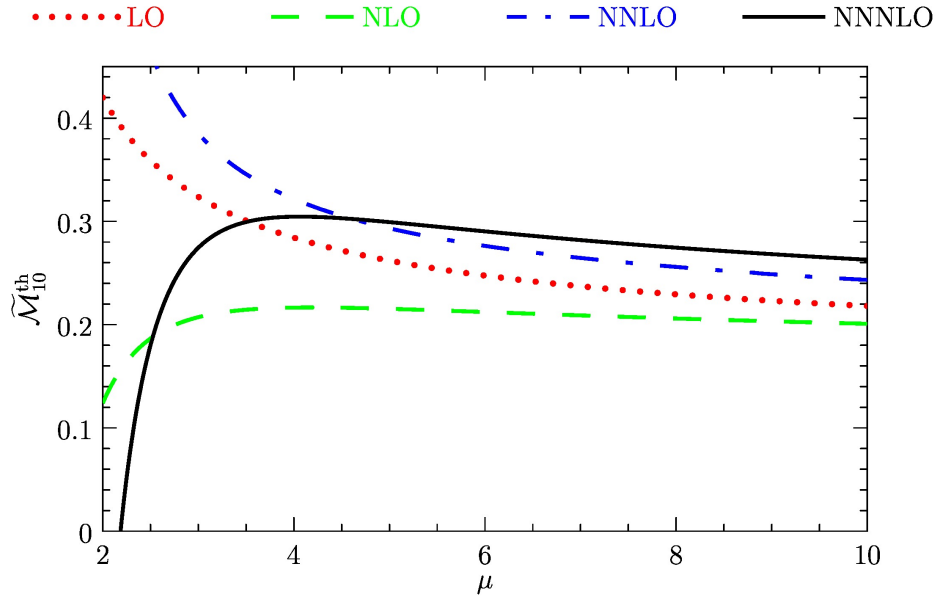
Bottom-quark mass from sum rules

- Consider moments for large n (around 10)

$$\mathcal{M}_n \equiv \int_0^\infty ds\, \frac{R_b(s)}{s^{n+1}} = \frac{12\pi^2}{n!}\left(\frac{d}{dq^2}\right)^n \Pi_b(q^2)\bigg|_{q^2=0}.$$

- Known at NNNLO   [Beneke, Maier, Piclum, TR '14]

- Depend strongly on the mass: $\mathcal{M}_n \propto m_b^{-2n}$

- Saturated by the Upsilon resonances

Bottom-quark mass from sum rules

$$m_b^{\mathrm{PS}}(2\,\mathrm{GeV}) = \left[4.532^{+0.002}_{-0.035}(\mu) \pm 0.010(\alpha_s)^{+0.003}_{-0}(\mathrm{res}) \pm 0.001(\mathrm{conv})\right.$$

$$\left. \pm\, 0.002(\mathrm{charm})^{+0.007}_{-0.013}(n) \pm 0.003(\mathrm{exp})\right]\,\mathrm{GeV}$$

$$= 4.532^{+0.013}_{-0.039}\,\mathrm{GeV}\,.$$

$$m_b(m_b) = \left[4.203^{+0.002}_{-0.031}(\mu) \pm 0.002(\alpha_s)^{+0.003}_{-0}(\mathrm{res})^{+0.013}_{-0.004}(\mathrm{conv})\right.$$

$$\left. \pm\, 0.002(\mathrm{charm})^{+0.006}_{-0.012}(n) \pm 0.003(\mathrm{exp})\right]\,\mathrm{GeV}$$

$$= 4.203^{+0.016}_{-0.034}\,\mathrm{GeV}\,.$$

[Beneke, Piclum, Maier, TR '14, '16]

