Discrete anomaly matching and high-T "center vortices" in QCD(adj)

Erich Poppitz U of Toronto

with Mohamed Anber, Lewis & Clark College, 1807.00093[hep-th]
& in progress

Discrete anomaly matching and high-T "center vortices" in QCD(adj)

Our work is inspired by a series of papers of Gaiotto, Kapustin, Komargodski, Seiberg, Willet (in combinations) during 2014-2017. Features well-known to the lattice community are put in a new light, leading to constraints on the IR behavior of strongly coupled theories: new "'t Hooft anomaly matching" conditions.

Existence of center-symmetry ("1-form" symmetry) at the same time as other discrete ("0-form") symmetries [CP, for pure-YM at $\theta = \pi$; discrete chiral symmetry for QCD(adj)] lead to new mixed 1-form/0-form 't Hooft anomalies. Like continuous 't Hooft anomalies, these can be computed in the UV and must be matched by the IR theory. Through "anomaly inflow" they also affect the domain walls (between vacua with broken discrete chiral or center symmetry), leading to rather nontrivial behavior on their worldvolume.

New "'t Hooft anomaly matching" conditions don't appear all the time, so I decided to focus on them: new nontrivial constraints on IR behavior!

(tangential: new phases of QCD(adj)? Anber, EP 1805.12990, Cordova, Dumitrescu 1806.09592 - **not this talk)**

Discrete anomaly matching and high-T "center vortices" in QCD(adj)

The choice of QCD(adj) for this study of new 't Hooft anomalies motivated by:

- As zero-N-ality, QCD(adj) has obvious 1-form center symmetry

  (because there are fermions, a step more complex than pure YM with $\theta = \pi$ [Gaiotto et al])

- QCD(adj) offers a **rare regime** where confinement and chiral symmetry breaking can be analytically understood in a theoretically controlled way: $R^3 x S^1$ small-S^1 limit, periodic fermions - so anomaly matching implications can be seen explicitly! [Unsal+… 2007-]

  on a personal level in '15 we found properties - now understood as due to these anomaly matchings [Anber, Sulejmanpasic, EP, 1501.06673] - so revisit

- In short, a good "lab" for better understanding new anomaly matching!

- Finally: do these anomaly matchings apply only to such 'exotic' theories? - NO!

  but more involved: i.e. in massless QCD with fundamentals center symmetries w/ anomalies appear after gauging B, $SU(Nf)_V$… [eg Tanizaki '18] (…"two group" discrete version…)

Discrete anomaly matching and high-T "center vortices" in QCD(adj)

Outline:

1.) new "'t Hooft anomaly" between discrete chiral symmetry and center symmetry:
the simplest example in QFT is the charge-q massless 2d Schwinger model

2.) the charge-2 Schwinger model appears as the wordvolume theory of high-T domain walls - which are a type of center vortex - between center-breaking vacua in SU(2) QCD(adj) with a single Weyl massless adjoint (& multi-flavor generalizations if more adjoints)

3.) Schwinger model results imply that (& perhaps can be seen on lattice):
- fermion condensate nonzero on the high-T center vortex
- Wilson loop inside center vortex (spacelike!) obeys perimeter law

1,2,3 all in Anber, EP 1807.00093

Thus, high-T center vortex worldvolume (2d) mirrors behavior of low-T (4d) theory: 4d bulk chiral symmetry broken and quarks deconfined on domain walls!

seen earlier on $R^3 x S^1$ in Anber, Sulejmanpasic, EP 1501.06673

All tied by the anomalies!

1.) new "'t Hooft anomaly" between discrete chiral symmetry and center symmetry: the simplest example in QFT is the charge-q massless 2d Schwinger model

$$L = -\frac{1}{4e^2} f_{kl} f^{kl} + i\bar\psi_+(\partial_- + iqA_-)\psi_+ + i\bar\psi_-(\partial_+ + iqA_+)\psi_- \qquad \partial_\pm \equiv \partial_t \pm \partial_x,\ A_\pm \equiv A_t \pm A_x,$$

"0-form" symmetries:

topological charge: $T = \frac{1}{2\pi}\int f_{12} d^2x \in \mathbb{Z}$

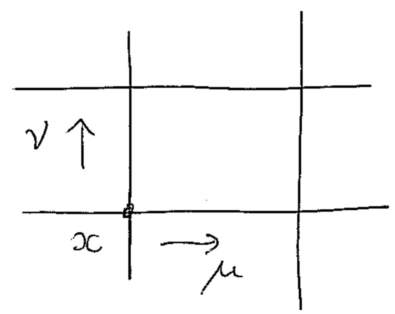
$U(1)_V$: $\psi_\pm \rightarrow e^{iq\alpha}\psi_\pm$ gauged

$U(1)_A$: $\psi_\pm \rightarrow e^{\pm i\chi}\psi_\pm$ anomalous: $Z_{\text{ferm.}} \rightarrow e^{i2q\chi T} Z_{\text{ferm.}}$

$\mathbb{Z}_{2q}^{d\chi}$: $\psi_\pm \rightarrow e^{\pm i\frac{\pi}{q}}\psi_\pm$ ($\chi = \frac{2\pi}{2q}$ gives anomaly-free subgroup)

"1-form" symmetry:

$\mathbb{Z}_q^C$ center symmetry



$U_{x,\mu} \rightarrow z_\mu U_{x,\mu}$

a Z_q phase, one per spacetime direction (global symmetry)

easy to see on lattice: plaquette term in action invariant, fermion hopping as well, since integer charge q>1

