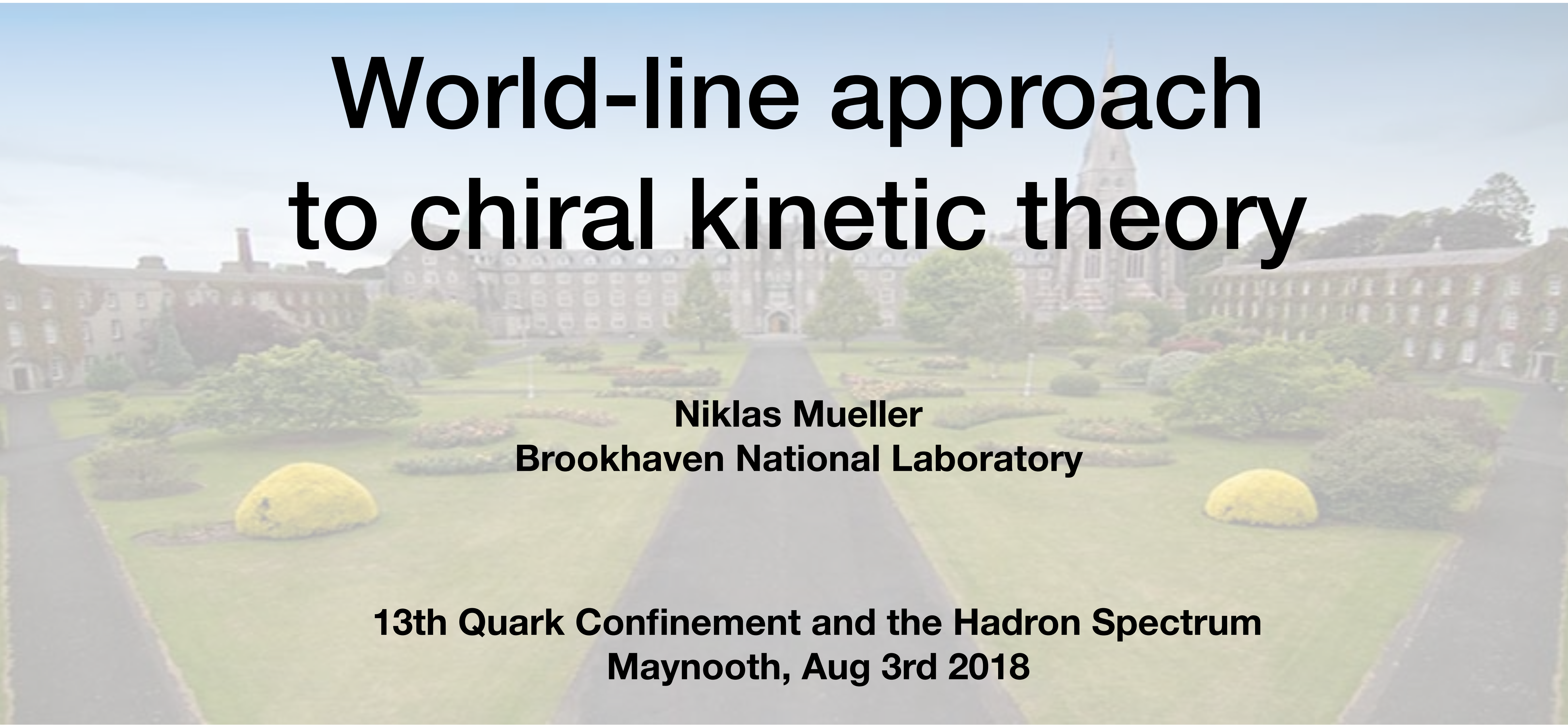


World-line approach to chiral kinetic theory

Niklas Mueller
Brookhaven National Laboratory

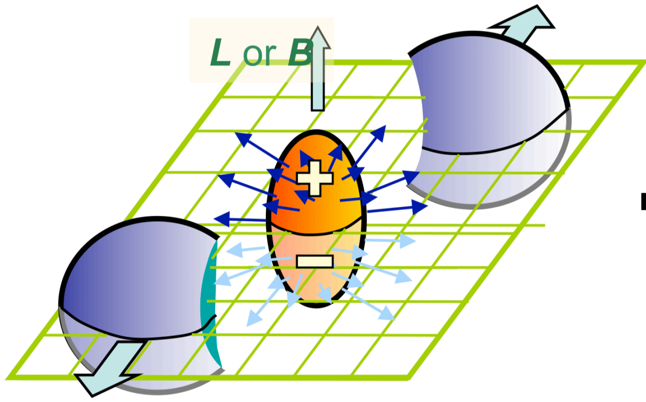
13th Quark Confinement and the Hadron Spectrum
Maynooth, Aug 3rd 2018

with Raju Venugopalan and Yi Yin

Phys.Rev. D96 (2017) no.1, 016023, Phys.Rev. D97 (2018) no.5, 051901

Motivation

Topological structure of QCD in ion-ion collisions



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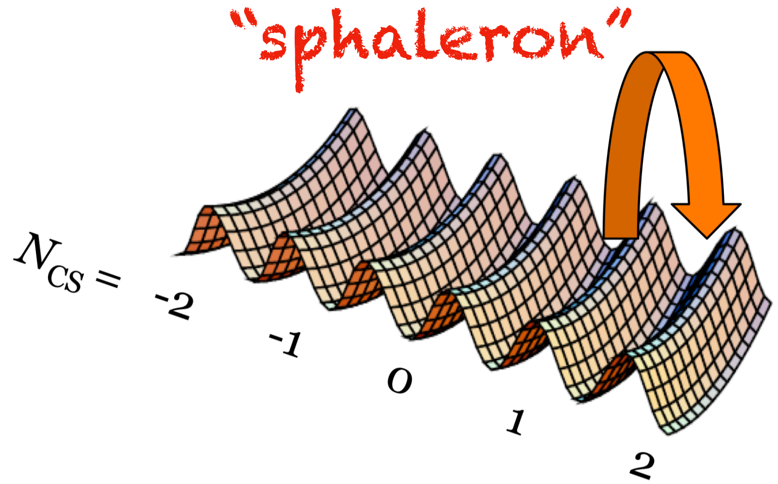

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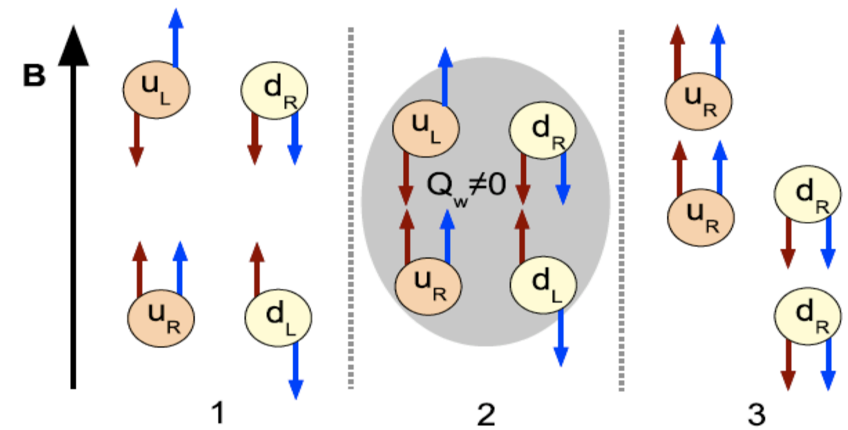

Chiral Magnetic Effect *Kharzeev, McLerran, Warringa*

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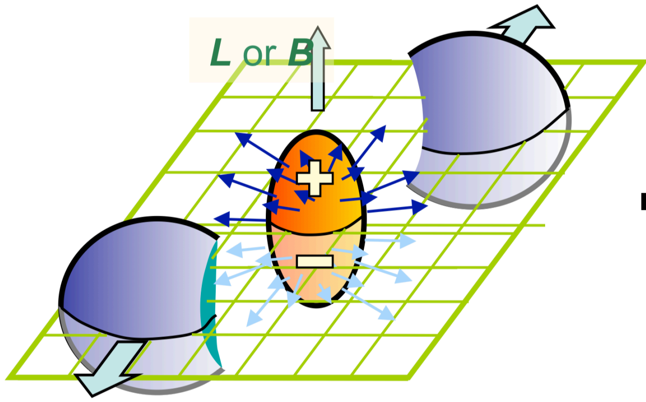