Leading order condensate corrections

| n	8	10	12	16	20	24
$\mathcal{M}_n^{\mathrm{exp},\Upsilon(1S)}/\mathcal{M}_n^{\mathrm{exp}}$	0.738	0.803	0.850	0.913	0.948	0.969
$\widetilde{\mathcal{M}}_n^{\mathrm{pert},\Upsilon(1S)}/\widetilde{\mathcal{M}}_n^{\mathrm{pert}}$	0.769	0.814	0.849	0.899	0.932	0.953
$\widetilde{\mathcal{M}}_n^{\mathrm{pert,rest}}/\widetilde{\mathcal{M}}_n^{\mathrm{pert}}$	0.231	0.186	0.151	0.101	0.068	0.047
$\delta_{\langle G^2\rangle}\widetilde{\mathcal{M}}_n^{\Upsilon(1S)}/\widetilde{\mathcal{M}}_n^{\mathrm{pert}}$	1.711	1.842	1.953	2.135	2.281	2.404
$\delta_{\langle G^2\rangle}\widetilde{\mathcal{M}}_n^{\mathrm{rest}}/\widetilde{\mathcal{M}}_n^{\mathrm{pert}}$	-1.713	-1.845	-1.957	-2.144	-2.296	-2.427
$\delta_{\langle G^2\rangle}\widetilde{\mathcal{M}}_n/\widetilde{\mathcal{M}}_n^{\mathrm{pert}}$	-0.002	-0.003	-0.005	-0.009	-0.015	-0.023

- Huge cancellations between the contribution to the 1S resonance and the rest (at scale mb)

- Corrections are small compared to the expectation from power counting

$$\delta_{\Lambda^4}\mathcal{M}_n/\mathcal{M}_n \sim \frac{1}{n}\left(\frac{n\Lambda_{\mathrm{QCD}}}{m_b}\right)^4$$

- From p.c. we expect a breakdown for $\mathrm{n} \sim m_b/\Lambda_{\mathrm{QCD}} \approx 16$ where

$$\delta_{\Lambda^4}\mathcal{M}_{16}/\mathcal{M}_{16}\big|_{\mathrm{p.c.\ expectation}} \approx 0.06$$

Dimension four at partial NLO
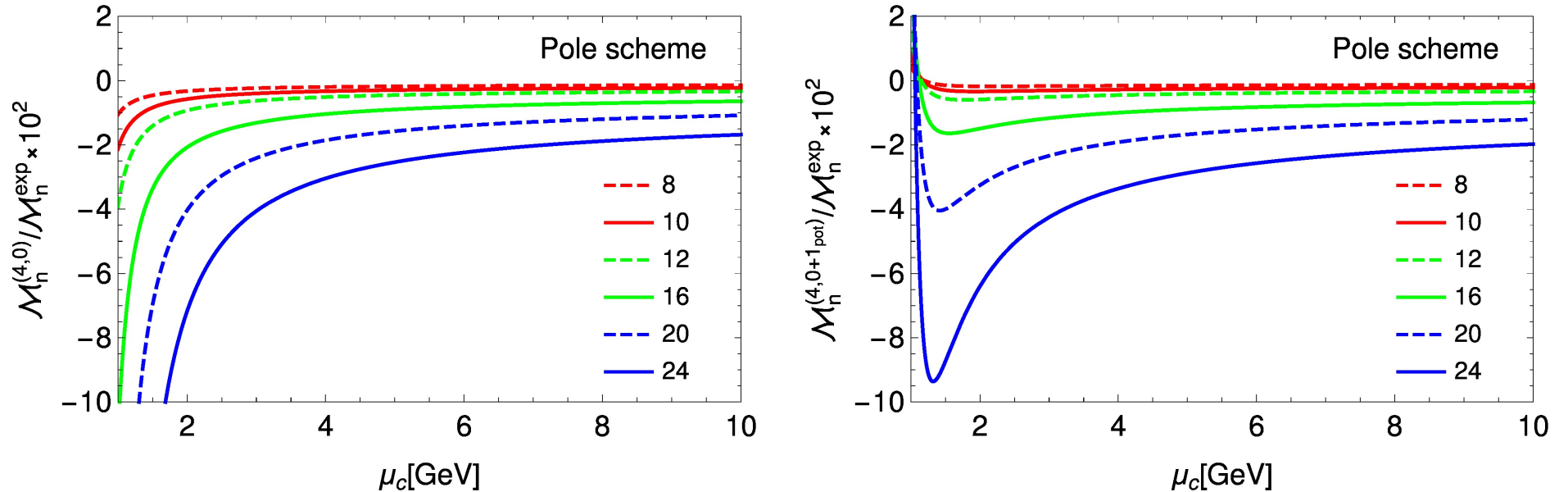
- Cancellations become more effective for larger scales and overcompensate the growth of the factor $\alpha_s^{-6}(\mu_c)$

