Nonequilibrium viscous correction to the phase space density and bulk viscosity in the relaxation time approximation

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based on: Phys. Rev. C97 (2018) no. 4, 044914

Quark Confinement and the Hadron Spectrum,
Maynooth University, Ireland, 1-6 August 2018

Hydrodynamics and heavy ion collisions

- ✓ Relativistic fluid dynamics describes very well the evolution of matter produced in heavy ion collisions after it achieves approximate local thermal equilibrium

- ✓ Hydrodynamics - macroscopic description of a system; transport coefficients - parameters

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - \Delta^{\mu\nu}(P + \Pi) + \pi^{\mu\nu}$$

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

ϵ - energy density

P - pressure

u^μ - four-velocity

- ✓ Viscous corrections – determined by transport coefficients

$\pi^{\mu\nu}$ - stress tensor determined by η, τ_π

Π - bulk pressure determined by ζ, τ_Π

Role of bulk viscosity in nuclear matter dynamics

- Bulk viscosity plays an important role in hydrodynamic modelling of strongly interacting matter

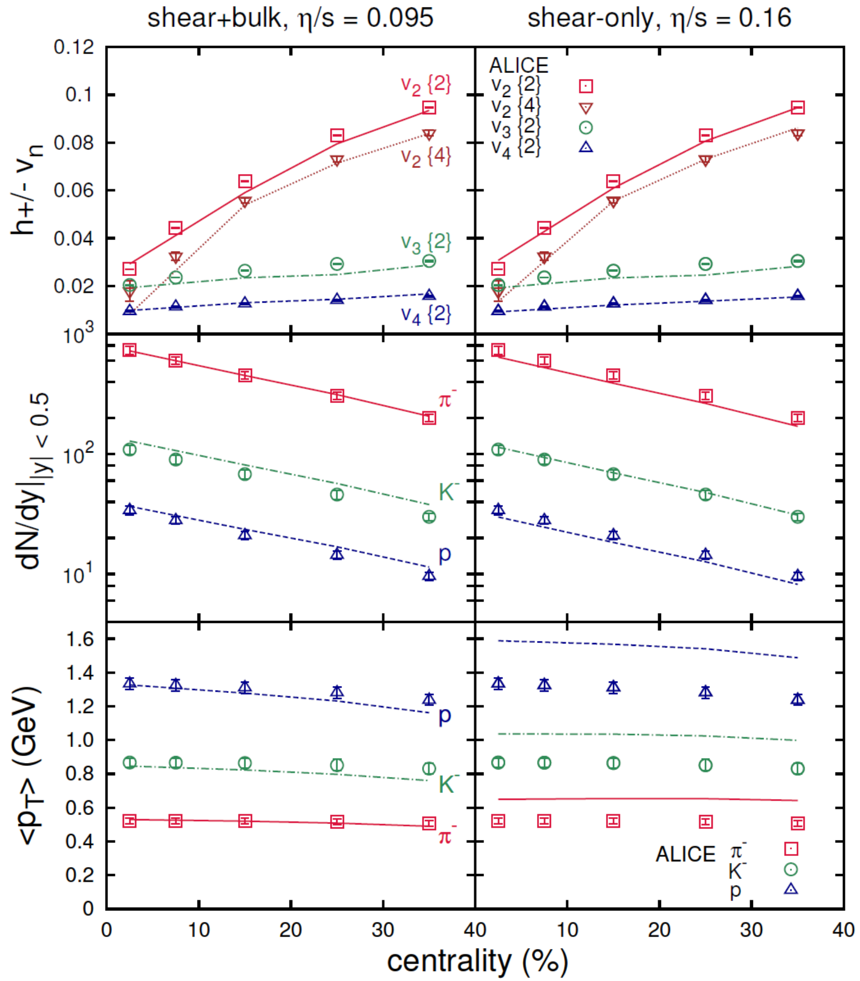
  Ryu et al: PRL 115 (2015) 132301, PRC 97 (2018) 034910

  Simultaneous description of the multiplicity and the mean transverse momentum

- Bulk viscosity over entropy density is large $\zeta/s \simeq 0.3$

- Relevant for IP-Glasma initial conditions

  Schenke, Shen, Tribedy: arXiv:1807.05205



Motivation

- need for consistent, well-defined equations of hydrodynamics which can provide reliable phenomenological estimates
- QCD – complex, multi-scale system
- lack of systematic microscopic methods to compute the bulk viscosity in the phase transition region

→ **Physics of bulk viscosity can be understood by studying „simpler" systems in the perturbative regime where analytical methods can be applied**

→ **Need for a consistent inclusion of mean field effects into hydrodynamics**
(Many attempts undertaken before, but none of them was able to fully capture all subtleties related to the mean field inclusion)

Outline

- Nonequilibrium deviation from the equilibrium distribution function

- Equations of hydrodynamics with mean field effects
  - Equilibrium hydrodynamics
  - Nonequilibrium hydrodynamics: Landau condition and viscous corrections

- Bulk viscosity computation
  - Anderson-Witting model of the Chapman-Enskog approach
  - 14-moment approximation

- Summary and conclusions

Basics

What do we study?