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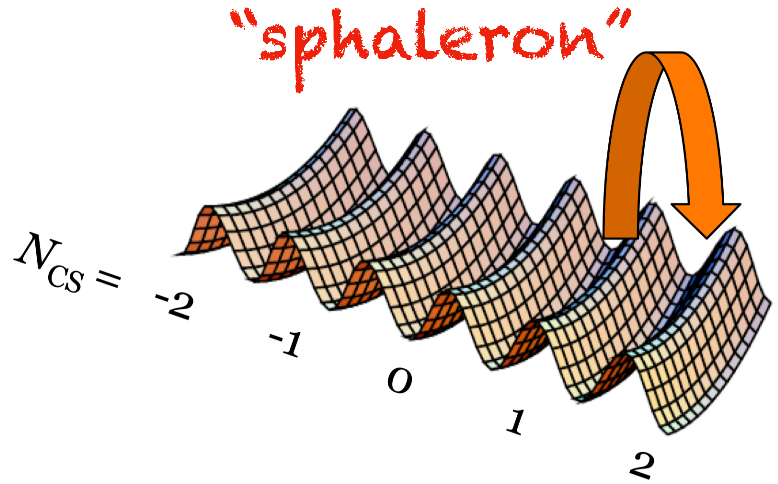

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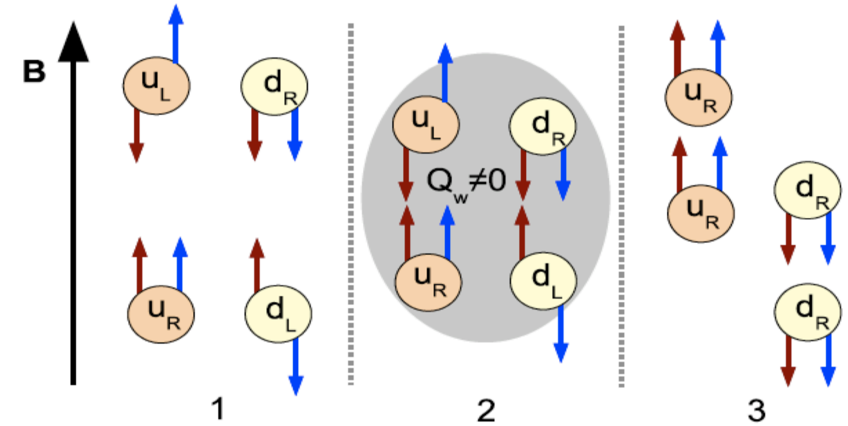

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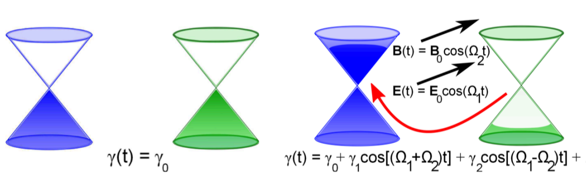
Beyond QCD: Macroscopic manifestations of quantum anomalies and topology!

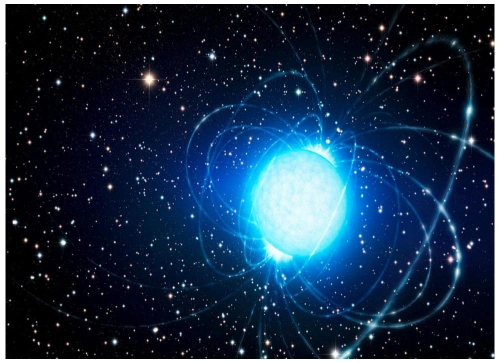
- **Condensed Matter**
  Weyl semimetals, high Tc superconductors, graphene…



Pavan Hosur

- **Cosmology**
  Electroweak Baryogenesis, …

- **Astrophysics**
  Neutron stars and supernovea, chiral instabilities, kicks etc.



- ….

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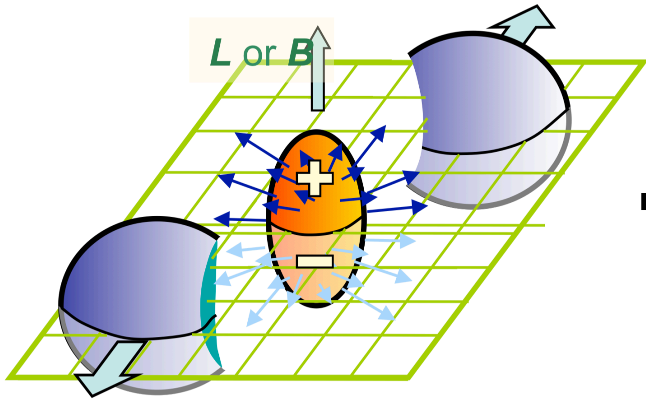

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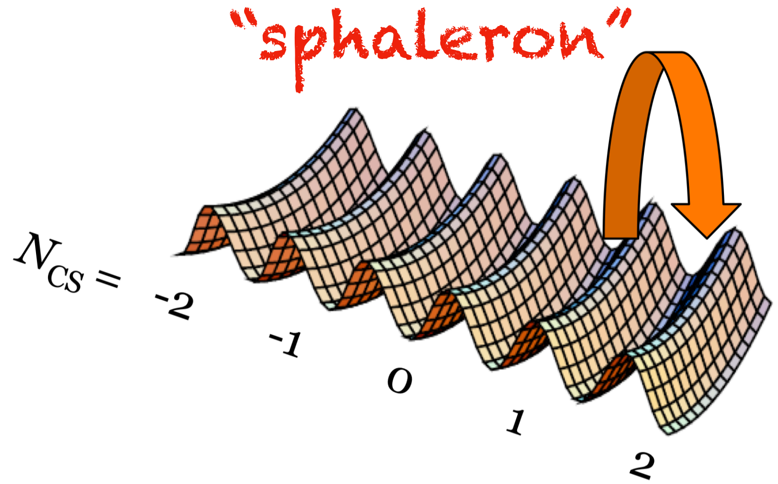

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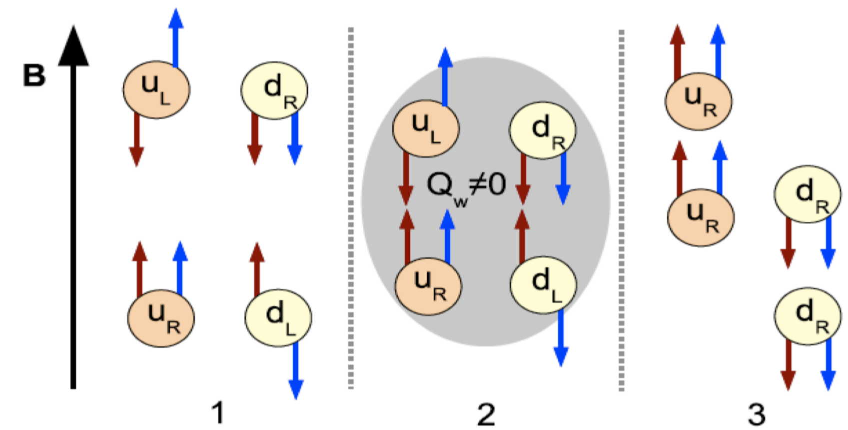

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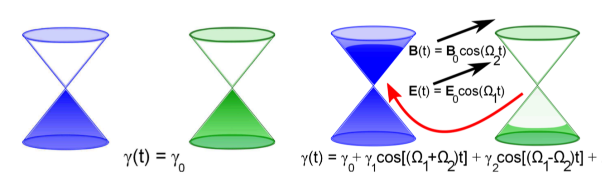


Pavan Hosur

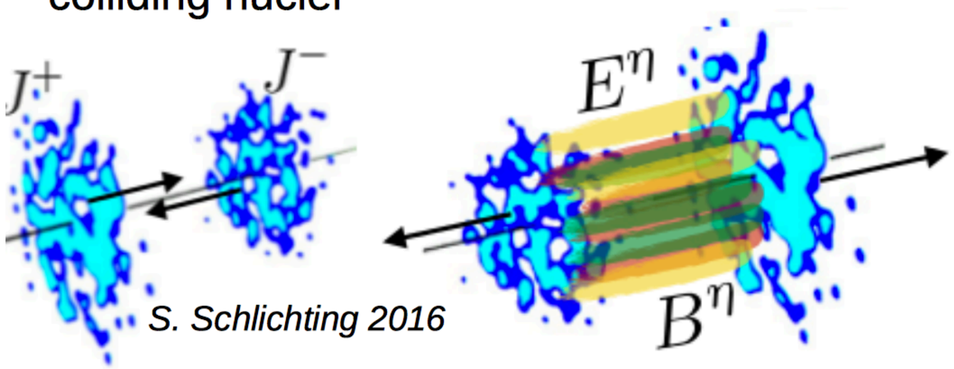
*Probing very profound and truly "**quantum**" aspects of nature*