- Stabilization of scale dependence from partial NLO corrections

- Taking a small scale ~1.5 GeV the results indicate a breakdown around n = 20 close to the p.c. expectation

Dimensions six and eight
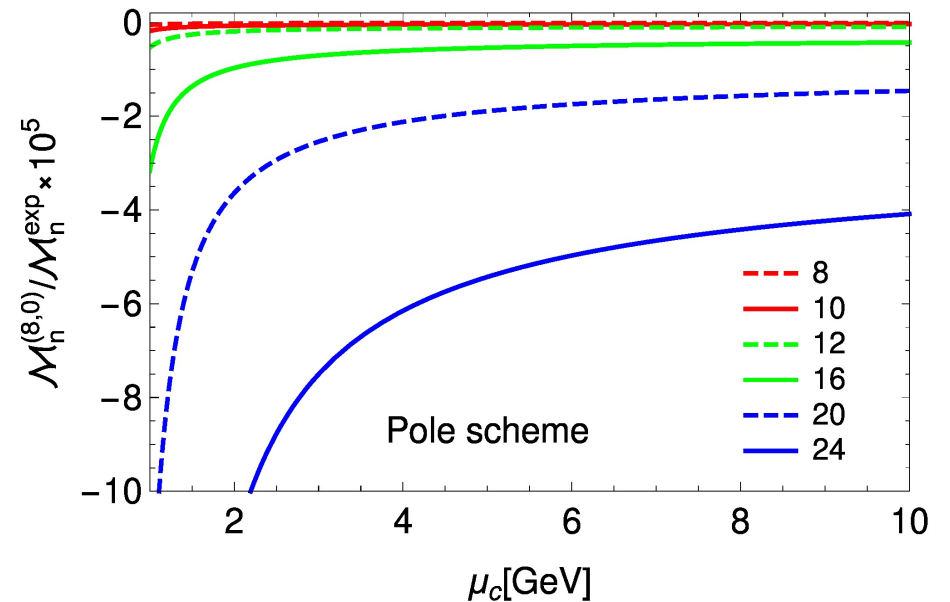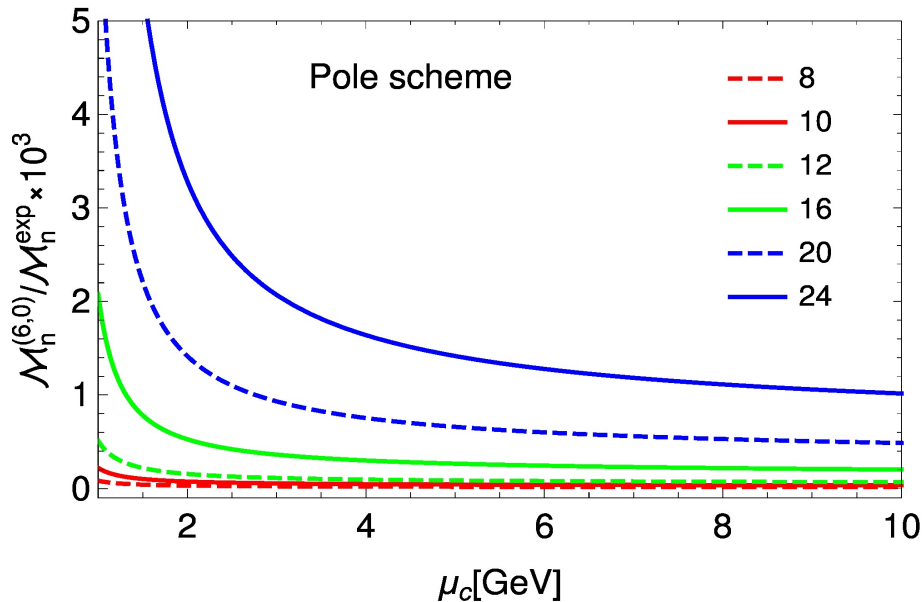
- Tiny compared to expectation

- Huge cancellations: at scale mb

  - Dimension six: one part in $3 \cdot 10^5 \ (n = 10), \ 5 \cdot 10^4 \ (n = 16), \ 10^4 \ (n = 24)$

