Discrete anomaly matching and high-T “center vortices” in QCD(adj)

Erich Poppitz

with Mohamed Anber, Lewis & Clark College, 1807.00093[hep-th]
& in progress
Discrete anomaly matching and high-T “center vortices” in QCD(adj)

Our work is inspired by a series of papers of Gaiotto, Kapustin, Komargodski, Seiberg, Willet (in combinations) during 2014-2017. Features well-known to the lattice community are put in a new light, leading to constraints on the IR behavior of strongly coupled theories: new ‘’t Hooft anomaly matching” conditions.

Existence of center-symmetry (“1-form” symmetry) at the same time as other discrete (“0-form”) symmetries [CP, for pure-YM at $\theta = \pi$; discrete chiral symmetry for QCD(adj)] lead to new mixed 1-form/0-form ‘’t Hooft anomalies. Like continuous ‘’t Hooft anomalies, these can be computed in the UV and must be matched by the IR theory. Through “anomaly inflow” they also affect the domain walls (between vacua with broken discrete chiral or center symmetry), leading to rather nontrivial behavior on their worldvolume.

New “’t Hooft anomaly matching” conditions don’t appear all the time, so I decided to focus on them: new nontrivial constraints on IR behavior!

(tangential: new phases of QCD(adj)? Anber, EP 1805.12990, Cordova, Dumitrescu 1806.09592 - not this talk)
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The choice of QCD(adj) for this study of new ’t Hooft anomalies motivated by:

- As zero-N-ality, QCD(adj) has obvious 1-form center symmetry
  (because there are fermions, a step more complex than pure YM with $\theta = \pi$ [Gaiotto et al])

- QCD(adj) offers a rare regime where confinement and chiral symmetry
  breaking can be analytically understood in a theoretically controlled way:
  $R^3 \times S^1$ small-$S^1$ limit, periodic fermions - so anomaly matching implications can
  be seen explicitly!
  on a personal level in ’15 we found properties - now understood as due to
  these anomaly matchings [Anber, Sulejmanpasic, EP, 1501.06673] - so revisit

- In short, a good “lab” for better understanding new anomaly matching!

- Finally: do these anomaly matchings apply only to such ‘exotic’ theories? - NO!
  but more involved: i.e. in massless QCD with fundamentals center symmetries w/ anomalies
  appear after gauging $B$, $SU(N_f)_V$... [eg Tanizaki ’18] (…“two group” discrete version…)
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Outline:

1.) new “‘t Hooft anomaly” between discrete chiral symmetry and center symmetry: the simplest example in QFT is the charge-q massless 2d Schwinger model

2.) the charge-2 Schwinger model appears as the wordvolume theory of high-T domain walls - which are a type of center vortex - between center-breaking vacua in SU(2) QCD(adj) with a single Weyl massless adjoint (& multi-flavor generalizations if more adjoints)

3.) Schwinger model results imply that (& perhaps can be seen on lattice):
   - fermion condensate nonzero on the high-T center vortex
   - Wilson loop inside center vortex (spacelike!) obeys perimeter law

Thus, high-T center vortex worldvolume (2d) mirrors behavior of low-T (4d) theory: 4d bulk chiral symmetry broken and quarks deconfined on domain walls!

All tied by the anomalies!
I.) new “t Hooft anomaly” between discrete chiral symmetry and center symmetry: the simplest example in QFT is the charge-q massless 2d Schwinger model

\[ L = -\frac{1}{4e^2} f_{kli} f^{kl} + i\bar{\psi}_+(\partial_- + iqA_-)\psi_+ + i\bar{\psi}_-(\partial_+ + iqA_+)\psi_- \quad \partial_\pm \equiv \partial_t \pm \partial_x, \ A_\pm \equiv A_t \pm A_x, \]

“0-form” symmetries:

**U(1)\(_V\):** \( \psi_\pm \rightarrow e^{iq\alpha} \psi_\pm \) gauged

**U(1)\(_A\):** \( \psi_\pm \rightarrow e^{\pm i\chi} \psi_\pm \) anomalous: \( \mathbb{Z}_{\text{ferm.}} \rightarrow e^{i2q\chi T} \mathbb{Z}_{\text{ferm.}} \)

**\(\mathbb{Z}_{2q}\) :** \( \psi_\pm \rightarrow e^{\pm i\frac{\pi}{q}} \psi_\pm \) \( (\chi = \frac{2\pi}{2q} \) gives anomaly-free subgroup \)

“1-form” symmetry:

**\(\mathbb{Z}_q\) :** center symmetry

\[
\begin{array}{c|c}
\mathcal{U}_{x,\mu} & \mathcal{U}_{x,\mu}^c \\
\hline
\mathcal{U}_{x,\mu} & \mathcal{U}_{x,\mu}^c \end{array}
\]

a \( \mathbb{Z}_q \) phase, one per spacetime direction (global symmetry)

“1-form” symmetry only acts on line operators (hence name):

\[ \mathbb{Z}_q^C : e^{i \int A_x dx} \rightarrow \omega_q e^{i \int A_x dx}, \ \omega_q \equiv e^{i \frac{2\pi}{q}} \]
1.) new “‘t Hooft anomaly” between discrete chiral symmetry and center symmetry: the simplest example in QFT is the charge-q massless 2d Schwinger model

“0-form” $\mathbb{Z}_{2q}^d$ & “1-form” $\mathbb{Z}_q^C$ have a mixed anomaly!

also easy(er!) to see on the lattice! Let us gauge the 1-form center symmetry:

for symmetries acting on links (1-form center symmetry), introduce plaquette-based (“2-form”) $\mathbb{Z}_q$ gauge field to make $\mathbb{Z}_q$ center symmetry local to see the anomaly a background 2-form field suffices; in 2d, there is no field strength of the 2-form (no cubes!); introduce a $\mathbb{Z}_q$ background phase on a single plaquette = “$\mathbb{Z}_q$ center vortex” [i.e. any 2-form $\mathbb{Z}_q$ background topological… can move around shaded square by changing link variables]

now recall:

$$U_{\text{plaquette}} = \prod_{\text{link } \in \text{plaquette}} U_{\text{link}} = e^{i a^2 F_{\text{plaquette}}}$$

$\text{flux thru } T^2 = 2\pi n, \ n \in \mathbb{Z} = \text{integer } T \ (\text{top. charge)} \text{ from before!}$

But in the background of a single $\mathbb{Z}_q$ center vortex, we have instead

$$\prod_{\text{all plaquettes}} U_{\text{plaquette}} e^{i \frac{2\pi}{q}} = e^{i \text{flux thru } T^2} = e^{i \frac{2\pi}{q}} = \text{fractional } T \ (\text{top. charge)}!$$
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So, after we introduce a center 2-form background (center vortex), we have

$$\prod_{\text{all plaquettes}} U_{\text{plaquette}} e^{i \frac{2\pi}{q} q} = e^{i \text{flux thru } T^2} = e^{i \frac{2\pi}{q} T} = \text{fractional } T \text{ (top. charge)}!$$

But recall that under a 0-form discrete chiral $\mathbb{Z}^d_{2q}$ we have that

$$Z_{\text{ferm.}} \to e^{i 2\pi T} Z_{\text{ferm.}} \text{ and } Z_{\text{ferm.}} \text{ is invariant if } T \text{ is integer, but not otherwise.}$$

We conclude that if center 1-form symmetry is gauged, the discrete chiral symmetry ceases to be a symmetry, in other words, we have a $\mathbb{Z}^d_{2q} - \mathbb{Z}^C_q$ ‘‘t Hooft anomaly!