Motivation

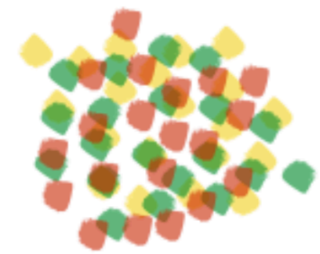
Theoretical descriptions extremely challenging

CGC
colliding nuclei

flux tubes

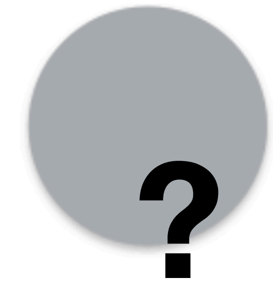


over-occupied
plasma



kinetic regime

hydrodynamic
regime



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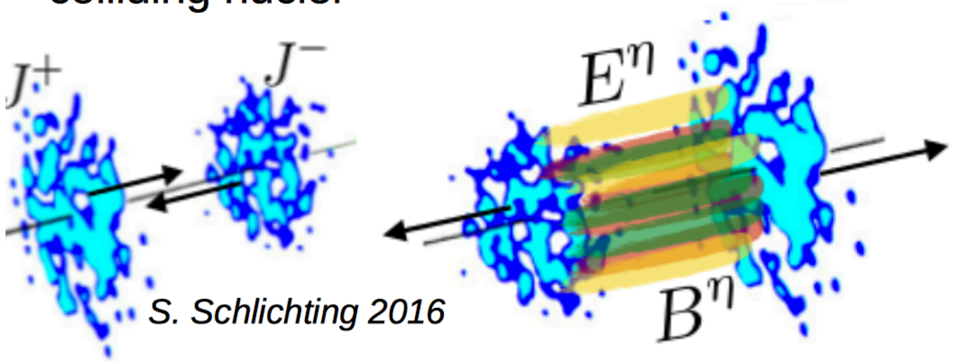

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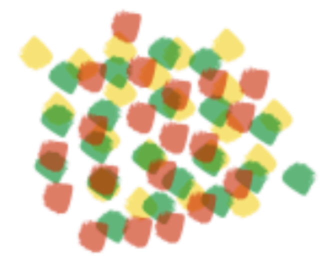

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hydrodynamic
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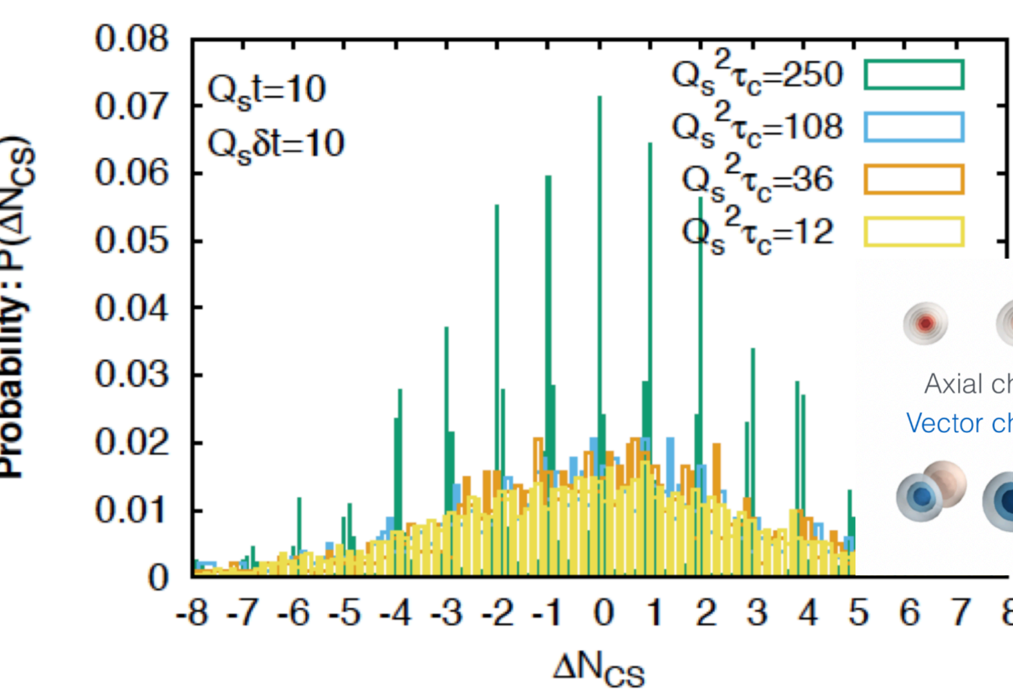
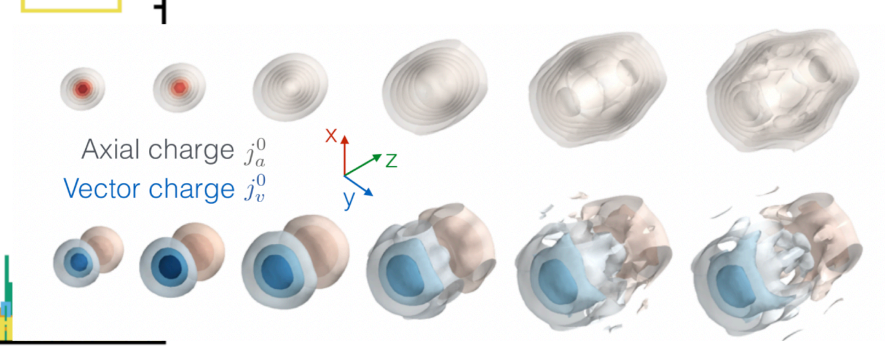
ab-initio QFT approaches



Mace,Schlichting,Venugopalan, PRD93 (2016), 074036



Mace, NM, Schlichting, Sharma, 2016

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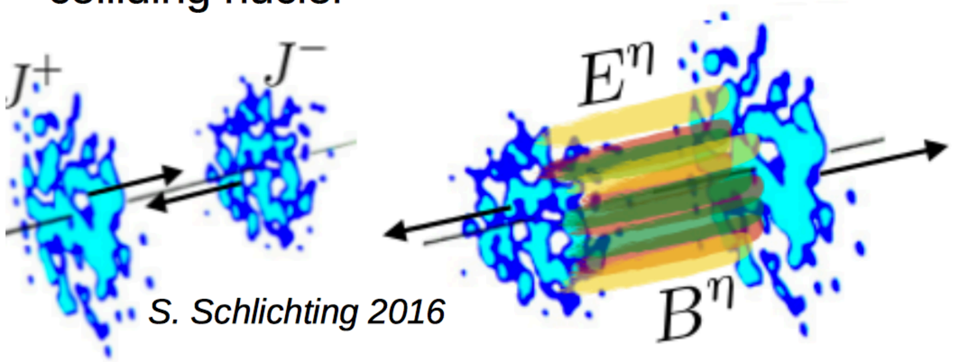

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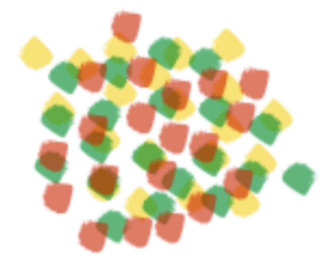