  - Dimension eight: one part in $10^8 \ (n = 10), \ 10^7 \ (n = 16), \ 2 \cdot 10^6 \ (n = 24)$

- Only looking at the convergence from dimension four to six and eight, we would naively conclude that we can calculate the 50th moment reliably

Higher dimensions vs duality violations

- Recall that the moments are an off-shell quantity

$$\mathcal{M}_n = \frac{12\pi^2}{n!} \left(\frac{d}{dq^2} \right)^n \Pi_b(q^2) \Big|_{q^2=0} .$$

- Off-shellness acts as very efficient IR cutoff, cf. also the smallness of charm-quark mass effects which affect the extracted PS mass by only 1 MeV

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- The assumption of quark-hadron duality must be questioned when the moments are saturated by the 1S resonance (95% for n=20)

- Corrections of the form $\exp(-m_b/(n\Lambda_{\mathrm{QCD}}))$ are not captured in the condensate expansion (trivial Taylor expansion)

- Originates from "coherent soft fluctuations" [Shifman '00]:

  – Emission of many soft lines

  – Off-shellness can be distributed among soft lines pushing the $b\bar{b}$ on-shell

  – Therefore not/less affected by effective IR cutoff mechanism

- Can affect mb determination at a relevant level for n~20.
  For n~10 duality violations are exponentially suppressed.

Conclusions

- Computed local condensates up to dimension eight and partial NLO corrections at dimensions four

- Partial NLO corrections provide preferred scale choice

- Good convergence for $M_{\Upsilon(1S)}$ allows the determination of the bottom-quark mass with a non-perturbative uncertainty of about 20 MeV

- No convergence for excited states, non-local condensates?

- Description of moments is not limited by convergence of condensate expansion, but our knowledge (or rather lack thereof) about violations of quark-hadron duality

- Conservative approaches should use $n \lesssim 15$

- Sum rule for n~10 very clean, most reliable method for mb determination from the Upsilon system

Thank you!

Upsilon(2S) mass

- A more promising approach is $\Lambda_{\mathrm{QCD}} \sim m_b v^2 \ll m_b v \ll m_b$

- The dipole interaction $\hat{H}_D$ can still be treated as a perturbation, but not the non-perturbative Hamiltonian $\hat{H}_{\mathrm{np}}$

- The non-perturbative contribution takes the form of a non-local condensate

$$\delta M_n^{\mathrm{non-pert}} = \frac{T_F}{3N_c} \int_0^\infty dt \langle n | \mathbf{r} e^{-t(H_o - E_n)} \mathbf{r} | n \rangle \langle g\mathbf{E}^a(t)\phi(t,0)_{ab}^{\mathrm{adj}} g\mathbf{E}^b(0) \rangle$$

[Voloshin '79; Brambilla, Pineda, Soto, Vairo '99, Pineda '01]

