Exploring Non-Abelian Gauge Theory with Energy-Momentum Tensor

~Stress, Thermodynamics and Correlations~

Masakiyo Kitazawa

for FlowQCD / WHOT-QCD Collaborations

FlowQCD: M. Asakawa, T. Hatsuda, T. Iritani, H. Suzuki, R. Yanagihara
PRD\textbf{94}, 114512(2016); PRD\textbf{96}, 111502(2017); arXiv:1803.05656

WHOT-QCD: S. Ejiri, K. Kanaya, H. Suzuki, Y. Taniguchi, T. Umeda, ...
Energy-Momentum Tensor

One of the most fundamental quantities in physics

\[ T_{\mu\nu} \]

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \]

\[ \partial_{\mu} T_{\mu\nu} = 0 \]
Energy-Momentum Tensor

\[ T_{\mu\nu} = \begin{bmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{bmatrix} \]

All components are important physical observables!
Definition of the operator is nontrivial because of the explicit breaking of Lorentz symmetry.

\[ T_{\mu \nu} = F_{\mu \rho} F_{\nu \rho} - \frac{1}{4} \delta_{\mu \nu} F F \]

Its measurement is extremely noisy due to high dimensionality and etc.
Thermodynamics

direct measurement of expectation values

\[ \langle T_{00} \rangle, \langle T_{ii} \rangle \]

Fluctuations and Correlations

viscosity, specific heat, ...

\[ c_V \sim \langle \delta T_{00}^2 \rangle \]
\[ \eta = \langle T_{12}; T_{12} \rangle \]

If we have

\[ T_{\mu\nu} \]

- flux tube / hadrons
- stress distribution

Hadron Structure

- vacuum configuration
- mixed state on 1\textsuperscript{st} transition

Vacuum Structure
Contents

Constructing EMT on the lattice

Thermodynamics
- direct measurement of expectation values
  \( \langle T_{00} \rangle, \langle T_{ii} \rangle \)

EMT Correlation Function

Stress distribution in \( \bar{q}q \) system

- fluctuations and correlations
  \( \eta = \int_0^\infty dt \langle T_{12}; T_{12} \rangle \)
  \( c_V \sim \langle \delta T_{00}^2 \rangle \)

- hadron structure
  - flux tube / hadrons
  - stress distribution
EMT on the Lattice: Conventional

Lattice EMT Operator

\[ T_{\mu\nu} = Z_6 T_{\mu\nu}^{[6]} + Z_3 T_{\mu\nu}^{[3]} + Z_1 \left( T_{\mu\nu}^{[1]} - \langle T_{\mu\nu}^{[1]} \rangle \right) \]

\[
T_{\mu\nu}^{[6]} = (1 - \delta_{\mu\nu}) F_{\mu\rho}^a F_{\nu\rho}^a, \quad T_{\mu\nu}^{[3]} = \delta_{\mu\nu} \left( F_{\mu\rho}^a F_{\nu\rho}^a - \frac{1}{4} F_{\rho\sigma}^a F_{\rho\sigma}^a \right), \quad T_{\mu\nu}^{[1]} = \delta_{\mu\nu} F_{\rho\sigma}^a F_{\rho\sigma}^a
\]

- Fit to thermodynamics: \( Z_3, Z_1 \)
- Shifted-boundary method: \( Z_6, Z_3 \) Giusti, Meyer, 2011; 2013; Giusti, Pepe, 2014; Borsanyi+, 2018

Multi-level algorithm

- effective in reducing statistical error of correlator Meyer, 2007; Borsanyi, 2018; Astrakhantsev+, 2018
Yang-Mills Gradient Flow

\[
\frac{\partial}{\partial t} A_\mu(t, x) = -\frac{\partial S_{YM}}{\partial A_\mu}
\]