kinetic regime

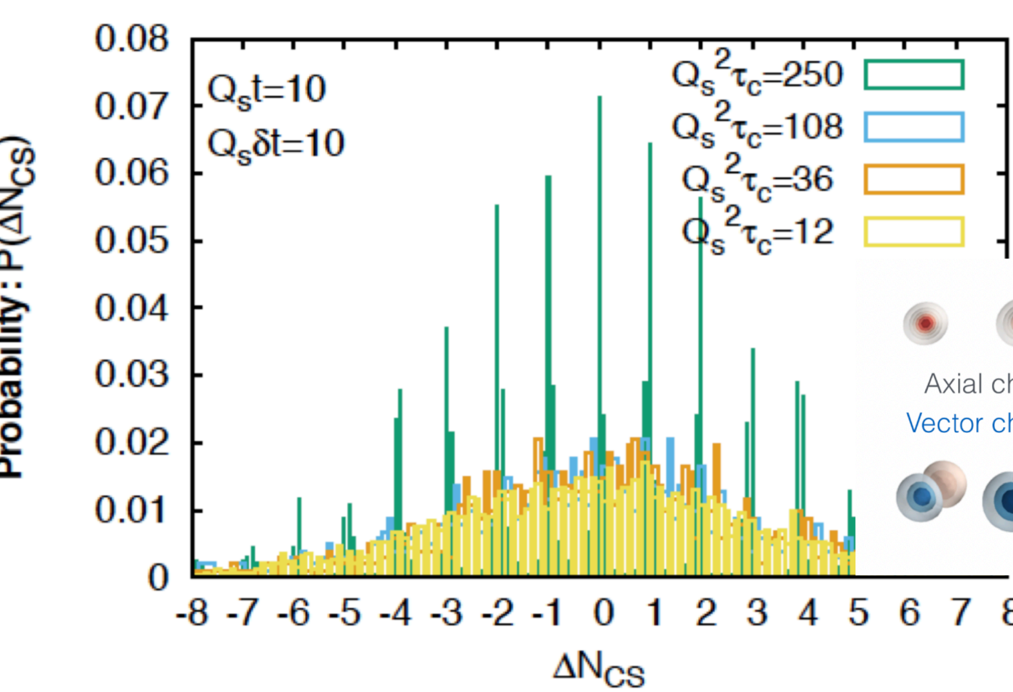
hydrodynamic regime

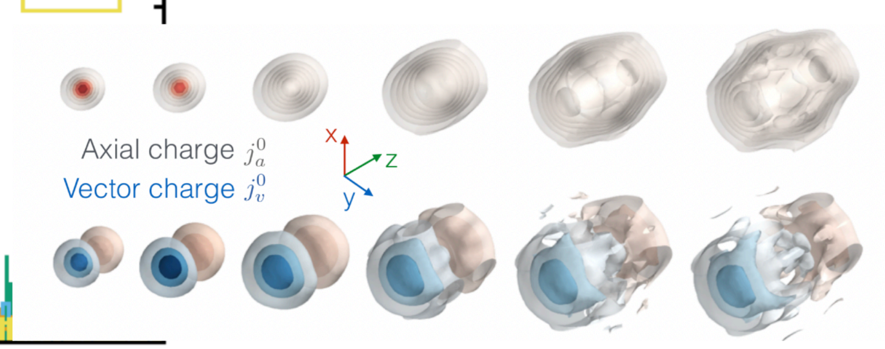
ab-initio QFT approaches

effective descriptions

Chiral Kinetic Theory

Anomalous Hydrodynamics





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The need for a Chiral Kinetic Theory

- **How do chirality and anomalies arise in effective descriptions?**

- **What do we learn of real time dynamics and topological structure of QCD?**

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Significant work on CKT

Son,Yamamoto, PRL109 (2012), 181602; PRD87 (2013) 085016
Stephanov, Yin, PRL109 (2012) 162001
Chen, Son, Stephanov, Yee, Yin, PRL 113 (2014) 182302
Chen, Son, Stephanov, PRL115 (2015) 021601
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Gao,Liang,Pu,Wang,Wang, PRL109 (2012) 232301
Stone,Dwivedi,Zhou, PRD91 (2015) 025004
Zahed, PRL109 (2012) 091603;
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Stephanov,Yee,Yin, PRD91 (2015) 125014
Fukushima, PRD92 (2015) 054009
Manuel, Torres-Rincon, PRD90 (2014) 074018
Hidaka,Pu,Yang, arXiv:1612.04630

Huang, Jiang, Shi, Liao, Zuhang, PLB777 (2018), 177
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Huang, Shi, Liao, Zhuang, arxiv:1801.03640
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Carigano, Manuel, Torres-Rincon, arxiv:1806.01684
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In most of this work Berry's phase conjectured to be the origin of the chiral anomaly

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Will discuss a novel approach here, based on world-line formulation of QCD

1. World-line approach to QFT

- **One-loop effective action**

$$\Gamma[A] = -\log\left[\det(-D^2)\right] \equiv -\mathrm{Tr}\left(\log(-D^2)\right) \qquad \mathcal{L} = \Phi^\dagger D^2 \Phi$$

Polyakov 80's, Strassler 1992, D'Hoker & Gagne, Schubert

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- **Integral representation of log (heat-kernel)**

$$\log(\sigma) = \int_1^\sigma \frac{dy}{y} \equiv \int_1^\sigma dy \int_0^\infty dt\, e^{-yt} = -\int_0^\infty \frac{dt}{t}\left(e^{-\sigma t} - e^{-t}\right)$$

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- **Effective action: QM path integral of particle on circle**

$$\Gamma[A] \;=\; \int_0^\infty \frac{dT}{T}\mathcal{N}\int \mathcal{D}x\,\mathcal{P}\exp\left[-\int_0^T d\tau\,\left(\frac{1}{2\varepsilon}\dot{x}^2 + igA[x(\tau)]\cdot\dot{x}\right)\right]$$

no approximations!

Polyakov 80's, Strassler 1992, D'Hoker & Gagne, Schubert

1. World-line approach to QFT

- **Effective action for fermions (D'Hoker and Gagne)**

$$S[A,B] = \int d^4x\, \bar{\psi}\left(i\partial\!\!\!/ + A\!\!\!/ + \gamma_5 B\!\!\!/\right)\psi \longrightarrow -W[A,B] = \log\,\det\left(i\partial\!\!\!/ + A\!\!\!/ + \gamma_5 B\!\!\!/\right)$$

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$$W_{\mathbb{R}} = \frac{1}{8}\int_0^\infty \frac{dT}{T}\,\mathrm{Tr}_{16}\, e^{-\frac{\varepsilon}{2}T\tilde{\Sigma}^2}$$

$$\tilde{\Sigma}^2 = (p-\mathcal{A})^2\, \mathbb{I}_8 + \frac{i}{2}\Gamma_\mu \Gamma_\nu F_{\mu\nu}[\mathcal{A}] \qquad \mathcal{A} = \begin{pmatrix} A+B & 0 \\ 0 & A-B \end{pmatrix}$$

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- **dimensional extension**

$$\Gamma_\mu = \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}, \quad \Gamma_6 = \begin{pmatrix} 0 & i\mathbb{I}_4 \\ i\mathbb{I}_4 & 0 \end{pmatrix} \qquad \Gamma_7 = -i\prod_{A=1}^{6}\Gamma_A = \begin{pmatrix} \mathbb{I}_4 & 0 \\ 0 & -\mathbb{I}_4 \end{pmatrix}$$

1. World-line approach to QFT

- **2. Ingredient: Grassmann coherent states for internal symmetry groups**

$$a_r^{\pm} = \frac{1}{2}(\Gamma_r \pm i\Gamma_{r+3}), \qquad \{a_r^+, a_s^-\} = \delta_{rs}, \qquad \{a_r^+, a_s^+\} = \{a_r^-, a_s^-\} = 0$$

$$\langle\theta|a_r^- = \langle\theta|\theta_r \qquad a_r^-|\theta\rangle = \theta_r|\theta\rangle \qquad \langle\bar\theta|a_r^+ = \langle\bar\theta|\bar\theta_r \qquad a_r^+|\bar\theta\rangle = \bar\theta_r|\bar\theta\rangle$$

Berezin, Marinov;
D'Hoker, Gagne;
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Berezin, Marinov; D'Hoker, Gagne; Ohnuki, Kashiwa

$$\mathrm{Tr}_{16}\, e^{-\frac{\mathcal{E}}{2}T\tilde\Sigma^2} = \mathrm{tr}\int d^4z\; d^3\theta\; \langle z, -\theta|e^{-\frac{\mathcal{E}}{2}T\tilde\Sigma^2}|z.\theta\rangle$$

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- **Working it out yields …**

$$W_{\mathbb{R}} = \frac{1}{8}\int_0^\infty \frac{dT}{T}\mathscr{N}\int_P \mathscr{D}x \int_{AP}\mathscr{D}\psi\, \mathrm{tr\,exp}\left\{-\int_0^T d\tau\, \mathscr{L}(\tau)\right\}$$

- **world line action**

$$\mathcal{L}(\tau) = \frac{\dot x^2}{2\mathcal{E}} + \frac{1}{2}\psi_a\dot\psi_a - i\dot x_\mu \mathcal{A}_\mu + \frac{i\mathcal{E}}{2}\psi_\mu F_{\mu\nu}[\mathcal{A}]\psi_\nu$$

(seen before) SUSY QM & spinning particles:
Berezin & Marinov, Barducci, Balachandran, Casalbuoni, Brink, Howe, DiVecchia (70s-80s)

2. Towards chiral kinetic theory

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- **Consider simpler scalar case**

$$\Gamma[A] \;=\; \int_0^\infty \frac{dT}{T}\mathcal{N}\int \mathcal{D}x\,\mathcal{P}\exp\left[-\int_0^T d\tau\;\left(\frac{1}{2\varepsilon}\dot{x}^2 + igA[x(\tau)]\cdot\dot{x}\right)\right]$$