The system is made of scalar quasi-particles with zero-temperature mass m_0

We consider both the Boltzmann (classical) gas and the Bose-Einstein (quantum) gas

It is a weakly interacting system with the coupling constant λ

The coupling is running so $\beta_\lambda = T d\lambda / dT$

massless particles and no running coupling ⟶ the system is conformal ⟶ bulk viscosity vanishes

Nonequilibrium vs. equilibrium

Dictionary

	Equilibrium (well defined state)	**Nonequilibrium** (small deviations, perturbative corrections to equilibrium quantities)
Quasiparticle thermal mass	$m_{\text{eq}} \equiv m_{\text{eq}}(x)$	$m_{\text{th}} \equiv m_{\text{th}}(x)$
Quasiparticle mass	$m_x = \sqrt{m_0^2 + m_{\text{eq}}^2}$	$\tilde{m}_x = \sqrt{m_0^2 + m_{\text{th}}^2}$
Quasiparticle energy	$E_k = \sqrt{\mathbf{k}^2 + m_x^2}$	$\mathcal{E}_k = \sqrt{\mathbf{k}^2 + \tilde{m}_x^2}$
Quasiparticle four-momentum	$k^\mu \equiv (k_0, \mathbf{k}) = (E_k, \mathbf{k})$	$\tilde{k}^\mu \equiv (\tilde{k}_0, \mathbf{k}) = (\mathcal{E}_k, \mathbf{k})$
Lorentz invariant measure	$dK = d^3\mathbf{k}/[(2\pi)^3 E_k]$	$d\mathcal{K} = d^3\mathbf{k}/[(2\pi)^3 \mathcal{E}_k]$
Distribution function	$f_0 = 1/[e^{\beta E_k} - 1]$	$f = f_0 + \Delta f$

Nonequilibrium deviation from the equilibrium distribution function

Boltzmann equation with the mean field effect

$$(\tilde{k}^\mu \partial_\mu - \underbrace{\mathcal{E}_k \nabla \mathcal{E}_k \cdot \nabla_k}_{\text{term involving force}}) f = \underbrace{C[f]}_{\text{collision kernel}}$$

All quantities entering the equation are x-dependent

$$f(x,k) = \underbrace{f_{\rm th}(x,k)}_{\text{retains equilibrium form}} + \delta f(x,k) = f_0(x,k) + \delta f_{\rm th}(x,k) + \delta f(x,k)$$

$$\Delta f(x,k) = \delta f_{\rm th}(x,k) + \delta f(x,k)$$

$$f_{\rm th}(x,k) \equiv f_0(x,k)|_{m_0^2+m_{\rm eq}^2(x) \to m_0^2+m_{\rm eq}^2(x)+\Delta m_{\rm th}^2(x)} = \left[\exp\left(\sqrt{\mathbf{k}^2 + m_0^2 + m_{\rm eq}^2(x) + \underbrace{\Delta m_{\rm th}^2(x)}_{\text{correction from the nonequilibrium thermal mass}}}\,\beta(x)\right) - 1\right]^{-1}$$

$$\Delta f = \delta f - \beta f_0 (1+f_0) \frac{\Delta m_{\rm th}^2}{2E_k}$$

Nonequilibrium deviation from the equilibrium distribution function

Equilibrium vs. nonequilibrium thermal mass

Equilibrium:

$$m_{\rm eq}^2 = \frac{\lambda(q_0)}{2} q_0$$

$$q_0 = \int dK f_0$$

Nonequilibrium:

$$m_{\rm th}^2(q) = m_{\rm th}^2(q_0 + \Delta q) = m_{\rm eq}^2(q_0) + \Delta m_{\rm th}^2$$

$$\Delta m_{\rm th}^2 = \frac{dm_{\rm eq}^2}{dq_0} \Delta q$$

$$q = \int d\mathcal{K} f$$

$$\Delta q = \int dK \delta f + \frac{\partial q_0}{\partial m_{\rm eq}^2} \Delta m_{\rm th}^2$$