$$\mathbb{Z}^d_{2q} : \quad Z_{\text{ferm.}} \to e^{i \frac{2\pi}{q} T} Z_{\text{ferm.}}.$$
I.) New \textit{\textquoteleft\textquoteleft}t Hooft anomaly\textit{\textquoteright\textquoteright} between discrete chiral symmetry and center symmetry: the simplest example in QFT is the charge-\textit{q} massless 2d Schwinger model

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\begin{itemize}
  \item like for continuous symmetry \textquoteleft\textquoteleft t Hooft anomalies must be matched in IR:
    \begin{itemize}
      \item IR CFT, or
      \item one or both symmetries should be broken (\textit{\textquoteleft\textquoteleft}Goldstone\textit{\textquoteright\textquoteright} mode), or/and
      \item IR TQFT
    \end{itemize}
\end{itemize}

Gaiotto et al, \textquoteleft\textquoteleft14-	extquoteleft\textquoteleft17

The 0-form/1-form mixed anomaly was computed by Gaiotto et al by turning on discrete gauge backgrounds (as I showed you in the 2d case). A \textquoteleft\textquoteleft t Hooft anomaly, however, should be a property of the theory without any backgrounds; it does not require turning on fields. Continuous symmetry \textquoteleft\textquoteleft t Hooft anomalies are seen in $\langle j j j \rangle$ three-point global symmetry current correlators, as $1/q^2$ poles [Frishman et al, Coleman et al, 1980s].

Expect the \textit{\textquoteleft\textquoteleft}same\textit{\textquoteright\textquoteright} should be true here. The anomalies should involve properties of the quantum operators representing the discrete symmetries. General statements are so far not known, but examples exist, notably in QM [Gaiotto et al] and 2d QFT [our work].
I will spare you the details and simply state the result:
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  - IR TQFT

In the 2d QFT case (Schwinger model [our work]) and various QM examples [Gaiotto et al] the anomaly can be seen at the operator level and the IR matching decided upon:

\[
\hat{X}_{2q} \hat{Y}_{q} = \omega_q \hat{Y}_{q} \hat{X}_{2q} \quad (\omega_q = e^{i\frac{2\pi}{q}}).
\]

Classically, center and discrete chiral symmetries commute, but they do not in the QFT: anomaly!
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This is the famous ‘t Hooft-loop/Wilson-loop algebra for SU(q): q-dimensional reps!

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Gaiotto et al., ’14-’17
**1.** New "'t Hooft anomaly" between discrete chiral symmetry and center symmetry: the simplest example in QFT is the charge-q massless 2d Schwinger model

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**Finally, we also showed that both discrete chiral symmetry is broken (q discrete vacua, fermion condensate) and the 1-form center is broken (=perimeter law for q=1 Wilson loop).**

Used Manton '86, Iso-Murayama '90 Hamiltonian solution
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1,2,3 in Anber, EP 1807.00093
2.) 3.): ... wordlvolume theory of high-T domain walls - which are a type of center vortex - between center-breaking vacua...

$SU(2)$ Yang-Mills theory endowed with $n_f$ adjoint Weyl fermions $T \gg \Lambda_{QCD}$

\[
S_{3D}^{boson} = \frac{\beta}{g^2} \int_{\mathbb{R}^3} \left( \frac{1}{2} \text{tr} (F_{ij} F_{ij}) + \text{tr} (D_i A_4)^2 + g^2 V(A_4) + \mathcal{O}(g^4) \right) V(A_4) = -\frac{1}{12\pi \beta^4} \left[ -6\pi (\beta A_4^3)^2 + 4 (\beta A_4^3)^3 \right], \quad \text{for } \beta A_4^3 \in [0, \pi] \]

(shown for $n_f=1$ SYM)

- **two vacua** $\beta A_4^3 = 0, 2\pi$ broken center (“0-form”, along $x_4$) $\frac{1}{2} \langle \text{Tr}_F \exp \left[ i \oint_{S_{1\beta}} A_4 \right] \rangle = \pm 1$

- $\mathbb{Z}_2$ “domain walls”, or “interfaces”, or “center vortices” of width $\sim 1/gT$ Bhattacharya et al 1991