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- **World-line instanton (saddle point), EOM for semi-classical particle**

$$\frac{m\ddot{x}_\mu}{\sqrt{-\dot{x}^2}} = \dot{x}_\nu F^{\mu\nu}$$

Dunne & Schubert, 2005;
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- **Generalization to Schwinger-Keldysh (SK) out-of-equilibrium path integral!**

$$Z = \int [d\xi]\,\exp\left(-G[\xi]\right)\int_{\mathcal{C}}[dA]\,\exp\left(iS_{\text{eff}}\right)$$

$$S_{\text{eff}}[A,\xi] = -\frac{1}{4}\int_{\mathcal{C}} d^4x\,F_{\mu\nu}F^{\mu\nu} + \Gamma[A,\xi]$$

2. Towards chiral kinetic theory

- **Real-time saddle point and SK: Truncated Wigner Approximation**

$$\dot{x}^\mu = \varepsilon P^\mu + \frac{i}{2}\psi^\mu\bar{\chi} - \frac{1}{6}\epsilon^{\mu\nu\alpha\beta}\psi_\nu\psi_\alpha\psi_\beta\tilde{\chi}$$

$$\dot{P}^\mu = \varepsilon P_\alpha F^{\mu\alpha} - \frac{i\varepsilon}{2}\psi^\alpha\partial^\mu F_{\alpha\beta}\psi^\beta + \frac{i}{2}F^{\mu\alpha}\psi_\alpha\bar{\chi}$$

$$\dot{\psi}^\mu = \varepsilon F^{\mu\alpha}\psi_\alpha + \frac{P^\mu}{2}\bar{\chi} + \frac{i}{4}\epsilon^{\mu\nu\alpha\beta}P_\beta\psi_\nu\psi_\alpha\tilde{\chi} - \frac{1}{12}\epsilon_{\alpha\beta\lambda\sigma}F^{\mu\alpha}\psi^\beta\psi^\lambda\psi^\sigma\tilde{\chi}$$

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- **Lorentz covariant, gauge invariant** **(+ world line reparametrization and SUSY)**

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$$\dot{\psi}^\mu = \varepsilon F^{\mu\alpha}\psi_\alpha + \frac{P^\mu}{2}\bar{\chi} + \frac{i}{4}\epsilon^{\mu\nu\alpha\beta}P_\beta\psi_\nu\psi_\alpha\tilde{\chi} - \frac{1}{12}\epsilon_{\alpha\beta\lambda\sigma}F^{\mu\alpha}\psi^\beta\psi^\lambda\psi^\sigma\tilde{\chi}$$

- **Lorentz covariant, gauge invariant** (+ world line reparametrization and SUSY)

- **Equivalent representation in terms of Pauli-Lubanski spin vector**

$$S_{\alpha\beta} = -i\psi_\alpha\psi_\beta \qquad\qquad \nu^\mu = \frac{1}{2P^0}\epsilon^{\mu\nu\alpha\beta}P_\nu S_{\alpha\beta}$$

2. Towards chiral kinetic theory

- **Real-time saddle point and SK: Truncated Wigner Approximation**

$$\dot{x}^\mu = \varepsilon P^\mu + \frac{i}{2}\psi^\mu\bar{\chi} - \frac{1}{6}\epsilon^{\mu\nu\alpha\beta}\psi_\nu\psi_\alpha\psi_\beta\tilde{\chi}$$

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- **Reduces to covariant generalization of Bargmann-Michel-Telegdi equation (and Wong's equation for color)**

2. Towards chiral kinetic theory

Grassmann phase space $(x^\mu, p^\mu) \to (x^\mu, p^\mu, \psi^\mu)$ *in preparation*

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$$\{f, H\} = f\left(\frac{\overleftarrow{\partial}}{\partial x^\mu}\dot{x}^\mu + \frac{\overleftarrow{\partial}}{\partial P^\mu}\dot{P}^\mu + \frac{\overleftarrow{\partial}}{\partial \psi^\mu}\dot{\psi}^\mu\right) = 0$$

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—> Integration = trace over Dirac matrix

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- **Chiral distributions are analytically known** (see PRD 97 (2018), 051901, Barducci et al. 1981)

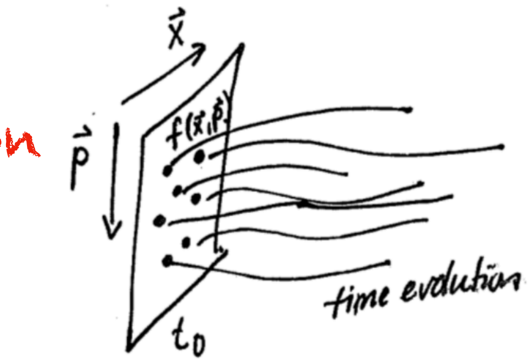
$$\frac{1}{2}\left(\pm P_\mu \psi^\mu + \frac{i}{3}\epsilon^{\mu\nu\alpha\beta}P_\mu\psi_\nu\psi_\alpha\psi_\beta\right) \longrightarrow \frac{1}{2}(1 \pm \gamma^5)\,\gamma\cdot p$$

2. Towards chiral kinetic theory

- **SK path integral specifies (quantum and statistical) ensemble, through initial density matrix / Wigner distribution. Fluctuations understood naturally.**

$$f \equiv \bar{f} + \delta f$$

1-particle distribution function



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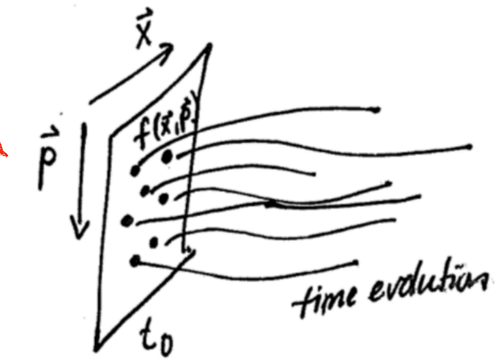

$$\bar{f}\Big(\frac{\overleftarrow{\partial}}{\partial x^\mu}\Big[\varepsilon P^\mu + \frac{i}{2}\psi^\mu\bar{\chi} - \frac{\epsilon^{\mu\nu\alpha\beta}}{6}\psi_\nu\psi_\alpha\psi_\beta\tilde{\chi}\Big] + \frac{\overleftarrow{\partial}}{\partial\psi^\mu}\Big[\varepsilon\bar{F}^{\mu\alpha}\psi_\alpha + \frac{P^\mu}{2}\bar{\chi} + \frac{i}{4}\epsilon^{\mu\nu\alpha\beta}P_\beta\psi_\nu\psi_\alpha\tilde{\chi}\Big]\Big)$$

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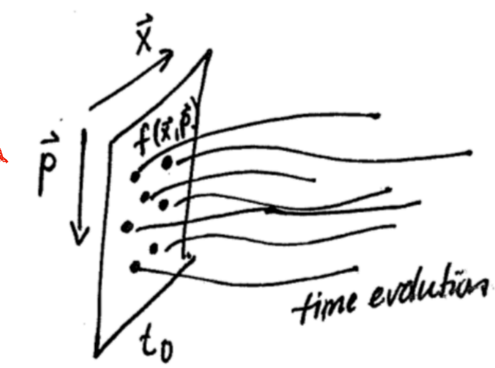

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- **Collision terms from integrating out fluctuations**

$$C[\delta f, \delta F] \equiv -\varepsilon\langle\delta f\frac{\overleftarrow{\partial}}{\partial\psi^\mu}\delta F^{\mu\nu}\rangle\psi_\nu + \frac{i\varepsilon}{2}\langle\delta f\frac{\overleftarrow{\partial}}{\partial P^\mu}\partial^\mu\delta F_{\alpha\beta}\rangle\psi^\alpha\psi^\beta$$

$$-\langle\delta f\frac{\overleftarrow{\partial}}{\partial P^\mu}\delta F^{\mu\alpha}\rangle\big(\varepsilon P_\alpha + \frac{i}{2}\psi_\alpha\bar{\chi} - \frac{1}{12}\epsilon_{\alpha\beta\lambda\sigma}\psi^\beta\psi^\lambda\psi^\sigma\tilde{\chi}\big)$$

3. The origin of anomalies and Berry’s phase

Berry’s phase conjectured to be the origin of the chiral anomaly

- **Phase of fermion determinant well known to be origin of anomaly**
  Fujikawa 1979, Alvarez-Gaume & Witten, Polyakov 80s

$$W[A,B] = W_{\mathbb{R}}[A,B] + iW_{\mathbb{I}}[A,B]$$

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D'Hoker, Gagne

$$W_{\mathbb{I}} = -\frac{1}{2}\arg\det[\Omega]$$

$$\Omega = \Gamma_\mu(p_\mu - A_\mu) - i\Gamma_7\Gamma_\mu\Gamma_5\Gamma_6 B_\mu$$