Luscher, Narayanan, Neuberger, 2006
Luscher, Weiss, 2011

- diffusion equation in 4-dim space
- diffusion distance \( d \sim \sqrt{8t} \)
- "continuous" cooling/smearing
- No UV divergence at \( t > 0 \)

\[
\partial_t A_\mu = D_\nu G_{\mu\nu} = \partial_\nu \partial_\nu A_\mu + \cdots
\]
Yang-Mills Gradient Flow

\[ \frac{\partial}{\partial t} A_\mu(t, x) = - \frac{\partial S_{\text{YM}}}{\partial A_\mu} \]

leading

\[ A_\mu(0, x) = A_\mu(x) \]

Applications

scale setting / topological charge / running coupling
noise reduction / defining operators / ...

Luscher 2010
Narayanan, Neuberger, 2006
Luscher, Weiss, 2011
Small Flow-Time Expansion

\[ \tilde{O}(t, x) \xrightarrow{t \to 0} \sum_i c_i(t) \mathcal{O}_i^R(x) \]

- an operator at \( t > 0 \)
- remormalized operators of original theory

Original 4-dim theory

\[ t \xrightarrow{t \to 0} 2\sqrt{2t} \]
Constructing EMT 1

$$\tilde{O}(t, x) \xrightarrow{t \to 0} \sum_i c_i(t) O_i^R(x)$$

- Gauge-invariant dimension 4 operators

$$U_{\mu\nu}(t, x) = G_{\mu\rho}(t, x)G_{\nu\rho}(t, x) - \frac{1}{4} \delta_{\mu\nu}G_{\mu\nu}(t, x)G_{\mu\nu}(t, x)$$

$$E(t, x) = \frac{1}{4} \delta_{\mu\nu}G_{\mu\nu}(t, x)G_{\mu\nu}(t, x)$$

Suzuki, 2013
Constructing EMT 2

$$U_{\mu\nu}(t, x) = \alpha_U(t) \left[ T_{\mu\nu}^R(x) - \frac{1}{4} \delta_{\mu\nu} T_{\rho\rho}^R(x) \right] + O(t)$$

$$E(t, x) = \langle E(t, x) \rangle + \alpha_E(t) T_{\rho\rho}^R(x) + O(t)$$

Suzuki coeff.:

$$\begin{cases} 
\alpha_U(t) = g^2 \left[ 1 + 2b_0 s_1 g^2 + O(g^4) \right] \\
\alpha_E(t) = \frac{1}{2b_0} \left[ 1 + 2b_0 s_2 g^2 + O(g^4) \right]
\end{cases}$$

Remormalized EMT

$$T_{\mu\nu}^R(x) = \lim_{t \to 0} \left[ \frac{1}{\alpha_U(t)} U_{\mu\nu}(t, x) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)} E(t, x) \right]_{\text{subt.}}$$

Suzuki, 2013
Gradient Flow Method

lattice regularized gauge theory

continuum theory

Smeared world by gradient flow

Measure on the lattice

Take Extrapolation \((t, a) \rightarrow (0, 0)\)

\[
\langle T_{\mu\nu}(t) \rangle_{\text{latt}} = \langle T_{\mu\nu}(t) \rangle_{\text{phys}} + C_{\mu\nu}t + D_{\mu\nu} \left( \frac{a^2}{t} \right) + \cdots
\]

\(O(t)\) terms in SFTE lattice discretization
Contents

Constructing EMT on the lattice

Thermodynamics
- Thermodynamics
  - direct measurement of expectation values
    \(\langle T_{00}\rangle, \langle T_{ii}\rangle\)

EMT Correlation Function
- Fluctuations and Correlations
  - viscosity, specific heat, ...
  - \(\eta = \int_0^\infty dt \langle T_{12}; T_{12} \rangle\)
  - \(c_V \sim \langle \delta T_{00}^2 \rangle\)