Solving self-consistently:

$$\Delta m_{\rm th}^2 = 2T^2 \frac{dm_{\rm eq}^2}{dT^2} \frac{\int dK \delta f}{\beta \int dK E_k f_0 (1 + f_0)}$$

Full form of the correction to the distribution function:

$$\Delta f = \delta f - T^2 \frac{dm_{\rm eq}^2}{dT^2} \frac{f_0(1+f_0)}{E_k} \frac{\int dK \delta f}{\int dK E_k f_0 (1 + f_0)}$$

Nonequilibrium deviation from the equilibrium distribution function

Thermal mass – temperature dependence

For any theory with $m_{\mathrm{eq}}^2 = \mathcal{O}(\lambda T^2)$ $\lambda \ll 1$

$$m_{\mathrm{eq}}^2 = \frac{\lambda(q_0)}{2} q_0 \quad \Rightarrow \quad \frac{dm_{\mathrm{eq}}^2}{dT} = \frac{\lambda(q_0)}{2}\frac{dq_0}{dT} + \frac{q_0}{2}\frac{d\lambda(q_0)}{dT}$$

$$\beta_\lambda \equiv \beta(\lambda) = T\frac{d\lambda(q_0)}{dT} \qquad \beta_\lambda = \mathcal{O}(\lambda^2)$$

Temperature dependence of the thermal mass in weakly interacting systems:

$$T^2\frac{dm_{\mathrm{eq}}^2}{dT^2} = m_{\mathrm{eq}}^2 + aT^2\beta_\lambda$$

Equations of hydrodynamics

Local equilibrium hydrodynamics

Energy-momentum tensor:

$$T_0^{\mu\nu} = \int dK k^\mu k^\nu f_0 - g^{\mu\nu} U_0$$

$$T_0^{\mu\nu} = \epsilon_0 u^\mu u^\nu - P_0 \Delta^{\mu\nu}$$

mean-field contribution

- thermodynamic consistency of hydrodynamic equations
- conservation of energy and momentum

$$dU_0 = \frac{q_0}{2} dm_{\rm eq}^2$$

Energy density and pressure:

$$\epsilon_0 = \bar{\epsilon}_0 - U_0, \qquad \bar{\epsilon}_0 = \int dK (u_\mu k^\mu)^2 f_0$$

$$P_0 = \bar{P}_0 + U_0 \qquad \bar{P}_0 = -\frac{1}{3} \int dK \Delta^{\mu\nu} k_\mu k_\nu f_0$$

- Enthalpy not changed: $\bar{\epsilon}_0 + \bar{P}_0 = \epsilon_0 + P_0$
- Thermodynamic relation satisfied: $T s_0 = T \frac{dP_0}{dT} = \epsilon_0 + P_0$

Equations of hydrodynamics

Nonequilibrium hydrodynamics

Energy-momentum tensor:

Non-equilibrium mean-field contribution

$$T^{\mu\nu} = \int d\mathcal{K} \tilde{k}^\mu \tilde{k}^\nu f - g^{\mu\nu} U$$

$$U = U_0 + \Delta U \qquad \Delta U = \frac{q_0}{2} \Delta m_{\mathrm{th}}^2$$

All quantities contain nonequilibrium thermal mass correction

$$T^{\mu\nu} = T_0^{\mu\nu} + \Delta T^{\mu\nu}$$

Particular components:

$$\Delta T^{00} = \int dK E_k^2 \Delta f$$

$$\Delta T^{0i} = \int dK E_k k^i \Delta f$$

$$\Delta T^{ij} = \int dK k^i k^j \Delta f - \frac{\Delta m_{\mathrm{th}}^2}{2} \int dK \frac{k^i k^j}{E_k^2} f_0 + \delta^{ij} \frac{\Delta m_{\mathrm{th}}^2}{2} \int dK f_0$$

Equations of hydrodynamics

Nonequilibrium hydrodynamics – local rest frame

Landau matching is defined by the eigenvalue problem $u_\mu T^{\mu\nu} = \epsilon u^\nu$

Local rest frame: $u^\mu = (1,0,0,0)$ ⇒ $T^{00} = \epsilon$, $T^{0i} = 0$ ⇒ $\Delta T^{00} = 0$, $\Delta T^{0i} = 0$

Landau matching conditions:

$$\int dK E_k k^i \delta f = 0 \qquad \int dK \left[E_k^2 - T^2 \frac{dm_{\rm eq}^2}{dT^2} \right] \delta f = 0$$