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- **(consistent) anomaly from variation**

$$\partial_\mu\langle j^5_\mu(y)\rangle \equiv \partial_\mu \frac{i\delta W_{\mathbb{I}}}{\delta B_\mu(y)}\Big|_{B=0} = -\frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}(y)F_{\rho\sigma}(y)$$

3. The origin of anomalies and Berry's phase

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- **In non-relativistic, adiabatic limit**

$$H \equiv mc^2 + \frac{\left(\boldsymbol{p} - \frac{\boldsymbol{A}}{c}\right)^2}{2m} + A^0(x) - \frac{\boldsymbol{S} \cdot \left(\left[\boldsymbol{v}/c - \boldsymbol{A}/(mc^2)\right] \times \boldsymbol{E}\right)}{2mc} - \frac{\boldsymbol{B} \cdot \boldsymbol{S}}{m}$$

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- **while the anomaly is due to the imaginary part…**

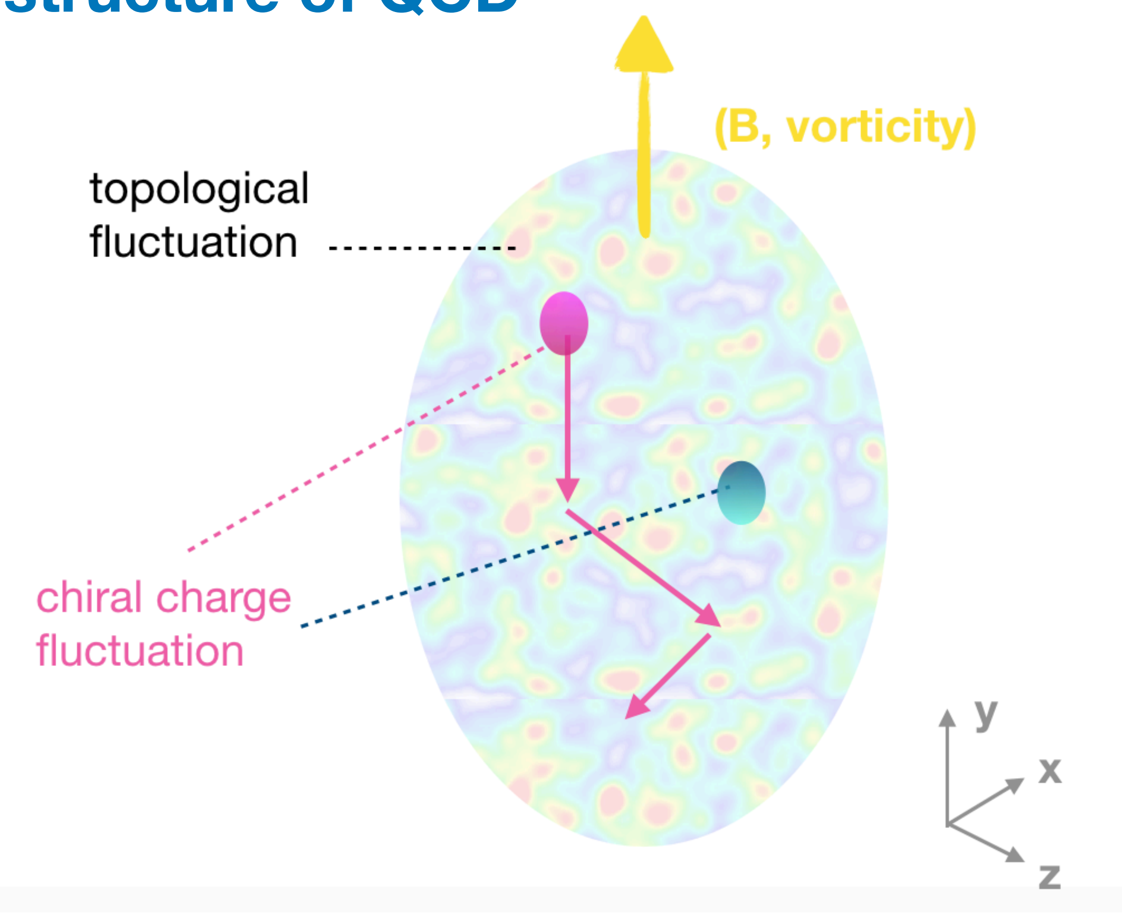
Berry's phase as the origin of the chiral anomaly
in Chiral Kinetic Theory (8+ PRLs and many more papers)
is misconception!

see also K. Fujikawa Phys.Rev. D97 (2018) no.1, 016018

4. Real-time Sphaleron transitions

Chiral Transport as probe of topological structure of QCD

$$C(t,\delta t) = \frac{1}{V}\left\langle \left(N_{CS}(t+\delta t) - N_{CS}(t)\right)^2 \right\rangle$$



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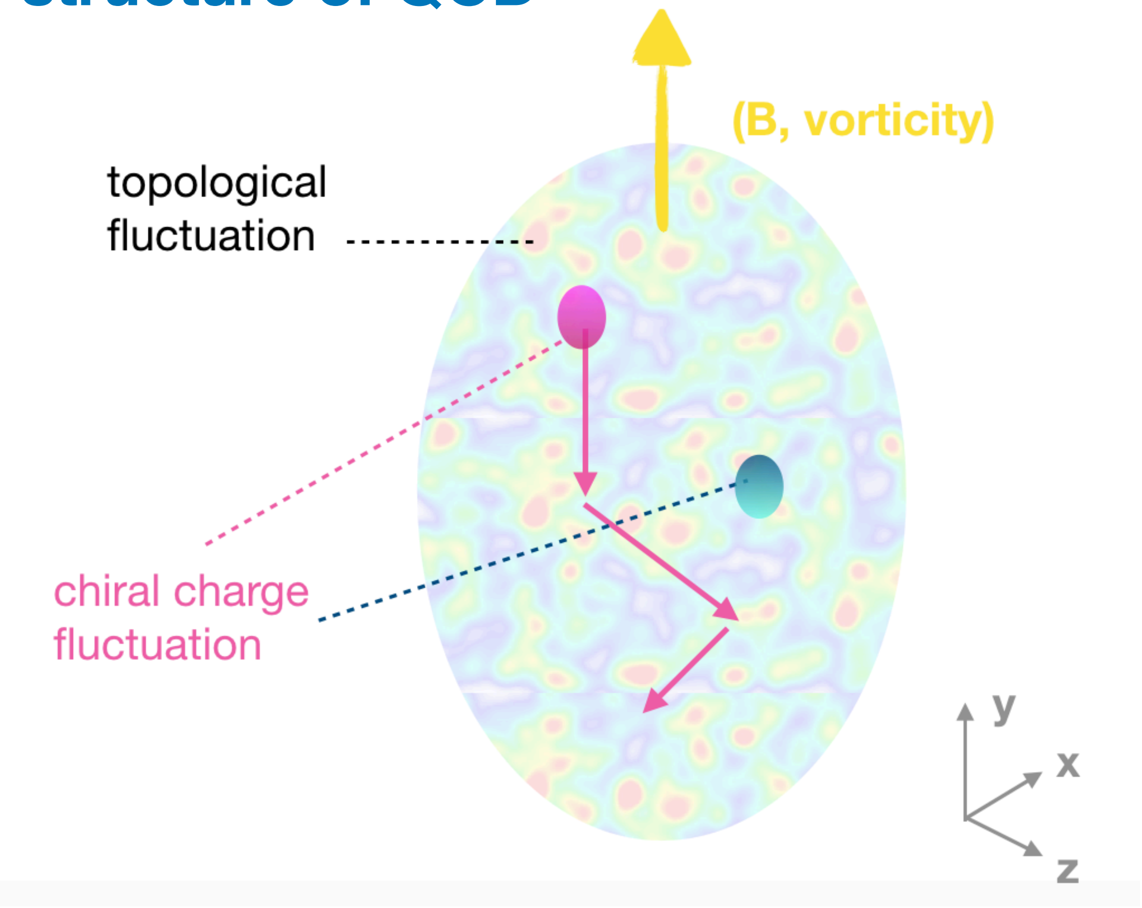