"1-form" symmetry only acts on line operators (hence name):

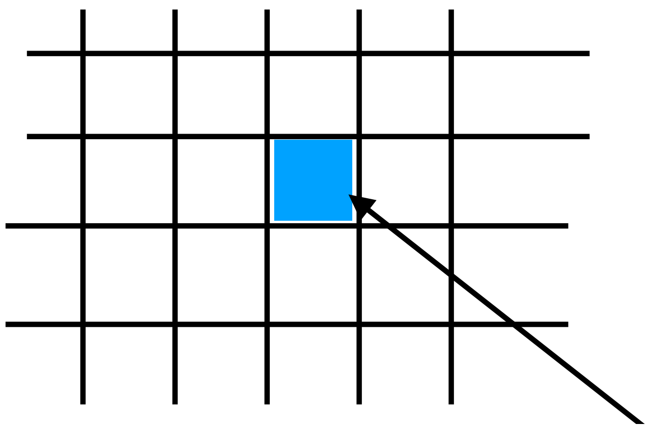
$$\mathbb{Z}_q^C : \ e^{i\oint A_x dx} \rightarrow \omega_q\, e^{i\oint A_x dx}, \quad \omega_q \equiv e^{i\frac{2\pi}{q}}.$$

1.) new "'t Hooft anomaly" between discrete chiral symmetry and center symmetry:

the simplest example in QFT is the charge-q massless 2d Schwinger model

"0-form" $\mathbb{Z}_{2q}^{d\chi}$ & "1-form" $\mathbb{Z}_q^C$ have a mixed anomaly!

also easy**(er!)** to see on the lattice! Let us gauge the 1-form center symmetry:



for symmetries acting on links (1-form center symmetry), introduce plaquette-based ("2-form") Z_q gauge field to make Z_q center symmetry local

to see the anomaly a background 2-form field suffices;
in 2d, there is no field strength of the 2-form (no cubes!);
introduce a Z_q background phase on a single plaquette =
"Z_q center vortex" [i.e. any 2-form $Z_q$background topological…
can move around shaded square by changing link variables]

now recall:

$$U_{plaquette} = \prod_{link \in plaquette} U_{link} = e^{ia^2 F_{plaquette}} \longrightarrow$$

$$\longrightarrow \; flux\ thru\ T^2 = 2\pi n, \;\; n \in Z$$ = integer T (top. charge) from before!

But in the background of a single Z_q center vortex, we have instead

$$\prod_{all\ plaquettes} U_{plaquette} e^{i\frac{2\pi}{q}} = e^{i flux\ thru\ T^2} = e^{i\frac{2\pi}{q}}$$ = fractional T (top. charge)!

1.) new "'t Hooft anomaly" between discrete chiral symmetry and center symmetry:
the simplest example in QFT is the charge-q massless 2d Schwinger model

"0-form" $\mathbb{Z}_{2q}^{d\chi}$ & "1-form" $\mathbb{Z}_q^C$ have a mixed anomaly!

So, after we introduce a center 2-form background (center vortex), we have

$$\prod_{all\ plaquettes} U_{plaquette} e^{i\frac{2\pi}{q}} = e^{i flux\ thru\ T^2} = e^{i\frac{2\pi}{q}}$$ = fractional T (top. charge)!

But recall that under a 0-form discrete chiral $\mathbb{Z}_{2q}^{d\chi}$ we have that

$Z_{ferm.} \to e^{i2\pi T} Z_{ferm.}$ and $Z_{ferm.}$ is invariant if T is integer, but not otherwise.

We conclude that if center 1-form symmetry is gauged, the discrete chiral symmetry ceases to be a symmetry, in other words, we have a $\mathbb{Z}_{2q}^{d\chi}$-$\mathbb{Z}_q^C$ 't Hooft anomaly!

$$\mathbb{Z}_{2q}^{d\chi}: \quad Z_{ferm.} \to e^{i\frac{2\pi}{q}} Z_{ferm.}$$

- phase in the chiral transform (in the center vortex bckgd) **IS** mixed 't Hooft anomaly
- phase independent on T^2 volume, RG invariant, same on all scales: UV & IR
- like for continuous symmetry 't Hooft anomalies must be matched by IR theory:
  - IR CFT, or
  - one or both symmetries should be broken ("Goldstone" mode), or/and
  - IR TQFT

Gaiotto et al, '14-'17

1.) new "'t Hooft anomaly" between discrete chiral symmetry and center symmetry:
the simplest example in QFT is the charge-q massless 2d Schwinger model

"0-form" $\mathbb{Z}_{2q}^{d\chi}$ & "1-form" $\mathbb{Z}_q^C$ have a mixed anomaly!

- like for continuous symmetry 't Hooft anomalies must be matched in IR:
  - IR CFT, or
  - one or both symmetries should be broken ("Goldstone" mode), or/and
  - IR TQFT

Gaiotto et al, '14-'17

The 0-form/1-form mixed anomaly was computed by Gaiotto et al by turning on discrete gauge backgrounds (as I showed you in the 2d case). A 't Hooft anomaly, however, should be a property of the theory without any backgrounds; it does not require turning on fields. Continuous symmetry 't Hooft anomalies are seen in <j j j > three-point global symmetry current correlators, as $1/q^2$ poles
[Frishman et al, Coleman et al, 1980s].