Contains the medium correction

Viscous corrections:

$$\Delta T^{ij} = \int dK k^i k^j \delta f \Rightarrow \pi^{ij} = \int dK k^{\langle i} k^{j \rangle} \delta f, \quad \Pi = \frac{1}{3} \int dK \mathbf{k}^2 \delta f$$

Known structures but x-dependent mass enters the equations

Equations of hydrodynamics

Nonequilibrium hydrodynamics – general frame

Energy-momentum tensor:

$$T^{\mu\nu} = \int dK k^\mu k^\nu f_0 - g^{\mu\nu} U_0 + \int dK \left[k^\mu k^\nu - u^\mu u^\nu T^2 \frac{dm_{\rm eq}^2}{dT^2} \right] \delta f$$

Landau matching condition:

$$\int dK \left[(u_\mu k^\mu) k^\nu - u^\nu T^2 \frac{dm_{\rm eq}^2}{dT^2} \right] \delta f = 0$$

Viscous corrections:

$$\pi^{\mu\nu} = \int dK k^{\langle\mu} k^{\nu\rangle} \delta f$$

$$\Pi = -\frac{1}{3} \int dK \Delta_{\mu\nu} k^\mu k^\nu \delta f$$

$$A^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta} A^{\alpha\beta}$$

$$\Delta^{\mu\nu}_{\alpha\beta} \equiv (\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha - 2/3 \Delta^{\mu\nu} \Delta_{\alpha\beta})/2$$

Transport coefficients

Methods

QUANTUM FIELD THEORY

- **Kubo formulas**

Jeon: PRD 52 (1995) 3591
Moore, Sohrabi: PRL 106 (2011) 122302, JHEP 1211 (2012) 148

KINETIC THEORY

- **Chapman-Enskog approach** $f = f_0 + f_1 + f_2 + \dots$

  Solving Boltzmann equation with a full collision kernel

  Anderson-Witting model (relaxation time approximation)

Jeon, Yaffe: PRD 53 (1996) 5799
Arnold, Moore, Yaffe: JHEP 0011 (2000) 001, JHEP 0301 (2003) 030, JHEP 0305 (2003) 051,
York, Moore: PRD 79 (2009) 054011

- **methods of moments (14-moment approx.)**

Denicol et al: PRL 105 (2010) 162501, EPJA 48 (2012) 170, PRD 85 (2012) 114047, PRC 90 (2014) 024912

In this work we employ kinetic theory approaches in the relaxation time approximation

On the way to compute bulk viscosity (and other transport coefficients)

Anderson-Witting model

Boltzmann equation in the Anderson-Witting model

$$\left(\tilde{k}^{\mu}\partial_{\mu} - \mathcal{E}_k \nabla \mathcal{E}_k \cdot \nabla_k\right) f = -\frac{(u\cdot\tilde{k})}{\tau_R}\Delta f$$

LHS of the Boltzmann equation dictates the form of RHS

$$\delta f(k) = f_0(k)(1+f_0(k))\phi(k)$$

$$\Delta f(k) = f_0(k)(1+f_0(k))\left(\phi(k) - \frac{T^2}{E_k}\frac{dm^2_{\mathrm{eq}}}{dT^2}\frac{\int dK\phi(k)f_0(k)(1+f_0(k))}{\int dK E_k f_0(k)(1+f_0(k))}\right)$$

$\phi = \phi_{\mathrm{s}} + \phi_{\mathrm{b}}$ (shear part + bulk part)

Transport coefficients

Anderson-Witting model: shear viscosity

Solution of the A-W model for the shear part: $\phi_{\rm s}(k) = -\frac{\tau_R}{TE_k} k^{\langle j} k^{i\rangle} \partial_j u_i$ $\quad \delta f = f_0(1+f_0)\phi$

Shear viscosity can be computed using: $\pi^{ij} = 2\eta\sigma^{ij}$ $\quad \pi^{ij} = \int dK k^{\langle i} k^{j\rangle} \delta f$

$$\frac{\eta}{\tau_R} = \beta J_{3,2} \qquad \frac{\eta}{\tau_R} = \frac{\epsilon_0 + P_0}{5}$$

Shear viscosity is not influenced by the mean field in the leading order

For quantum gas: $J_{n,q} = a_q \int dK (u\cdot k)^{n-2q} (-\Delta_{\mu\nu} k^\mu k^\nu)^q \, f_0(k)(1+f_0(k))$

For classical gas: $J_{n,q} \to I_{n,q}$ $\quad I_{n,q} = a_q \int dK (u\cdot k)^{n-2q} (-\Delta_{\mu\nu} k^\mu k^\nu)^q f_{0,c}(k)$