- Results for this are currently not available

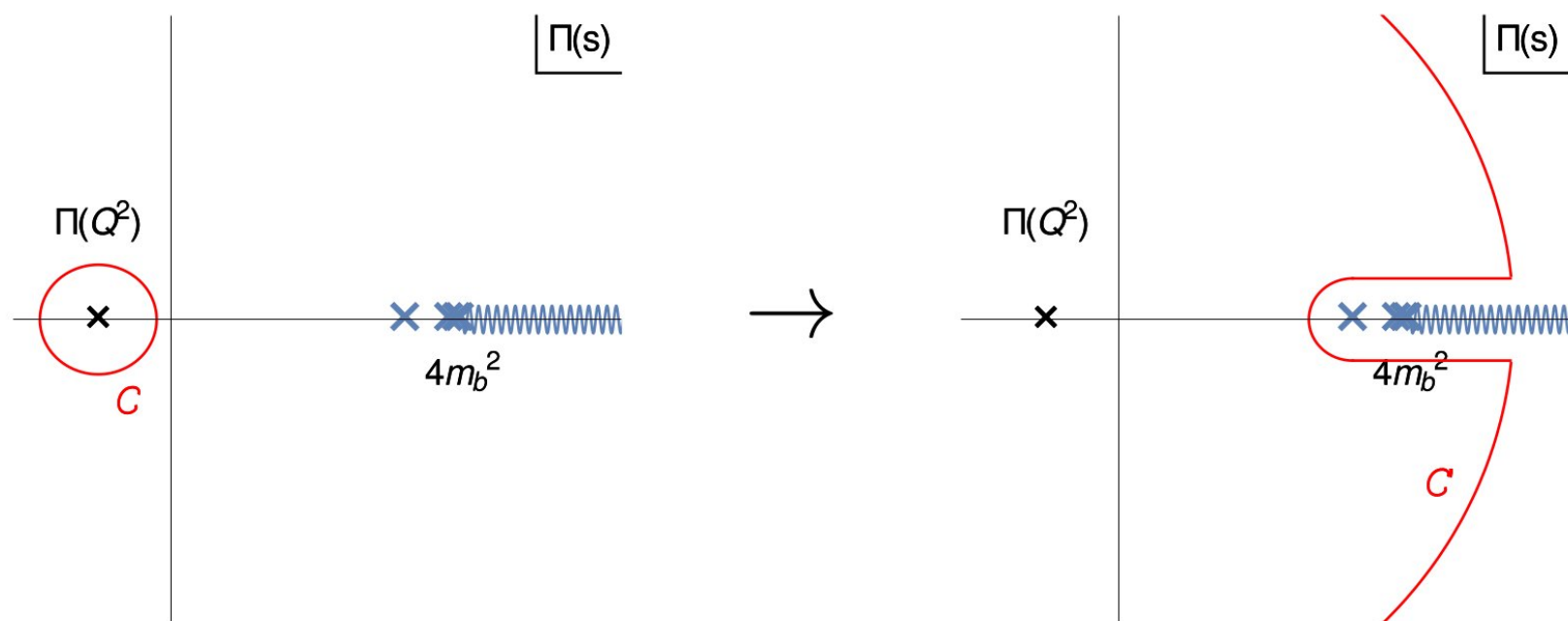- Estimate for the size of $\delta M_2^{\mathrm{non-pert}}$ from power counting gives

$$M_{\Upsilon(2S)}^{\mathrm{exp}} = 10023.26 \pm 0.31 \, \mathrm{MeV},$$

$$M_{\Upsilon(2S)} = 9\,886 \, {}^{+195}_{-122}\,(\mu) \, {}^{+25}_{-76}\,(m_b) \, {}^{+28}_{-26}\,(\alpha_s) \pm \mathcal{O}(100)\,(\text{non-pert.}) \, \mathrm{MeV}.$$

Upsilon sum rules

- Derive a dispersion relation using analyticity

$$\Pi_b(Q^2) = \frac{1}{2\pi i} \oint_{\mathcal{C}} dz \frac{\Pi_b(z)}{z - Q^2}$$

$$\Pi_b(Q^2) = \frac{1}{\pi} \int_{s_0}^{\infty} ds \frac{\operatorname{Im} \Pi_b(s)}{s - Q^2} + \frac{1}{2\pi i} \oint_{\bigcirc} dz \frac{\Pi_b(z)}{z - Q^2}.$$

Upsilon sum rules

- Derive a dispersion relation using analyticity
- Sum rule follows from derivatives at $Q^2 = 0$

$$\mathcal{M}_n = \int_0^\infty ds \, \frac{R_b(s)}{s^{n+1}} = \frac{12\pi^2}{n!} \left(\frac{d}{dq^2} \right)^n \Pi_b\left(q^2\right)\bigg|_{q^2=0}.$$

- Left hand side is experimental observable
- Right hand side can be computed within condensate expansion

  - For n~10 dominated by threshold region
  - $1/\sqrt{n}$ plays the role of the velocity
  - Strong dependence on the bottom quark mass: $\mathcal{M}_n \sim m_b^{-2n}$

- Assuming quark-hadron duality, the bottom quark mass can be determined by fitting the RHS to the LHS

[Novikov, Okun, Shifman, Vainshtein, Voloshin, Zakharov '77-'78]