Expect the "same" should be true here. The anomalies should involve properties of the quantum operators representing the discrete symmetries. General statements are so far not known, but examples exist, notably in QM [Gaiotto et al] and 2d QFT [our work].

I will spare you the details and simply state the result:

1.) new "'t Hooft anomaly" between discrete chiral symmetry and center symmetry:
the simplest example in QFT is the charge-q massless 2d Schwinger model

"0-form" $\mathbb{Z}_{2q}^{d\chi}$ & "1-form" $\mathbb{Z}_{q}^{C}$ have a mixed anomaly!

- like for continuous symmetry 't Hooft anomalies must be matched in IR:
  - IR CFT, or
  - one or both symmetries should be broken ("Goldstone" mode), or
  - IR TQFT

Gaiotto et al, '14-'17

In the 2d QFT case (Schwinger model [our work]) and various QM examples [Gaiotto et al] the anomaly can be seen at the operator level and the IR matching decided upon:

$\mathbb{Z}_{2q}^{d\chi}$ operator $\longrightarrow \hat{X}_{2q}$

$\mathbb{Z}_{q}^{C}$ operator $\longrightarrow \hat{Y}_{q}$

$$\hat{X}_{2q}\,\hat{Y}_q = \omega_q\,\hat{Y}_q\,\hat{X}_{2q} \quad \left(\omega_q = e^{i\frac{2\pi}{q}}\right).$$

Classically, center and discrete chiral symmetries commute, but they do not in the QFT: anomaly!

1.) new "'t Hooft anomaly" between discrete chiral symmetry and center symmetry:
the simplest example in QFT is the charge-q massless 2d Schwinger model

"0-form" $\mathbb{Z}_{2q}^{d\chi}$ & "1-form" $\mathbb{Z}_{q}^{C}$ have a mixed anomaly!

- like for continuous symmetry 't Hooft anomalies must be matched in IR:
  - IR CFT, or
  - one or both symmetries should be broken ("Goldstone" mode), or
  - IR TQFT

Gaiotto et al, '14-'17

In the 2d QFT case (Schwinger model [our work]) and various QM examples [Gaiotto et al] the anomaly can be seen at the operator level and the IR matching decided upon:

$\mathbb{Z}_{2q}^{d\chi}$ operator

$\mathbb{Z}_{q}^{C}$ operator

$$\hat{X}_{2q}\,\hat{Y}_q = \omega_q\,\hat{Y}_q\,\hat{X}_{2q} \quad \left(\omega_q = e^{i\frac{2\pi}{q}}\right).$$

This is the famous 't Hooft-loop/Wilson-loop algebra for SU(q): q-dimensional reps!

1.) new "'t Hooft anomaly" between discrete chiral symmetry and center symmetry:
the simplest example in QFT is the charge-q massless 2d Schwinger model

"0-form" $\mathbb{Z}_{2q}^{d\chi}$ & "1-form" $\mathbb{Z}_q^C$ have a mixed anomaly!

- like for continuous symmetry 't Hooft anomalies must be matched in IR:
  - IR CFT, or
  - one or both symmetries should be broken ("Goldstone" mode), or
  - IR TQFT

Gaiotto et al, '14-'17

In the 2d QFT case (Schwinger model [our work]) and various QM examples [Gaiotto et al] the anomaly can be seen at the operator level and the IR matching decided upon:

$\mathbb{Z}_{2q}^{d\chi}$ operator

$\mathbb{Z}_q^C$ operator

$$\hat{X}_{2q}\,\hat{Y}_q = \omega_q\,\hat{Y}_q\,\hat{X}_{2q} \quad (\omega_q = e^{i\frac{2\pi}{q}}).$$

This is the famous 't Hooft-loop/Wilson-loop algebra for SU(q): q-dimensional reps!

Finally, we also showed that both discrete chiral symmetry is broken (q discrete vacua, fermion condensate) and the 1-form center is broken (=perimeter law for q=1 Wilson loop).

Used Manton '86, Iso-Murayama '90 Hamiltonian solution

Outline:

1.) new "'t Hooft anomaly" between discrete chiral symmetry and center symmetry:
✔ the simplest example in QFT is the charge-q massless 2d Schwinger model

2.) the charge-2 Schwinger model appears as the wordvolume theory of high-T domain walls - which are a type of center vortex - between center-breaking vacua in SU(2) QCD(adj) with a single Weyl massless adjoint (& multi-flavor generalizations if more adjoints)

3.) Schwinger model results imply that (& perhaps can be seen on lattice):
- fermion condensate nonzero on the high-T center vortex
- Wilson loop inside center vortex (spacelike!) obeys perimeter law

1,2,3 in Anber, EP 1807.00093

2.), 3.): ... wordlvolume theory of high-T domain walls - which are a type of center vortex - between center-breaking vacua...