- $\mathbb{Z}_2$ 0-form center restored on DW

- $U(1)$ unbroken on wall (Polyakov loop not $\sim 1$) Cartan of $SU(2)$ massless; W-boson mass $\sim T$

  localized 2d $U(1)$ on wall not very interesting except $\theta = \pi$ pure YM! Gaiotto et al 2017
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- SYM has $\mathbb{Z}_2$ center 1-form and $\mathbb{Z}_4$ chiral 0-form w/ mixed 't Hooft anomaly!

  story similar as in Schwinger model: \textbf{fractional topological charge in 4d (here: 1/2)}

  \textbf{background of two center vortices intersecting at a point} (one along $x_1$-$x_2$, the other along $x_3$-$x_4$)

  - at $T<T_c$, $\mathbb{Z}_4$ chiral broken to $\mathbb{Z}_2$, matching the anomaly (assume $T_{\text{chi}}=T_c$)
  - at $T>T_c$, $\mathbb{Z}_2$ center broken, matching the anomaly ($\ldots >$ or $=$)

- domain walls, in either phase, are "nontrivial": anomaly inflow!

  Gaiotto et al 2014-17

- high-T center vortices have mixed $\mathbb{Z}_4$ chiral/$\mathbb{Z}_2$ center anomaly on 2d worldvolume

- can be seen in the high-T theory quite explicitly:
2.), 3.): ... wordlvolume theory of high-T domain walls - which are a type of center vortex - between center-breaking vacua...

- **U(1)** unbroken on wall (Polyakov loop not \(\sim\)1) Cartan of SU(2) massless; W-boson mass \(\sim\)T localized 2d U(1) on wall not very interesting except \(\theta = \pi\) pure YM! Gaiotto et al 2017 & even richer in QCD(adj)!

- Adjoint fermions at high-T have zero modes on the wall for \(n_f=1\), two normalizable:
  \[
  L_{DW}^{\text{axial}} = \frac{1}{4e^2} F_{kl} F_{kl} + i \bar{\lambda}_+ \left[ \partial_1 + i \partial_2 - i2(A_1 + iA_2) \right] \lambda_+ \\
  + i \bar{\lambda}_- \left[ \partial_1 - i \partial_2 + i2(A_1 - iA_2) \right] \lambda_-
  \]
  axial Schwinger model of charge-2! L and R have opposite charge

- In 2d axial and vector easily mapped to each other: \(Z_4\) chiral symmetry and \(Z_2\) center.

From q=2 Schwinger model results, chiral and center broken, so:
- **nonzero fermion condensate + Wilson loop perimeter law on the high-T “center vortex”!**
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... high-T center vortex worldvolume (2d) mirrors behavior of low-T (4d) theory:
4d bulk chiral symmetry broken and quarks deconfined on domain walls!

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<th>low T</th>
<th>assumed T_{chi} = T_{dec.} (&gt; or =)</th>
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explicit confinement on \(\mathbb{R}^3 \times S^1\) in Anber, Sulejmanpasic, EP 1501.06673

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low T

in bulk:
- $Z_4$ chiral broken
- $Z_2$ center unbroken
  (area law for W loop in bulk)

on DWs:
- quarks are deconfined
- $Z_2$ center broken on DW
  (perimeter law for W loop on DW while area law in bulk)

high T

in bulk:
- $Z_4$ chiral unbroken
- $Z_2$ 0-form center broken
- $Z_2$ 1-form center unbroken
  (area law for W loop in bulk)

on DWs:
- $Z_4$ chiral broken
- $Z_2$ 1-form center broken
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High-T DW properties mirror low-T bulk and low-T DW properties in a way really required by (rather, consistent with) anomaly...
One wonders if $n_f > 1$ high-T DWs teach us about QCD(adj) $n_f > 1$ unknown low-T phase? on DW: multi-flavor Schwinger models with classically marginal 4-fermi terms (reducing symmetries, generated by W-boson loops)? symmetry realization vs 4d low-T?