Stress distribution in \(\bar{q}q\) system
- Hadron Structure
  - flux tube / hadrons
  - stress distribution
\[ T/T_c = 1.67. \]

\[ \frac{(\varepsilon - 3P)/T^4}{(\varepsilon + P)/T^4} \]

\[ \sqrt{8t} < a \quad \text{: strong discretization effect} \]
\[ \sqrt{8t} > 1/(2T) \quad \text{: over smeared} \]
\[ a < \sqrt{8t} < 1/(2T) \quad \text{: Linear t dependence} \]
Double Extrapolation  
$t \to 0, a \to 0$

\[
\langle T_{\mu\nu}(t) \rangle_{\text{latt}} = \langle T_{\mu\nu}(t) \rangle_{\text{phys}} + C_{\mu\nu}t + D_{\mu\nu}(t)\frac{a^2}{t}
\]

**O(t) terms in SFTE  lattice discretization**

**Continuum extrapolation**
\[
\langle T_{\mu\nu}(t) \rangle_{\text{cont}} = \langle T_{\mu\nu}(t) \rangle_{\text{lat}} + C(t)a^2
\]

**Small t extrapolation**
\[
\langle T_{\mu\nu} \rangle = \langle T_{\mu\nu}(t) \rangle + C't
\]
Double Extrapolation

Black line: continuum extrapolated
Double Extrapolation

Fitting ranges:
- **range-1**: $0.01 < tT^2 < 0.015$
- **range-2**: $0.005 < tT^2 < 0.015$
- **range-3**: $0.01 < tT^2 < 0.02$

Systematic error from the choice of fitting range $\approx$ statistical error
Error includes
- statistical error
- choice of t range for $t \to 0$ limit
- uncertainty in $a\Lambda_{\text{MS}}$

**total error <1.5% for $T>1.1T_c$**

- Excellent agreement with integral method
- High accuracy only with ~2000 confs.
Thermodynamics on the Lattice
recent progress in SU(3) YM

- **Integral method**
  - Most conventional / established
  - Use themodynamic relations
    Boyd+ 1995; Borsanyi, 2012

  \[ p = \frac{T}{V} \ln Z \]

  \[ T \frac{\partial (p/T^4)}{\partial T} = \frac{\varepsilon - 3p}{T^4} \]

- **Gradient-flow method**
  - Take expectation values of EMT

  \[ \varepsilon = \langle T_{00} \rangle \]

  \[ p = \langle T_{11} \rangle \]

- **Moving-frame method**
  Giusti, Pepe, 2014~

- **Non-equilibrium method**
  - Use Jarzynski’s equality Caselle+, 2016; 2018

- **Differential method**
  Shirogane+(WHOT-QCD), 2016~
Measurement of thermodynamics with various methods.
All results are in good agreement.
But, non-negligible discrepancy at $T/T_c \approx 1.3$?
Gradient Flow for Fermions

\[ \partial_t \psi(t, x) = D_\mu D_\mu \psi(t, x) \]
\[ \partial_t \bar{\psi}(t, x) = \psi(t, x) \bar{D}_\mu \bar{D}_\mu \]
\[ D_\mu = \partial_\mu + A_\mu(t, x) \]

- Not “gradient” flow but a “diffusion” equation.
- Divergence in field renormalization of fermions.
- All observables are finite at \( t>0 \) once \( Z(t) \) is fixed.

\[ \tilde{\psi}(t, x) = Z(t)\psi(t, x) \]
Agreement with integral method except for $N_t=4, 6$

No stable extrapolation for $N_t=4, 6$

Statistical error is substantially suppressed!