$SU(2)$ Yang-Mills theory endowed with n_f adjoint Weyl fermions $T \gg \Lambda_{QCD}$

$$S_{3D}^{\text{boson}} = \frac{\beta}{g^2} \int_{\mathbb{R}^3} \left(\frac{1}{2} \text{tr}\left(F_{ij} F_{ij}\right) + \text{tr}\left(D_i A_4\right)^2 + g^2 V(A_4) + \mathcal{O}\left(g^4\right) \right)$$

$$V(A_4) = -\frac{1}{12\pi\beta^4}\left[-6\pi\left(\beta A_4^3\right)^2 + 4\left(\beta A_4^3\right)^3\right], \quad \text{for } \beta A_4^3 \in [0, \pi]$$

(shown for nf=1 SYM)

- two vacua $\beta A_4^3 = 0, 2\pi$ broken center ("0-form", along x_4) $\frac{1}{2}\langle \text{Tr}_F \exp\left[i \oint_{\mathbb{S}^1_\beta} A_4\right]\rangle = \pm 1$
- Z_2 "domain walls", or "interfaces", or "center vortices" of width ~1/gT   Bhattacharya et al 1991 ...
- Z_2 0-form center restored on DW
- U(1)unbroken on wall (Polyakov loop not ~1) Cartan of SU(2) massless; W-boson mass ~T
  localized 2d U(1) on wall   not very interesting except $\theta = \pi$ pure YM!   Gaiotto et al 2017

2.), 3.): ... wordlvolume theory of high-T domain walls - which are a type of center vortex - between center-breaking vacua...

$SU(2)$ Yang-Mills theory endowed with n_f adjoint Weyl fermions $T \gg \Lambda_{QCD}$

$$S_{3D}^{\text{boson}} = \frac{\beta}{g^2}\int_{\mathbb{R}^3}\left(\frac{1}{2}\mathrm{tr}\left(F_{ij}F_{ij}\right) + \mathrm{tr}\left(D_i A_4\right)^2 + g^2 V(A_4) + \mathcal{O}\left(g^4\right)\right)$$

$$V(A_4) = -\frac{1}{12\pi\beta^4}\left[-6\pi\left(\beta A_4^3\right)^2 + 4\left(\beta A_4^3\right)^3\right], \quad \text{for } \beta A_4^3 \in [0,\pi]$$

(shown for nf=1 SYM)

- two vacua $\beta A_4^3 = 0, 2\pi$ broken center ("0-form", along x_4) $\frac{1}{2}\langle \mathrm{Tr}_F \exp\left[i\oint_{\mathbb{S}^1_\beta} A_4\right]\rangle = \pm 1$
- Z_2 "domain walls", or "interfaces", or "center vortices" of width ~1/gT — Bhattacharya et al 1991 ...
- Z_2 0-form center restored on DW
- U(1)unbroken on wall (Polyakov loop not ~1) Cartan of SU(2) massless; W-boson mass ~T
  localized 2d U(1) on wall — not very interesting except $\theta = \pi$ pure YM! — Gaiotto et al 2017
- SYM has Z_2 center 1-form and Z_4 chiral 0-form w/ mixed 't Hooft anomaly!
  story similar as in Schwinger model: fractional topological charge in 4d (here: 1/2) background of two center vortices intersecting at a point *(one along x1-x2, the other along x3-x4)*
  - at $T<T_c$, Z_4 chiral broken to Z_2, matching the anomaly — (assume Tchi= Tc)
  - at $T>T_c$, Z_2 center broken, matching the anomaly — (...> or =)
- domain walls, in either phase, are "nontrivial": anomaly inflow! — Gaiotto et al 2014-17

- high-T center vortices have mixed Z_4chiral/Z_2 center anomaly on 2d worldvolume
- can be seen in the high-T theory quite explicitly:

2.), 3.): ... wordlvolume theory of high-T domain walls - which are a type of center vortex - between center-breaking vacua...

- U(1)unbroken on wall (Polyakov loop not ~1) Cartan of SU(2) massless; W-boson mass ~T
  localized 2d U(1) on wall   not very interesting except $\theta = \pi$ pure YM!   Gaiotto et al 2017
  & even richer in QCD(adj)!

- adjoint fermions at high-T have zero modes on the wall   Anber EP 2018
  for nf=1, two normalizable:

$$\mathcal{L}_{DW}^{\rm axial} = \frac{1}{4e^2} F_{kl} F_{kl} + i\bar\lambda_+ \left[\partial_1 + i\partial_2 - i2(A_1 + iA_2)\right]\lambda_+ + i\bar\lambda_- \left[\partial_1 - i\partial_2 + i2(A_1 - iA_2)\right]\lambda_-$$

axial Schwinger model of charge-2!
L and R have opposite charge

- In 2d axial and vector easily mapped to each other: Z_4 chiral symmetry and Z_2 center.

From q=2 Schwinger model results, chiral and center broken, so:

- **nonzero fermion condensate + Wilson loop perimeter law on the high-T “center vortex”!**

Outline:

1.) new "'t Hooft anomaly" between discrete chiral symmetry and center symmetry:
✔ the simplest example in QFT is the charge-q massless 2d Schwinger model

2.) the charge-2 Schwinger model appears as the wordvolume theory of high-T domain walls - which are a type of center vortex - between center-breaking vacua in SU(2) QCD(adj) with a single Weyl massless adjoint (& multi-flavor generalizations if more adjoints) ✔

3.) Schwinger model results imply that (& perhaps can be seen on lattice):
✔ - fermion condensate nonzero on the high-T center vortex
- Wilson loop inside center vortex (spacelike!) obeys perimeter law

1,2,3 in Anber, EP 1807.00093

Finally:

... high-T center vortex worldvolume (2d) mirrors behavior of low-T (4d) theory:
4d bulk chiral symmetry broken and quarks deconfined on domain walls!

seen earlier on $R^3 x S^1$ in
Anber, Sulejmanpasic, EP 1501.06673

All tied by the anomalies!