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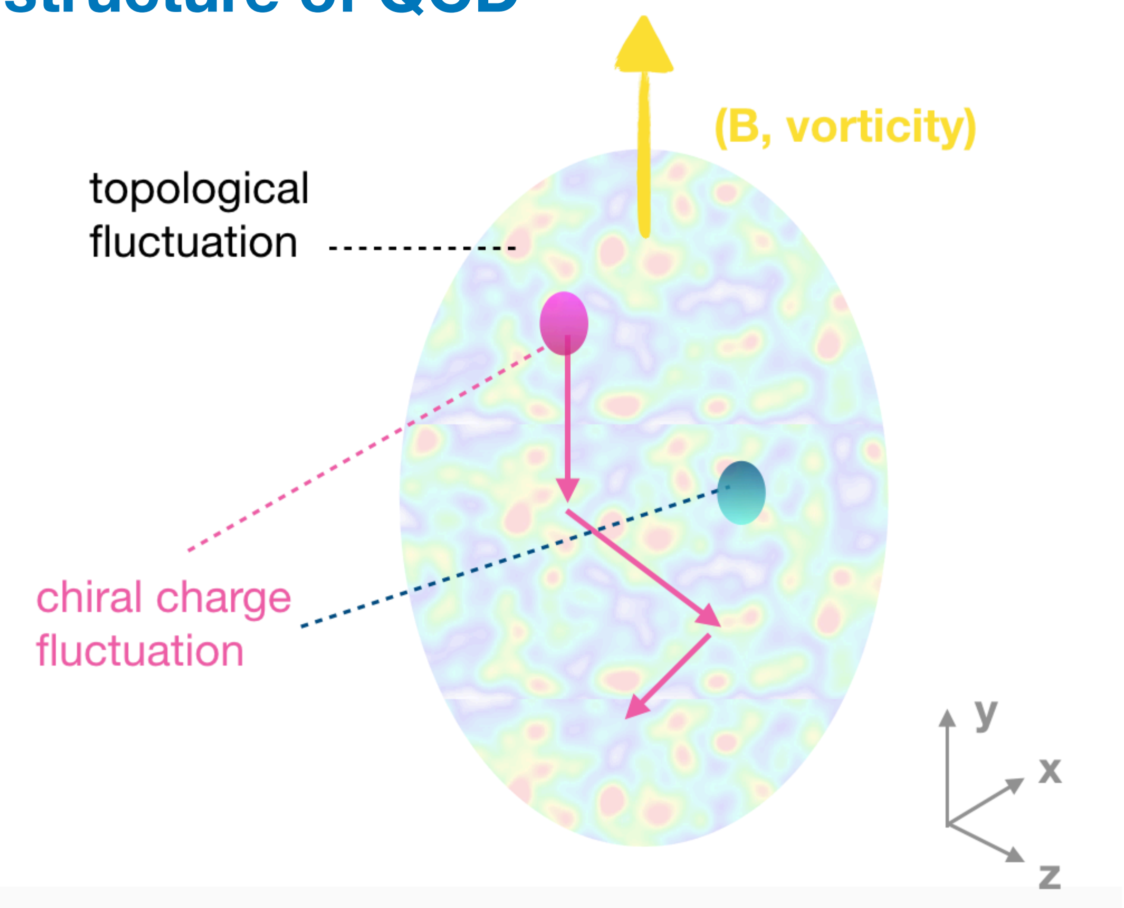

- **Real-time dynamics for topological transitions, via stochastic Boltzmann-Vlasov equations** *Arnold, Son, Yaffe; Bodecker*

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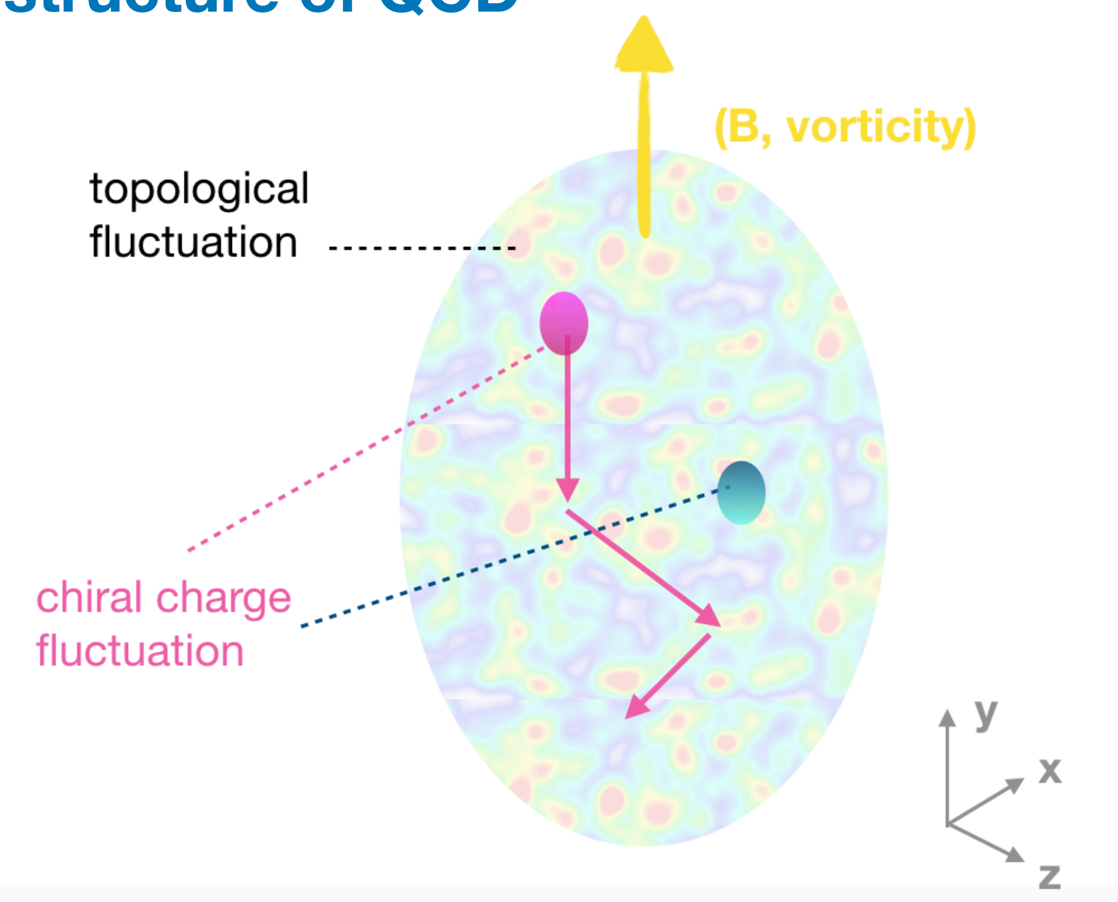

- **Real-time dynamics for topological transitions, via stochastic Boltzmann-Vlasov equations** *Arnold, Son, Yaffe; Bodecker*

- **Without fermions, derived from world-lines (Jalilian-Marian, Jeon, Venugopalan, Wirstam) —> generalize to out-of-equilibrium**

Summary

- Effective descriptions of ‘quantum’ phenomena, such as chiral anomalies, topological transitions and spin transport challenging

- Possible ab-initio approach: World-line representation of QFT

- Origin of anomaly clear in world-line framework: clarification of role of Berry’s phase

- Saddle-Point limit (real-time SK!): generalized Grassmann extended semi-classical phase space

- Lorentz-covariant, Gauge covariant

- Fluctuations from initial density matrix, naturally understood

Backup

More details on Schwinger Keldysh formulation of World-lines

- **Important work by S. Mathur 1993: SK world-line representation for scalar propagator in thermal equilibrium**

$$G(X_2, X_1) \equiv N \int \frac{D[X]D[P]D[\lambda]}{\mathrm{Vol[Diff]}} e^{i\int_0^1 d\tau [P_\mu(\tau)X^\mu,_\tau(\tau) - \lambda/2(\tau)(p^2(\tau)-m^2)]}$$

$$G_{\Lambda>0}(p) = \frac{1}{4\pi}\int_0^\infty d\lambda(\tau) e^{-i\lambda/2(p^2(\tau)-m^2)} = \frac{i}{p^2 - m^2 + i\epsilon},$$

$$G_{\Lambda<0}(p) = \frac{1}{4\pi}\int_{-\infty}^0 d\lambda(\tau) e^{-i\lambda/2(p^2(\tau)-m^2)} = \frac{-i}{p^2 - m^2 - i\epsilon}$$

- **Generalization:**

$$Z = \int [d\xi] \exp\left(-G[\xi]\right) \int_{\mathcal{C}} [dA] \exp(iS_{\mathrm{eff}}),$$

where

$$S_{\mathrm{eff}}[A, \xi] = -\frac{1}{4}\int_{\mathcal{C}} d^4x F_{\mu\nu}F^{\mu\nu} + \Gamma[A, \xi].$$

in preparation

Backup

Chiral Phase Space

Weyl equation $\frac{1}{2}(\gamma \cdot p)(1 \pm \gamma^5)\Psi = 0\,.$

Weyl Hamiltonian $H = \frac{\varepsilon}{2}\left[P^2 + i\psi^\mu F_{\mu\nu}(x)\psi^\nu\right] + \frac{i}{2}c_+\chi_+ - \frac{i}{2}c_-\chi_-\,,$

$$c_\pm \equiv \tfrac{1}{2}\left(\pm P_\mu\psi^\mu + \tfrac{i}{3}\epsilon^{\mu\nu\alpha\beta}P_\mu\psi_\nu\psi_\alpha\psi_\beta\right).$$