Transport coefficients

Anderson-Witting model: bulk viscosity

Solution of the A-W model for the bulk part:

$$\phi_{\rm b}(k) = \beta\tau_R(\partial_i u^i)(c_s^2 - 1/3)\left(E_k - \frac{1}{E_k}\frac{J_{3,0} - T^2(dm_{\rm eq}^2/dT^2)J_{1,0}}{J_{1,0} - T^2(dm_{\rm eq}^2/dT^2)J_{-1,0}}\right)$$

$$\delta f = f_0(1+f_0)\phi$$

Bulk viscosity can be computed using: $\Pi = -\zeta\partial_i u^i$ $\quad$ $\Pi = M\int dK\delta f$

Nonconformality parameter:

Microscopic: $M = -\frac{1}{3}\left(m_0^2 - a\beta_\lambda T^2\right)$ $\quad$ $a - \mathrm{const}$

Macroscopic: $c_s^2 = \frac{dP_0/dT}{d\epsilon_0/dT}$ ➡ $\frac{1}{3} - c_s^2 = -\frac{MJ_{1,0}}{J_{3,0} - T^2(dm_{\rm eq}^2/dT^2)J_{1,0}}$

Anderson-Witting model: bulk viscosity

Bulk viscosity of the Boltzmann (classical) gas:

$$\frac{\zeta_{\rm Boltz}}{\tau_R} \approx T^4 \left(\frac{1}{3} - c_s^2\right)^2 \left(\frac{60}{\pi^2} - \frac{36 m_x}{\pi T}\right)$$

Bulk viscosity of the Bose-Einstein (quantum) gas:

$$\frac{\zeta}{\tau_R} \approx T^4 \left(\frac{1}{3} - c_s^2\right)^2 \left(\underbrace{\frac{2\pi^3 T}{25 m_x}} - \frac{4\pi^2}{75}\left(1 - \frac{9 m_{\rm eq}^2}{8 m_x^2}\right)\right)$$

Effect of the cut-off of infrared divergencies

Relaxation time approximation can be too crude to obtain a reliable form of bulk viscosity

Transport coefficients

14-moment approximation

We need equation of motion for Π

$$\Pi = M \int dK \delta f \equiv \tilde{M} \int dK \, \Delta f \quad \Rightarrow \quad u^\mu \partial_\mu \Pi = u^\mu \partial_\mu \Big[\tilde{M} \int dK \, \Delta f \Big]$$

$$\tilde{M} = M \frac{J_{1,0}}{J_{1,0} - T^2 (dm_{\mathrm{eq}}^2 / dT^2) J_{-1,0}}$$

- Boltzmann equation dictates the form of $u^\mu \partial_\mu \Delta f$
- Collision term in relaxation time approximation: $C[f] = -(u \cdot k) \dfrac{\Delta f}{\tau_R}$
- 14-moment approximation needed to close the equation in terms of Π and $\pi^{\mu\nu}$

$$\int dK (u^\alpha k_\alpha)^n \delta f \quad \rightarrow \quad \gamma_n^{(0)} \Pi$$

$$\int dK (u^\alpha k_\alpha)^n k^{\langle \mu} k^{\nu \rangle} \delta f \quad \rightarrow \quad \gamma_n^{(2)} \pi^{\mu\nu}$$

$\gamma_n^{(0)}$, $\gamma_n^{(2)}$ - combinations of thermal integrals

Transport coefficients

14-moment approximation

The structure of EoM:

$$\dot{\Pi} + \frac{\Pi}{\tau_R} = -\frac{\zeta\theta}{\tau_R} - \frac{\delta_{\Pi\Pi}}{\tau_R}\theta\Pi + \frac{\lambda_{\Pi\pi}}{\tau_R}\pi^{\mu\nu}\sigma_{\mu\nu}$$

$\frac{\zeta}{\tau_R}$ - the same as evaluated within Anderson-Witting model

$\frac{\delta_{\Pi\Pi}}{\tau_R}$ - has a compilcated structure but strongly affected by factor T/m_x coming from IR cut-off for Bose-Einstein gas

$$\frac{\lambda_{\Pi\pi}}{\tau_R} \propto 1/3 - c_s^2$$

Conclusions

- Fully consistent incorporation of thermal mean field in the hydrodynamical description of the dynamics of one-component systems
- The form of the nonequilibrium correction to the distribution function found
- The physics of bulk viscosity studied for the Boltzmann and Bose-Einstein gases
- Bulk viscosity is of the expected parametric form for the classical gas in the relaxation time approximation
- Relaxation time approximation can be too crude to study bulk viscosity of the quantum gases with Bose-Einstein distribution