Physical mass: Kanaya+ (WHOT-QCD), 1710.10015
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Stress distribution in \( \bar{q} q \) system

- Hadron Structure
  - flux tube / hadrons
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EMT Correlator: Motivation

- Transport Coefficient
  
  Kubo formula $\Rightarrow$ viscosity
  
  $$
  \eta = \int_0^\infty dt \int_0^{1/T} d\tau \int d^3x \langle T_{12}(x, -i\tau) T_{12}(0, t) \rangle
  $$

  Karsch, Wyld, 1987
  Nakamura, Sakai, 2005
  Meyer; 2007, 2008
  ...
  Borsanyi+, 2018
  Astrakhantsev+, 2018

- Energy/Momentum Conservation
  
  $$
  \langle \bar{T}_{0\mu}(\tau) \bar{T}_{\rho\sigma}(0) \rangle : \tau\text{-independent constant}
  $$

- Fluctuation-Response Relations
  
  $$
  c_V = \frac{\langle \delta E^2 \rangle}{VT^2}
  \quad E + p = \frac{\langle \bar{T}_{01}^2 \rangle}{VT} = \frac{\langle \bar{T}_{11} \bar{T}_{00} \rangle}{VT}
  $$
EMT Euclidean Correlator

FlowQCD, PR D96, 111502 (2017)

\[ \langle \mathcal{T}_{44}(\tau) \mathcal{T}_{44}(0) \rangle \quad \langle \mathcal{T}_{44}(\tau) \mathcal{T}_{11}(0) \rangle \quad \langle \mathcal{T}_{41}(\tau) \mathcal{T}_{41}(0) \rangle \]

- \( \tau \)-independent plateau in all channels \( \rightarrow \) conservation law
- Confirmation of fluctuation-response relations
- New method to measure \( c_V \)
  - Similar result for (41;41) channel: Borsanyi+, 2018
  - Perturbative analysis: Eller, Moore, 2018
Contents

Constructing EMT on the lattice

Thermodynamics

EMT Correlation Function

Stress distribution in $\bar{q}q$ system
Stress = Force per Unit Area
Stress = Force per Unit Area

\[ \vec{P} = \frac{\vec{F}}{S} \]

\[ \vec{P} = P \hat{n} \]
Pressure

\[ \vec{P} = \frac{\vec{F}}{S} \]

Generally, \( F \) and \( n \) are not parallel

In thermal medium

\[ T_{ij} = P \delta_{ij} \]

Stress Tensor

\[ \frac{F_i}{S} = \sigma_{ij} n_j \]

\[ \sigma_{ij} = -T_{ij} \]

Landau Lifshitz
**Force**

**Action-at-a-distance**

*Newton*  
1687

\[ F = -G \frac{m_1 m_2}{r^2} \]

**Local interaction**

*Faraday*  
1839

\[ F = -\frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{r^2} \]
Maxwell Stress
(in Maxwell Theory)

\[ \sigma_{ij} = \varepsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2} \delta_{ij} \left( \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \]

\[ \vec{E} = (E, 0, 0) \]

\[ T_{ij} = \begin{pmatrix} -E^2 & 0 & 0 \\ 0 & E^2 & 0 \\ 0 & 0 & E^2 \end{pmatrix} \]

- Parallel to field: **Pulling**
- Vertical to field: **Pushing**
Maxwell Stress
(in Maxwell Theory)

\[ T_{ij} v_j^{(k)} = \lambda_k v_i^{(k)} \]

\((k = 1, 2, 3)\)

length: \(\sqrt{|\lambda_k|}\)

Definite physical meaning
- Distortion of field, line of the force
- Propagation of the force as local interaction
Stress Tensor in $Q\bar{Q}$ System

Lattice simulation
SU(3) Yang-Mills
$a=0.029$ fm
$R=0.69$ fm
$t/a^2=2.0$

Yanagihara+, 1803.05656

Definite physical meaning
- Distortion of field, line of the force
- Propagation of the force as local interaction
- Manifestly gauge invariant
Lattice Setup

- SU(3) Yang-Mills (Quenched)
- Wilson gauge action
- Clover operator
- APE smearing / multi-hit
- Fine lattices (a=0.029-0.06 fm)
- Continuum extrapolation

Simulation: bluegene/Q@KEK

Yanagihara+, 1803.05656
Continuum Extrapolation at mid-point

- $a \to 0$ extrapolation with fixed $t$
\( a \to 0 \) extrapolation with fixed \( t \)