Phase space measure $d^4\psi = (-i/(\sqrt{2})^4)d\psi^3 d\psi^2 d\psi^1 d\psi^0$

$$\varepsilon\tilde{f}_\pm = 2i(\pm P_\mu\psi^\mu + \frac{i}{3}\epsilon^{\mu\nu\alpha\beta}P_\mu\psi_\nu\psi_\alpha\psi_\beta)\epsilon^{ijk}\psi^i\psi^j\psi^k\,. \tag{22}$$

The above expression can be quantized by identifying $\psi^\mu \to \gamma^5\gamma^\mu/\sqrt{2}$. This gives

$$\varepsilon\tilde{f}_\pm \to \rho_\pm = \frac{1}{2}(\gamma \cdot P)(1 \pm \gamma^5)\gamma^0\,, \tag{23}$$

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu\,,$$
$$\{\psi^\mu, \psi_\nu\} = -i\delta^\mu_\nu$$
$$\{\psi_5, \psi_5\} = -i\,,$$
$$\{\psi^\mu, \psi_5\} = 0\,.$$

$$\{\mathcal{Q}, \mathcal{Q}\} = -2i\,\mathcal{H}$$

Backup

Non-relativistic limit

Large fermion mass or chemical potential

$$\mathcal{L} = -\frac{m_R c\, z}{2}\left(1+\frac{m^2}{m_R^2}\right) + \frac{i}{2}\left(\psi_\mu \dot{\psi}^\mu + \psi_5 \dot{\psi}_5\right) - \frac{i m_R\, c}{2}\left(\frac{\dot{x}_\mu \psi^\mu}{z}\left[1-\frac{m^2}{2m_R^2}\right] + \frac{m}{m_R}\psi_5\right)\chi$$
$$+ \frac{\dot{x}_\mu A^\mu(x)}{c} - \frac{i}{2m_R c} z\psi^\mu F_{\mu\nu}\psi^\nu .$$

$$m_R^2 = m^2 + i\psi^\mu F_{\mu\nu}\psi^\nu$$

Well know Weyl Hamiltonian is recovered.

$$\mathcal{L}_{NR} = -mc^2 + \frac{1}{2}m\boldsymbol{v}^2 + \frac{i}{2}\left(\boldsymbol{\psi}\dot{\boldsymbol{\psi}} - \psi_0\dot{\psi}_0\right) - A^0 + \frac{\boldsymbol{v}}{c}\cdot\boldsymbol{A} + \frac{\boldsymbol{S}\cdot\left(\left[\boldsymbol{v}/c - \boldsymbol{A}/(mc^2)\right]\times\boldsymbol{E}\right)}{mc} + \frac{\boldsymbol{S}\cdot\boldsymbol{B}}{m} .$$

$$H \equiv mc^2 + \frac{\left(\boldsymbol{p} - \frac{\boldsymbol{A}}{c}\right)^2}{2m} + A^0(x) - \frac{\boldsymbol{S}\cdot\left(\left[\boldsymbol{v}/c - \boldsymbol{A}/(mc^2)\right]\times\boldsymbol{E}\right)}{2mc} - \frac{\boldsymbol{B}\cdot\boldsymbol{S}}{m}$$

Backup

Son, Yamamto; Stephanov, Yin: Berry phase

Adiabatic limit, large chemical potential has a Berry phase

$$S = \int \mathrm{d}t \; [p^i \dot{x}^i + A_i(x)\dot{x}^i - \mathcal{A}_i(p)\dot{p}^i - H(p,x)]$$

$$i\boldsymbol{\mathcal{A}}_{\mathbf{p}} \equiv u_{\mathbf{p}}^{\dagger} \boldsymbol{\nabla}_{\mathbf{p}} u_{\mathbf{p}},$$

and a nonzero Berry curvature,

$$\boldsymbol{\Omega}_{\mathbf{p}} \equiv \boldsymbol{\nabla}_{\mathbf{p}} \times \boldsymbol{\mathcal{A}}_{\mathbf{p}} = \pm \frac{\hat{\mathbf{p}}}{2|\mathbf{p}|^2},$$

$$\{p_i,\, p_j\} = -\frac{\epsilon_{ijk} B_k}{1 + \mathbf{B}\cdot\boldsymbol{\Omega}}$$

$$\{x_i,\, x_j\} = \frac{\epsilon_{ijk} \Omega_k}{1 + \mathbf{B}\cdot\boldsymbol{\Omega}},$$

$$\{p_i,\, x_j\} = \frac{\delta_{ij} + \Omega_i B_j}{1 + \mathbf{B}\cdot\boldsymbol{\Omega}}.$$

$$\dot{n}_{\mathbf{p}} + \frac{1}{1 + \mathbf{B}\cdot\boldsymbol{\Omega}_{\mathbf{p}}} \left[\left(\tilde{\mathbf{E}} + \tilde{\mathbf{v}} \times \mathbf{B} + (\tilde{\mathbf{E}}\cdot\mathbf{B})\boldsymbol{\Omega}_{\mathbf{p}} \right) \cdot \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}} + \left(\tilde{\mathbf{v}} + \tilde{\mathbf{E}} \times \boldsymbol{\Omega}_{\mathbf{p}} + (\tilde{\mathbf{v}}\cdot\boldsymbol{\Omega}_{\mathbf{p}})\mathbf{B} \right) \cdot \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{x}} \right] = 0.$$

$$\partial_t n + \boldsymbol{\nabla}\cdot\mathbf{j} = -\int \frac{d^3p}{(2\pi)^3} \left(\boldsymbol{\Omega}_{\mathbf{p}} \cdot \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}} \right) \mathbf{E}\cdot\mathbf{B}$$

What happens away from that limit?

Backup

Son, Yamamto; Stephanov, Yin: Berry phase

Fujikawa's lament… hep-ph/0501166

The notion of Berry's phase is known to be useful in various physical contexts [17]-[18], and the topological considerations are often crucial to obtain a qualitative understanding of what is going on. Our analysis however shows that the topological interpretation of Berry's phase associated with level crossing generally fails in practical physical settings with any finite T. The notion of "approximate topology" has no rigorous meaning, and it is important to keep this approximate topological property of geometric phases associated with level crossing in mind when one applies the notion of geometric phases to concrete physical processes. This approximate topological property is in sharp contrast to the Aharonov-Bohm phase [8] which is induced by the time-independent gauge potential and topologically exact for any finite time interval T. The similarity and difference between the geometric phase and the Aharonov-Bohm phase have been recognized in the early literature [1, 8], but our second quantized formulation, in which the analysis of the geometric phase is reduced to a diagonalization of the effective Hamiltonian, allowed us to analyze the topological properties precisely in the infinitesimal neighborhood of level crossing.

and… hep-ph/0511142

What we have shown in the present paper is that this expectation is not realized, and the similarity between the two is superficial.

Backup

Color

Jalilian-Marian, Jeon, Venugopalan, Wirstam, Phys.Rev. D62 (2000) 045020

$$\int \mathcal{D}\lambda^\dagger \mathcal{D}\lambda \, \mathcal{J}(\lambda^\dagger \lambda) \, \exp\left\{ -\int_0^T d\tau \, \left(\frac{\dot{x}^2}{2\mathcal{E}} + \frac{1}{2}\psi_a \dot{\psi}_a + \lambda^\dagger \dot{\lambda} - \lambda^\dagger \mathcal{L}_{\rm int} \lambda \right) \right\}$$

$$\mathcal{J}(\lambda^\dagger \lambda) = (\tfrac{\pi}{T})^N \textstyle\sum_\phi \exp[i\phi(\lambda^\dagger \lambda + N/2 - 1)].$$

Phase space and color: Wong's equations from world-lines

$$\begin{aligned} \dot{x}^\mu &= v^\mu \, , \\ \dot{P}^\mu &= g F^{a,\mu\nu} Q^a v_\nu \, , \\ \dot{Q}^a &= -g f^{abc} A^b_\mu Q^c v^\mu \end{aligned}$$

$$P^\mu D_\mu f(x,P,Q) = g P^\mu Q^a F_{\mu\nu} \frac{\partial}{\partial P_\nu} f(x,P,Q)$$

See also Litim and Manuel: Bodeker's effective theory is recovered!