Finally:

... high-T center vortex worldvolume (2d) mirrors behavior of low-T (4d) theory:
4d bulk chiral symmetry broken and quarks deconfined on domain walls!

assumed Tchi= Tdec.

(> or =)

low T	high T
in bulk:	in bulk:
Z_4 chiral broken	Z_4 chiral unbroken
Z_2 center unbroken (area law for W loop in bulk)	Z_2 0-form center broken; Z_2 1-form center unbroken (area law for W loop in bulk)
on DWs:	on DWs:
quarks are deconfined	Z_4 chiral broken
Z_2 center broken on DW (perimeter law for W loop on DW while area law in bulk)	Z_2 1-form center broken (perimeter law for W loop on DW while area law in bulk)

explicit confinement on $R^3 x S^1$ in Anber, Sulejmanpasic, EP 1501.06673

All tied by the anomalies!

Finally:

... high-T center vortex worldvolume (2d) mirrors behavior of low-T (4d) theory:
4d bulk chiral symmetry broken and quarks deconfined on domain walls!

low T

in bulk:

Z_4 chiral broken

Z_2 center unbroken
(area law for W loop in bulk)

on DWs:

quarks are deconfined

Z_2 center broken on DW
(perimeter law for W loop on DW while area law in bulk)

explicit confinement on $R^3 x S^1$ in Anber, Sulejmanpasic, EP 1501.06673

high T

in bulk:

Z_4 chiral unbroken

Z_2 0-form center broken

Z_2 1-form center unbroken
(area law for W loop in bulk)

on DWs:

Z_4 chiral broken

Z_2 1-form center broken
(perimeter law for W loop on DW while area law in bulk)

All tied by the anomalies!

Finally:

... high-T center vortex worldvolume (2d) mirrors behavior of low-T (4d) theory:
4d bulk chiral symmetry broken and quarks deconfined on domain walls!

low T

in bulk:

Z_4 chiral broken

Z_2 center unbroken
(area law for W loop in bulk)

on DWs:

quarks are deconfined

Z_2 center broken on DW
(perimeter law for W loop on DW while area law in bulk)

high T

in bulk:

Z_4 chiral unbroken

Z_2 0-form center broken

Z_2 1-form center unbroken
(area law for W loop in bulk)

on DWs:

Z_4 chiral broken

Z_2 1-form center broken

(perimeter law for W loop on DW while area law in bulk)

High-T DW properties mirror low-T bulk and low-T DW properties in a way really required by (rather, consistent with) anomaly...

FUTURE?

One wonders if n_f>1 high-T DWs teach us about QCD(adj) nf>1 unknown low-T phase?

on DW: multi-flavor Schwinger models with classically marginal 4-fermi terms (reducing symmetries, generated by W-boson loops)? symmetry realization vs 4d low-T?

Other theories with center symmetries: two-index S, AS flavors? eg Kuti et al on lattice '17

and some specific 3-, 4- index ones Anber, Vincent-Genot 1704.08277

(discrete chiral/$Z_{\{2,3,4\}}$ center symmetries not paid attention to, so far)

Is this nontrivial DW (low-T as well as high-T) physics accessible on lattice?

possible issues: twisting b.c. OK; but pinning down the DW? avoid averaging condensate over DWs (histograms?)

How are these new discrete anomalies, found by turning on sometimes very unusual backgrounds, (e.g. non-spin manifolds for n_f=2 QCD(adj)! Cordoba, Dumitrescu 1806.09592) **reflected in the operator algebra of the discrete symmetries involved?**

In other words, reaching the level of understanding we have for continuous symmetry 't Hooft anomalies would be nice!

e.g. continuum: Frishman, Schwimmer, Banks, Yankielowicz; Coleman, Grossman '80s

continuum "2-group" (0-form/1-form) case:
Cordoba, Dumitrescu, Intriligator; Benini, Cordoba, Hsin '18

THANK YOU!

Extra slides:

Extra technical slide 1.1: Discrete anomalies and anomaly inflow from "the bulk"

anomalous variation of partition function

= **variation of a 3d bulk CS term** (3d space w/ boundary = 2d spacetime); e.g. our Schwinger example:

$$S_{3d\,CS} = i\,\frac{2\pi}{q} \int\limits_{M_3(\partial M_3 = M_2)} \frac{2qA_\chi^{(1)}}{2\pi} \wedge \frac{qB_C^{(2)}}{2\pi}$$

continuum description of a Z_{2q} 1-form gauge field (gauging 0-form discrete symmetry)= pair A^1,A^0:

$2qA^{(1)} = dA^{(0)}\ ,\quad \oint dA^{(0)} \in 2\pi\mathbb{Z} \implies \oint A^{(1)} \in \frac{2\pi\mathbb{Z}}{2q}$ ensures Z_{2q} Wilson loop $e^{i\oint A^{(1)}} = e^{i\frac{2\pi}{2q}\mathbb{Z}}$

$\delta_{\mathbb{Z}_{2q}^{d\chi}} A_\chi^{(1)} = d\phi\ ,\quad \oint d\phi \in 2\pi\mathbb{Z}$ so that closed Wilson loop gauge Z_{2q} gauge invariant

continuum description of a Z_q 2-form gauge field (gauging 1-form discrete symmetry)= pair B^2,B^1:

$qB^{(2)} = dB^{(1)}\ ,\quad \oint dB^{(1)} \in 2\pi\mathbb{Z} \implies \oint B^{(2)} \in \frac{2\pi\mathbb{Z}}{q}$ ensures that for closed 2 surface $e^{i\oint B^{(2)}} = e^{i\frac{2\pi}{q}\mathbb{Z}}$

$\delta_{\mathbb{Z}_q^C} B_C^2 = d\lambda^{(1)}\ ,\quad \oint d\lambda^{(1)} \in 2\pi\mathbb{Z}$ so that $e^{i\oint B^{(2)}}$ is 1-form gauge invariant

On a closed M_3, S_CS is gauge invariant, but $e^{S_{CS}}$ is not unity.