Then, \( t \to 0 \) with three ranges
Stress Distribution on Mid-Plane

From rotational symm. & parity

EMT is diagonalized in Cylindrical Coordinates

\[
T_{c c'}(r) = \begin{pmatrix} T_{rr} & T_{\theta\theta} \\ T_{\theta\theta} & T_{zz} \end{pmatrix}
\]

\[
T_{rr} = \vec{e}_r^T T \vec{e}_r,
\]

\[
T_{\theta\theta} = \vec{e}_\theta^T T \vec{e}_\theta
\]

Degeneracy in Maxwell theory

\[
T_{rr} = T_{\theta\theta} = -T_{zz} = -T_{44}
\]
In Maxwell theory

\[ T_{rr} = T_{\theta\theta} = -T_{zz} = -T_{44} \]

- Degeneracy: \( T_{44} \approx T_{zz}, \quad T_{rr} \approx T_{\theta\theta} \)
- Separation: \( T_{zz} \neq T_{rr} \)
- Nonzero trace anomaly \( \sum T_{cc} \neq 0 \)
Degeneracy: $T_{44} \approx T_{zz}$, $T_{rr} \approx T_{\theta\theta}$
Separation: $T_{zz} \neq T_{rr}$
Nonzero trace anomaly: $\sum T_{cc} \neq 0$
Force

**Force from Potential**

\[ F_{\text{pot}} = -\frac{dV}{dR} \]

**Force from Stress**

\[ F_{\text{stress}} = \int_{\text{mid.}} d^2 x T_{zz}(x) \]
Force from Potential

\[ F_{\text{pot}} = -\frac{dV}{dR} \]

Force from Stress

\[ F_{\text{stress}} = \int_{\text{mid.}} d^2 x T_{zz}(x) \]

Newton
1687

Faraday
1839
Force from Potential

\[ F_{\text{pot}} = -\frac{dV}{dR} \]

Force from Stress

\[ F_{\text{stress}} = \int_{\text{mid.}} d^2x T_{zz}(x) \]

Good Agreement!
Abelian-Higgs Model

\[ \mathcal{L}_{\text{AH}} = -\frac{1}{4} F_{\mu\nu}^2 + |(\partial_\mu + igA_\mu)\phi|^2 - \lambda (\phi^2 - v^2)^2 \]

**GL parameter:** \( \kappa = \sqrt{\frac{\lambda}{g}} \)

- type-I: \( \kappa < 1/\sqrt{2} \)
- type-II: \( \kappa > 1/\sqrt{2} \)
- Bogomol’nyi bound: \( \kappa = 1/\sqrt{2} \)

**Infinitely long tube**

- degeneracy: \( T_{zz}(r) = T_{44}(r) \) - Luscher, 1981
- momentum conservation:
  \[ \frac{d}{dr} (rT_{rr}) = T_{\theta\theta} \]
Stress Tensor in AH Model
infinitely-long flux tube

Bogomol’nyi bound: \( \kappa = 1/\sqrt{2} \)

\[ T_{rr} = T_{\theta\theta} = 0 \]

Stress Tensor in AH Model
infinite-long flux tube

Type-I $\kappa = 0.1$

- $T_{rr}(r)$
- $T_{zz}(r)$
- $T_{rr}(r)$
- $T_{\theta\theta}(r)$

$T_{rr} > 0$

Type-II $\kappa = 3.0$

- $T_{44}(r)$
- $T_{zz}(r)$
- $T_{rr}(r)$
- $T_{\theta\theta}(r)$

$T_{rr} < 0$

- No degeneracy bw $T_{rr}$ & $T_{\theta\theta}$
- $T_{\theta\theta}$ changes sign

conservation law
\[
\frac{d}{dr} (rT_{rr}) = T_{\theta\theta}
\]
Stress Tensor in AH Model
ininitely-long flux tube

**Type-I**

\[ \kappa = 0.1 \]

\[ T_{rr} > 0 \]

- No degeneracy bw \( T_{rr} \) & \( T_{\theta\theta} \)
- \( T_{\theta\theta} \) changes sign

Inconsistent with lattice result

\[ T_{rr} \simeq T_{\theta\theta} \]
Finite-length effect of the flux tube is crucial!