PNRQCD

- Potential NRQCD is given by the Lagrangian

$$\mathcal{L}_{\text{PNRQCD}} = \psi^\dagger \left(i\partial_0 + \frac{\partial^2}{2m_t} + \frac{\partial^4}{8m_t^3} + g_s A_0(t, \mathbf{0}) - g_s \mathbf{x} \cdot \mathbf{E}(t, 0) \right) \psi + \left(\text{anti-quark} \right)$$

$$+ \int d^{d-1}\mathbf{r} \left[\psi_a^\dagger \psi_b \right] (x + \mathbf{r}) V_{ab;cd}(\mathbf{r}) \left[\chi_c^\dagger \chi_d \right] (x)$$

- Contains potential (anti)quark fields $\psi(\chi)$ with $p^0 \sim mv^2, \mathbf{p} \sim mv$ and heavy quark potentials $V_{ab;cd}$

- The ultrasoft gluon field is multipole expanded

- The colour-singlet projection of the potential has the form

$$V(\mathbf{p}, \mathbf{p}') = -\frac{4\pi\alpha_s C_F}{\mathbf{q}^2} \left[1 + \frac{\alpha_s}{4\pi} \mathcal{V}_C^{(1)} + \mathcal{O}(\alpha_s^2) \right] + \dots,$$

$$\mathcal{V}_C^{(1)} = \left[\left(\frac{\mu^2}{\mathbf{q}^2} \right)^\epsilon - 1 \right] \frac{\beta_0}{\epsilon} + \left(\frac{\mu^2}{\mathbf{q}^2} \right)^\epsilon a_1(\epsilon).$$
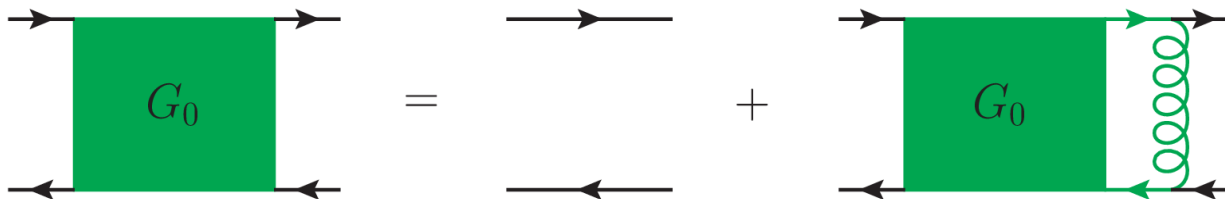
Non-relativistic Green function