Its $\mathbb{Z}_{2q}^{d\chi}$ variation is nonzero on the boundary where $\phi|_{M_2} = \frac{2\pi}{2q}$

$\delta_{\mathbb{Z}_{2q}^{d\chi}} S_{3dCS} = i\,\frac{2\pi}{q}\,\frac{2q\,\phi|_{M_2}}{2\pi} \int\limits_{\partial M_3 = M_2} \frac{qB_C^{(2)}}{2\pi} \in i\,\frac{2\pi}{q}\mathbb{Z}$ same as we found before Z_f $\to e^{i\frac{2\pi}{q}}$ Z_f

similar story for 5d CS relevant for 4d theories, one extra power of B^2; also useful to show anomaly inflow on DW

Extra technical slide I.II: Discrete anomalies and anomaly inflow from “the bulk”

anomaly ‘inflow’ 3d bulk to 2d boundary:

$$S_{3d\,CS} = i\,\frac{2\pi}{q} \int_{M_3(\partial M_3 = M_2)} \frac{2qA^{(1)}_\chi}{2\pi} \wedge \frac{qB^{(2)}_C}{2\pi}$$ its $\mathbb{Z}^{d\chi}_{2q}$ variation is nonzero on boundary where $\phi|_{M_2} = \frac{2\pi}{2q}$

$$\delta_{\mathbb{Z}^{d\chi}_{2q}} S_{3dCS} = i\,\frac{2\pi}{q}\,\frac{2q\;\phi|_{M_2}}{2\pi} \int_{\partial M_3 = M_2} \frac{qB^{(2)}_C}{2\pi} \in i\,\frac{2\pi}{q}\mathbb{Z}$$ same as we found before $Z\_f \to e^{i\frac{2\pi}{q}}\, Z\_f$

its $\mathbb{Z}^C_q$ variation is $\oint \frac{q\lambda^{(1)}}{2\pi}|_{M_2} \in \mathbb{Z}$ so a charge-1 Wilson line $e^{i\oint a} \to e^{i\oint \lambda^{(1)}} e^{i\oint a} = e^{i\frac{2\pi}{q}} e^{i\oint a}$)

$$\delta_{\mathbb{Z}^C_q} S_{3dCS} = i\,\frac{2\pi}{q}\,\mathbb{Z} \oint \frac{qA^{(1)}}{2\pi}$$ for nonzero $\oint \frac{qA^{(1)}}{2\pi}$ (in broken phase, such holonomy induces DW)

anomaly ‘inflow’ to DW: consider nonzero $\oint \frac{qA^{(1)}}{2\pi}$ (inside M2 boundary) in broken phase;

then 3d bulk term reduces to 2d CS, whose boundary is the DW worldvolume (1-dim)

$$S_{2dCS} = i\,\frac{2\pi}{q} \oint_{\Gamma_2;\partial\Gamma_2 = \Gamma_{DW}} \frac{qB^{(2)}}{2\pi}$$ whose center variation localizes on DW

$$\delta_{\mathbb{Z}^C_q} S_{2dCS} = i\,\frac{2\pi}{q} \oint_{\Gamma_{DW}} \frac{q\lambda^{(1)}}{2\pi} = i\,\frac{2\pi}{q}$$

and shows DW has center charge-1, exactly as a q=1 Wilson loop

Extra technical slide I.III: Discrete anomalies and anomaly inflow from "the bulk"

broken phase long distance theory and anomaly matching:

The long-distance theory of the $\mathbb{Z}_{2q}^{d\chi}$ - broken Schwinger model a sort of "Z_q sigma model": **"chiral lagrangian"** yielding q degenerate states and nothing else (as theory is gapped); this **is a topological theory, as no dynamics or d.o.f.: only captures vacuum states and symmetries in IR!** IR theory can be coupled to background gauge fields for the global symmetries, reproducing the 't Hooft anomalies of the UV theory.

lattice description: a Zq topological Ising model, equiv. Zq topological GT [w/ global 0-form Z_q and a 1-form Z_q]

continuum description: a Zq topological BF theory 0-form ϕ and 1-form $a^{(1)}$ both compact: $\phi \equiv \phi + 2\pi$

$$\oint da^{(1)} = 2\pi\mathbb{Z}$$

$$iS_{IR} = i\frac{q}{2\pi}\int_{\mathbb{R}_t\times\mathbb{S}^1} \phi \, da^{(1)}$$

0-form global Z_q: $\phi \to \phi + \frac{2\pi}{q}$

1-form global Z_q: $a^{(1)} \to a^{(1)} + \lambda^{(1)}$

(recall $\delta_{\mathbb{Z}_q^C} B_C^2 = d\lambda^{(1)}$ $\oint d\lambda^{(1)} \in 2\pi\mathbb{Z}$)