**Other theories with center symmetries: two-index S, AS flavors?**

and some specific 3-, 4- index ones

(discrete chiral/$\mathbb{Z}_{2,3,4}$ center symmetries not paid attention to, so far)

Is this nontrivial DW (low-T as well as high-T) physics accessible on lattice?

possible issues: twisting b.c. OK; but pinning down the DW? avoid averaging condensate over DWs (histograms?)

How are these new discrete anomalies, found by turning on sometimes very unusual backgrounds, (e.g. non-spin manifolds for $n_f=2$ QCD(adj)! Cordoba, Dumitrescu 1806.09592) reflected in the operator algebra of the discrete symmetries involved?

In other words, reaching the level of understanding we have for continuous symmetry 't Hooft anomalies would be nice!

(e.g. continuum: Frishman, Schwimmer, Banks, Yankielowicz; Coleman, Grossman ‘80s
continuum “2-group” (0-form/1-form) case: Cordoba, Dumitrescu, Intriligator; Benini, Cordoba, Hsin '18

THANK YOU!
Extra slides:
Extra technical slide I.I: Discrete anomalies and anomaly inflow from “the bulk”

anomalous variation of partition function

= variation of a 3d bulk CS term (3d space w/ boundary = 2d spacetime); e.g. our Schwinger example:

\[ S_{3d\, CS} = i \frac{2\pi}{q} \int_{M_3(\partial M_3=M_2)} 2q A^{(1)}_x/2\pi \wedge q B^{(2)}_C/2\pi \]

continuum description of a \( Z_{\{2q}\} \) 1-form gauge field (gauging 0-form discrete symmetry)= pair \( A^1, A^0 \):

\[ 2q A^{(1)} = dA^{(0)}, \int dA^{(0)} \in 2\pi \mathbb{Z} \implies \int A^{(1)} \in \frac{2\pi \mathbb{Z}}{2q} \]

ensures \( Z_{\{2q}\} \) Wilson loop \( e^{i \oint A^{(1)}} = e^{i \frac{2\pi}{2q} \mathbb{Z}} \)

\[ \delta_{Z_{2q}} A^{(1)} = d\phi , \int d\phi \in 2\pi \mathbb{Z} \text{ so that closed Wilson loop gauge } Z_{\{2q}\} \text{ gauge invariant} \]

continuum description of a \( Z_q \) 2-form gauge field (gauging 1-form discrete symmetry)= pair \( B^2, B^1 \):

\[ q B^{(2)} = dB^{(1)}, \int dB^{(1)} \in 2\pi \mathbb{Z} \implies \int B^{(2)} \in \frac{2\pi \mathbb{Z}}{q} \]

ensures that for closed 2 surface \( e^{i \oint B^{(2)}} = e^{i \frac{2\pi}{q} \mathbb{Z}} \)

\[ \delta_{Z_q} B^2_C = d\lambda^{(1)} , \int d\lambda^{(1)} \in 2\pi \mathbb{Z} \text{ so that } e^{i \oint B^{(2)}} \text{ is l-form gauge invariant} \]

On a closed \( M_3, S_{\text{CS}} \) is gauge invariant, but \( e^{S_{\text{CS}}} \) is not unity.

Its \( Z_{2q}^{d\chi} \) variation is nonzero on the boundary where \( \phi |_{M_2} = \frac{2\pi}{2q} \)

\[ \delta_{Z_{2q}} S_{3d\, CS} = i \frac{2\pi}{q} \frac{2q \phi |_{M_2}}{2\pi} \int_{\partial M_3=M_2} q B^{(2)}_C/2\pi \in i \frac{2\pi}{q} \mathbb{Z} \]

same as we found before \( Z_f \rightarrow e^{i \frac{2\pi}{q} \mathbb{Z}} Z_f \)