**Finite R, weak Type-I**

$k = 0.6$

**Lattice**

- $-\langle T_{44}^R(r) \rangle_{q\bar{q}}$ [GeV/fm$^3$]
- $-\langle T_{zz}^R(r) \rangle_{q\bar{q}}$ [GeV/fm$^3$]
- $\langle T_{rr}^R(r) \rangle_{q\bar{q}}$ [GeV/fm$^3$]
- $\langle T_{\theta\theta}^R(r) \rangle_{q\bar{q}}$ [GeV/fm$^3$]

$R = 0.91$ fm

$\xi_A = 0.71$ fm

$\xi_\phi = 0.84$ fm
The analysis of energy-momentum tensor on the lattice is now available, and various studies are ongoing!
- gradient flow method
- determination of $Z_6$, $Z_3$, $Z_1$ / multilevel algorithm

Summary
The analysis of energy-momentum tensor on the lattice is now available, and various studies are ongoing!
- gradient flow method
- determination of $Z_6$, $Z_3$, $Z_1$ / multilevel algorithm

So many future studies
- Flux tube at nonzero temperature
- EMT distribution inside hadrons
- viscosity / other operators / instantons / full QCD

Summary
backup
Numerical Simulation

- Expectation values of $T_{\mu\nu}$
- SU(3) YM theory
- Wilson gauge action
- Parameters:
  - $N_t = 12, 16, 20-24$
  - aspect ratio $5.3 < N_s/N_t < 8$
  - 1500~2000 configurations

- Scale from gradient flow
  $\rightarrow aT_c$ and $a\Lambda_{MS}$

FlowQCD, PRD94, 114512 (2016)
• $(44;11), (41;41)$ channels: confirmation of FRR
• $(44;44)$ channel: new measurement of $c_V$

$$c_V = \frac{\left\langle \delta E^2 \right\rangle}{VT^2}$$

2+1 QCD: Taniguchi+ (WHOT-QCD), 1711.02262

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<td>1.68</td>
<td>17.7(8)(^{+2.1}_{-0.4})</td>
<td>22.8(7)^*</td>
<td>17.7</td>
<td>21.06</td>
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<tr>
<td>2.24</td>
<td>17.5(0.8)(^{+0.6}_{-0.1})</td>
<td>17.9(7)^{**}</td>
<td>18.2</td>
<td>21.06</td>
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Quark—Anti-quark system

Formation of the flux tube $\Rightarrow$ confinement

Previous Studies on Flux Tube

- Potential
- Action density
- Color-electric field

so many studies...  
Cea+ (2012)  
Cardoso+ (2013)
Propagation of the force is clearly different in YM and Maxwell theories!
Preparing Static Q\bar{Q}

\[ V(R) = - \lim_{T \to \infty} \log \langle W(R, T) \rangle \]

\[ \langle O(x) \rangle_{Q\bar{Q}} = \lim_{T \to \infty} \frac{\langle \delta O(x) \delta W(R, T) \rangle}{\langle W(R, T) \rangle} \]

- APE smearing for spatial links
- Multi-hit for temporal links
- No gradient flow for \( W(R, T) \)

Potential at \( b=6.6 \)

(a=0.038 fm)
Appearance of plateau for $t/a^2<4$, $T/a>15$

Grand state saturation under control

$\beta=6.819$ ($a=0.029$ fm), $R=0.46$ fm
Grand state saturation under control

Appearance of plateau for $t/a^2<4, T/a>15$

$\beta=6.819$ ($a=0.029$ fm), $R=0.46$ fm

Wilson loop $W(R, T)$
Abelian-Higgs Model

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