EOMs imply both fields are flat (topological). On $\mathbb{R}_t\times\mathbb{S}^1$ fix a_0=0 gauge (Gauss law=constant ϕ on S^1).

a_x only d.o.f. (or rather $a = \oint_{S^1} a_x dx$). Lagrangian for ϕ , a is QM:

$$iS_{IR} = i\,\frac{q}{2\pi}\int_{\mathbb{R}_t} dt\; \phi\;\dot{a}$$

both variables are angular (2π period inherited from compactness)

Hamiltonian=0

quantize: $[\phi, a] = -i\frac{2\pi}{q}$ $\quad e^{i\phi}\, e^{ia} = e^{i\frac{2\pi}{q}}\; e^{ia}\; e^{i\phi}$ (+ $e^{iqa} = 1$ $\; e^{iq\phi} = 1$, trivial as operators)

treating ϕ as coordinate $\quad e^{i\phi}|P\rangle = |P\rangle e^{i\frac{2\pi P}{q}}$

$e^{ia}|P\rangle = |P+1 (\mathrm{mod}\; q)\rangle$

$e^{i\phi}$ order parameter of broken 0-form Z_q

e^{ia} "DW" between vacua (not local: spatial charge-1 Wilson loop)

q degenerate states = the q vacua, |P>, of the SM

(if a treated as coordinate: topological Z_q gauge theory)

Extra technical slide I.IV: Discrete anomalies and anomaly inflow from "the bulk"

broken phase long distance theory and anomaly matching:

$iS_{IR} = i\frac{q}{2\pi}\int \phi\,(da^{(1)} - B^{(2)})$: IR action with gauged 1-form center

recalling $\oint B^{(2)} \in \frac{2\pi\mathbb{Z}}{q}$ then: $\delta_{\mathbb{Z}_{2q}^{d\chi}} iS_{IR} = i\int\limits_{\mathbb{R}_t\times\mathbb{S}^1}(da^{(1)} - B^{(2)}) = i\frac{2\pi}{q}\mathbb{Z}$ anomaly matching!

(the anomaly matching part can be done rather nicely on the lattice, too)

---

continuum description: a Zq topological BF theory 0-form ϕ and 1-form $a^{(1)}$ both compact: $\phi \equiv \phi + 2\pi$ $\oint da^{(1)} = 2\pi\mathbb{Z}$

$iS_{IR} = i\frac{q}{2\pi}\int\limits_{\mathbb{R}_t\times\mathbb{S}^1}\phi\, da^{(1)}$

0-form global Z_q: $\phi \to \phi + \frac{2\pi}{q}$

1-form global Z_q: $a^{(1)} \to a^{(1)} + \lambda^{(1)}$

(recall $\delta_{\mathbb{Z}_q^C} B_C^2 = d\lambda^{(1)}$ $\oint d\lambda^{(1)} \in 2\pi\mathbb{Z}$)

EOMs imply both fields are flat (topological). On $\mathbb{R}_t\times\mathbb{S}^1$ fix a_0=0 gauge (Gauss law=constant ϕ on S^1).

a_x only d.o.f. (or rather $a = \oint\limits_{S^1} a_x dx$). Lagrangian for ϕ , a is QM:

$iS_{IR} = i\,\frac{q}{2\pi}\int\limits_{\mathbb{R}_t} dt\,\phi\,\dot{a}$ both variables are angular (2π period inherited from compactness)

quantize: $[\phi, a] = -i\frac{2\pi}{q}$ $e^{i\phi}\,e^{ia} = e^{i\frac{2\pi}{q}}\,e^{ia}\,e^{i\phi}$ (+ $e^{iqa} = 1$ $e^{iq\phi} = 1$, trivial as operators)

anomaly matching!

q degenerate states = the q vacua, |P>, of the SM

Extra technical slide II.I: Schwinger model discrete-symmetries operator algebra

Use Manton '86, Iso-Murayama '90 Hamiltonian solution. A_0 = 0 gauge, antiperiodic fermions on spatial circle, L:

$\oint A_x dx \equiv cL$ spatial holonomy only gauge variable

$$\mathbb{Z}_q^C : \ cL \to cL + \frac{2\pi}{q}$$

G: large gauge transforms shift cL by 2π

$|n\rangle$: Dirac sea states obeying Gauss' law (zero charge on circle) **explicit construction**

$Q_5|n\rangle = |n\rangle \left(2n - \frac{qcL}{\pi}\right)$ axial U(1) charge of Dirac sea depends on holonomy - anomaly!

$\tilde{Q}_5 \equiv Q_5 + \frac{qcL}{\pi}$ can define axial U(1) charge which does not depend on holonomy but not invariant under large gauge transforms

$G : \tilde{Q}_5 \to \tilde{Q}_5 + 2q$ but then, clearly:

$X_{2q} \equiv e^{i\frac{2\pi}{2q}\tilde{Q}_5}$ is invariant under G: operator performing $\mathbb{Z}_{2q}^{d\chi}$ chiral transforms

$G|n\rangle = |n+q\rangle$ under large gauge transforms n shifts by q units

$Y_q|n\rangle = |n+1\rangle$ center symmetry shifts n by 1 units

Extra technical slide II.II: Schwinger model discrete-symmetries operator algebra

$X_{2q} \equiv e^{i\frac{2\pi}{2q}\tilde{Q}_5}$ is invariant under large gauge transforms: $\mathbb{Z}_{2q}^{d\chi}$ chiral symmetry