- LO Coulomb potential is of the same order as the leading kinetic terms

  - Must be treated non-perturbatively

  - LO Lagrangian describes propagation of quark-antiquark pairs, where ladder diagrams with exchange of of potential gluons have been resummed
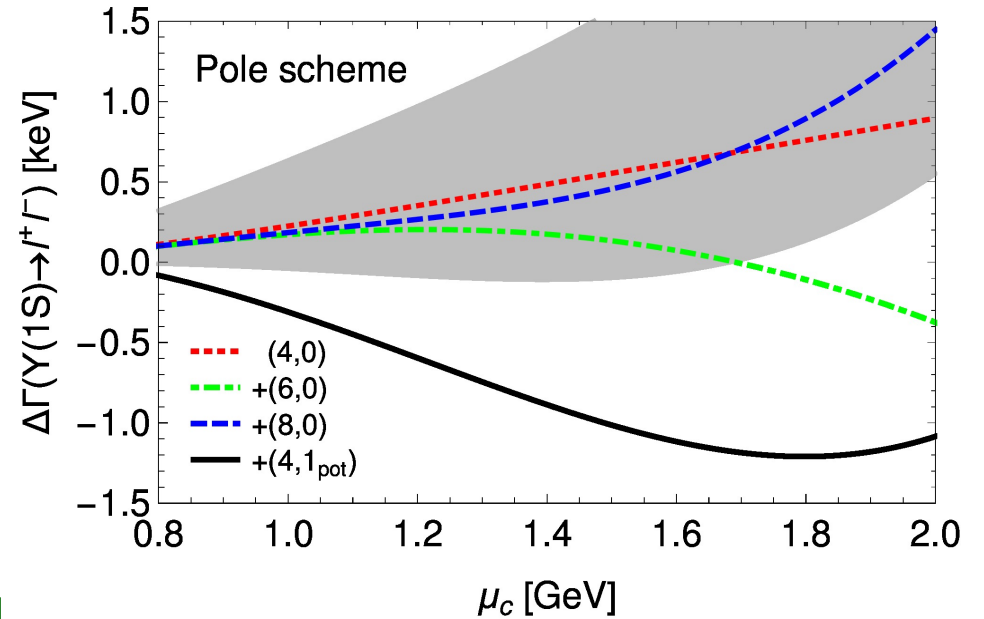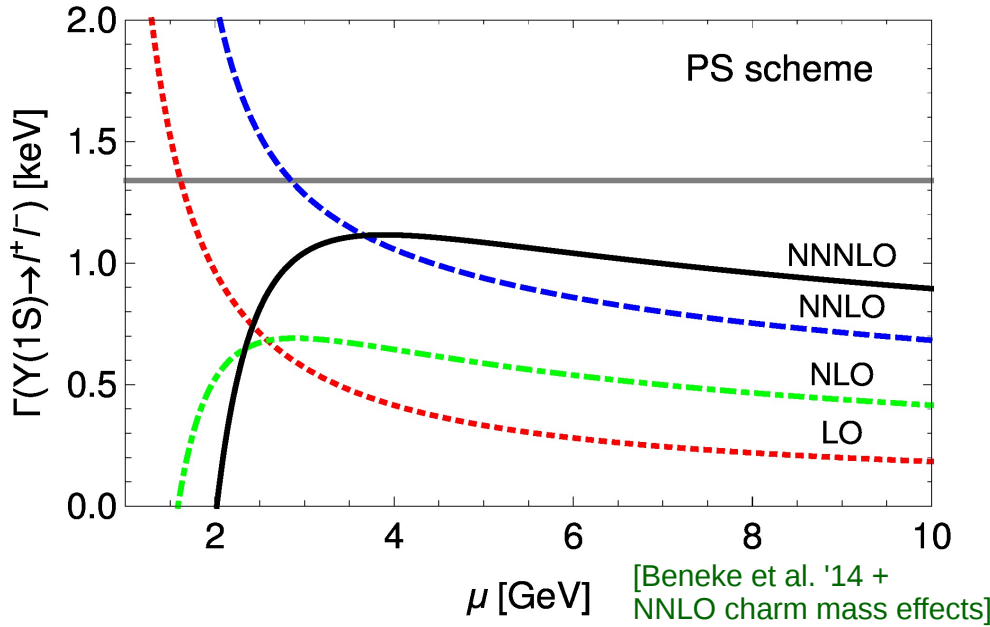

- Green function satisfies d-dimensional Lippmann-Schwinger equation

$$\left(\frac{\mathbf{p}^2}{m_t} - E \right) \tilde{G}_0(\mathbf{p}, \mathbf{p}'; E) - \tilde{\mu}^{2\epsilon} \int \frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}} \frac{4\pi C_F \alpha_s}{\mathbf{k}^2} \tilde{G}_0(\mathbf{p} - \mathbf{k}, \mathbf{p}'; E)$$
$$= (2\pi)^{d-1} \delta^{(d-1)}(\mathbf{p} - \mathbf{p}'),$$

Leptonic Upsilon(1S) width

[Beneke et al. '14 + NNLO charm mass effects]

$$\Gamma^{\text{exp}}(\Upsilon(1S) \to l^+ l^-) = 1.340 \pm 0.018 \,\text{keV}$$

$$\Gamma^{\text{pert}}(\Upsilon(1S) \to l^+ l^-)(3.5\,\text{GeV}) = \frac{4\pi\alpha^2}{9m_b^2} c_v \left[c_v - \left(c_v + \frac{d_v}{3} \right) \frac{E_1}{m_b} \right] |\psi_1(0)|^2 =$$

$$(0.48 + 0.19 + 0.47 - 0.04)\,\text{keV} = 1.11\,^{+0.01}_{-0.21}\,(\mu) \pm 0.00(m_b) \pm 0.04(m_c) \pm 0.05(\alpha_s)\,\text{keV}$$

$$\Gamma^{\text{cond}}(\Upsilon(1S) \to l^+ l^-)(1.2\,\text{GeV}) = \left[(352 - 862)\frac{O_0}{O_0^{\text{SVZ}}} - 149\frac{O_1}{O_1^{\text{naive}}} + 64\frac{O_2}{O_2^{\text{naive}}} \right]\,\text{eV}.$$