$G|n\rangle = |n+q\rangle$ under large gauge transforms n shifts by q units

$Y_q|n\rangle = |n+1\rangle$ $\mathbb{Z}_q^C$ center symmetry Y_q shifts n by 1 units

Construct q theta-vacua, eigenstates of G (theta shown for convenience, not observable with m=0):

$$|\theta, k\rangle \equiv \sum_{n\in\mathbb{Z}} e^{i(k+qn)\theta}|k+qn\rangle, \; k = 0, 1, \ldots, q-1$$

Iso-Murayama '90

$$\phi(x) \;\equiv\; \bar{\psi}_+(x)\psi_-(x)$$

$$\langle n'|\phi(x)|n\rangle = \delta_{n',n+1}\, C'\, e^{-i\frac{2\pi x}{L}}$$

The linear combinations

$$|P,\theta\rangle \equiv \frac{1}{\sqrt{q}}\sum_{k=0}^{q-1}\omega_q^{kP}|\theta,k\rangle, \; P = 0,\ldots,q-1,$$

have diagonal condensate and obey cluster decomposition:

$$\langle P',\theta|\phi(x)|P,\theta\rangle = e^{-i\theta}\;\omega_q^{-P}\;\delta_{P,P'}C'$$

further, from above, discrete chiral and center act as "shift" and "clock" matrices

$$X_{2q}\;|P,\theta\rangle = |P+1(\text{mod } q),\theta\rangle$$

$$Y_q\;|P,\theta\rangle = |P,\theta\rangle\;\omega_q^{-P}e^{-i\theta}$$

the famous q-dim representation of the 't Hooft algebra

$$X_{2q}\;Y_q = \omega_q\;Y_q\;X_{2q} \quad (\omega_q = e^{i\frac{2\pi}{q}})$$

Extra technical slide II.III: Schwinger model discrete-symmetries operator algebra

$$X_{2q}\,|P,\theta\rangle = |P+1(\text{mod } q),\theta\rangle$$

$$Y_q\,|P,\theta\rangle = |P,\theta\rangle\,\omega_q^{-P} e^{-i\theta}$$

the famous q-dim representation of the 't Hooft algebra

$$X_{2q}\,Y_q = \omega_q\,Y_q\,X_{2q} \quad (\omega_q = e^{i\frac{2\pi}{q}})$$

Discrete chiral symmetry: clearly broken, q-vacua (1st relation above).

Breaking of center symmetry:

perimeter law for q=1 Wilson loop - all charges are screened in the massless Schwinger model due to vacuum polarization.
Screening length at infinite L is ~ 1/e. Iso-Murayama '90

This establishes all claims in the 2d part of the talk.

Extra technical slide III: further properties of charge-q Schwinger model

$$X_{2q}\,|P,\theta\rangle = |P+1(\mathrm{mod}\ q),\theta\rangle$$

$$Y_q\,|P,\theta\rangle = |P,\theta\rangle\,\omega_q^{-P}e^{-i\theta}$$

1 Center symmetry and chiral symmetries broken, at any T in the 2d charge-q model:

- at T >> qe, condensate still nonzero, exponential e^(- T/qe) falloff

- at T>> qe, A_0 component has a center-breaking vev, "GPY" potential

2 Using Shifman/Smilga '94, one can see, in eL<<1 weak coupling semiclassical limit, quite explicitly fractional-I/fractional-I* pairs ("fractons") contributing to bilinear condensate

essentially: $\langle \bar\psi_+\psi_-(x)\ \bar\psi_-\psi_+(0)\rangle\big|_{x\to\infty} \neq 0$ + clustering: $\langle \bar\psi_+\psi_-\rangle \neq 0$

+ eL<<1 weak coupling:

fracton/anti-fracton (each of charge |1/q|) saddle point (x-apart)

3 There are no dynamical DWs between the |P> vacua (these would be new particles in this 1d world); a q=1 Wilson static loop instead serves the purpose (**)

[see also older work by Hansson, Nielsen, Zahed '94,
no mention of chiral symmetry or anomalies, but breaking of Z_q center discussed]

4 The story flows into the theta=Pi pure YM upon adding mass (with right phase).

[(**) also matches with discussion of Gaiotto et al '17]

Extra technical slide IV: nonabelian/multiflavor case - richness...

center breaking vacua for SU(N) QCD(adj) labeled by w_0=0, w_1,…w_r (r=N-1, w_k=k-th fund. weight)

DWs labeled by: k, k=1,…,N-1: w_0 —> w_k

		gauge group on k-wall			
	fermion	$SU(N-k)_{gauge}$	$SU(k)_{gauge}$	$U(1)_{gauge}$	$U(1)_{A,global}$ anomalous
matter	λ_+	$\square$	$\square$	N	1
	λ_-	$\overline{\square}$	$\overline{\square}$	-N	1

There is a Z_N 1-form center symmetry on the 2d k-wall (min. U(1) charge is 1)

Again, inherits bulk chiral/center 't Hooft anomaly…

[…centers of worldvolume gauge groups,
+ emergent 1-form 2d worldvolume center…]

Presumably saturated on the wall by

$$\langle \lambda_+ \lambda_- \rangle \neq 0$$ (for nf=1, as in the low-T 4d bulk)

but remains to be shown…

nf dependence? [2d-4d relations…?]