similar story for 5d CS relevant for 4d theories, one extra power of \( B^2 \); also useful to show anomaly inflow on DW
anomaly ‘inflow’ 3d bulk to 2d boundary:

$$S_{3d \, CS} = i \frac{2\pi}{q} \int_{M_3(\partial M_3=M_2)} 2q A^{(1)}_\chi \wedge \frac{q B^{(2)}_C}{2\pi}$$

its $Z_{2q}^{d\chi}$ variation is nonzero on boundary where $\phi|_{M_2} = \frac{2\pi}{2q}$

$$\delta_{Z_{2q}^{d\chi}} S_{3d \, CS} = \frac{2\pi}{q} \frac{2q}{2\pi} \int_{\partial M_3=M_2} q B^{(2)}_C \in i \frac{2\pi}{q} Z$$

same as we found before $Z \rightarrow e^{i \frac{2\pi}{q}} Z$

its $Z_q^C$ variation is $\int q \lambda^{(1)}|_{M_2} \in Z$ so a charge-1 Wilson line $e^{i \oint a} \rightarrow e^{i \oint \lambda^{(1)}} e^{i \oint a} = e^{i \frac{2\pi}{q}} e^{i \oint a}$

$$\delta_{Z_q^C} S_{3d \, CS} = i \frac{2\pi}{q} Z \int q A^{(1)} \quad \text{for nonzero} \quad \int q A^{(1)} \quad \text{(in broken phase, such holonomy induces DW)}$$

anomaly ‘inflow’ to DW: consider nonzero $\int q A^{(1)}$ (inside M2 boundary) in broken phase;

then 3d bulk term reduces to 2d CS, whose boundary is the DW worldvolume (1-dim)

$$S_{2d \, CS} = i \frac{2\pi}{q} \int_{\Gamma_2; \partial \Gamma_2=\Gamma_{DW}} q B^{(2)} \quad \text{whose center variation localizes on DW}$$

$$\delta_{Z_q^C} S_{2d \, CS} = i \frac{2\pi}{q} \int_{\Gamma_{DW}} q \lambda^{(1)} = i \frac{2\pi}{q}$$

and shows DW has center charge-1, exactly as a q=1 Wilson loop
broken phase long distance theory and anomaly matching:

The long-distance theory of the $\mathbb{Z}_q^{d\chi}$-broken Schwinger model is a sort of “$\mathbb{Z}_q$ sigma model”: “chiral lagrangian” yielding $q$ degenerate states and nothing else (as theory is gapped); this is a topological theory, as no dynamics or d.o.f.: only captures vacuum states and symmetries in IR! IR theory can be coupled to background gauge fields for the global symmetries, reproducing the ’t Hooft anomalies of the UV theory.

lattice description: a $\mathbb{Z}_q$ topological Ising model, equiv. $\mathbb{Z}_q$ topological GT [w/ global 0-form $\mathbb{Z}_q$ and a 1-form $\mathbb{Z}_q$]

continuum description: a $\mathbb{Z}_q$ topological BF theory 0-form $\phi$ and 1-form $a^{(1)}$ both compact: $\phi \equiv \phi + 2\pi$

$$iS_{IR} = i \frac{q}{2\pi} \int_{\mathbb{R}_t \times S^1} \phi \, da^{(1)}$$

0-form global $\mathbb{Z}_q$: $\phi \rightarrow \phi + \frac{2\pi}{q}$

$$\int da^{(1)} = 2\pi \mathbb{Z}$$

1-form global $\mathbb{Z}_q$: $a^{(1)} \rightarrow a^{(1)} + \lambda^{(1)}$

(recall $\delta_{\mathbb{Z}_q} B^2_C = d\lambda^{(1)}$)

EOMs imply both fields are flat (topological). On $\mathbb{R}_t \times S^1$ fix $a_0=0$ gauge (Gauss law=constant $\phi$ on $S^1$).

a_x only d.o.f. (or rather $a = \int a_x \, dx$). Lagrangian for $\phi, a$ is QM:

$$iS_{IR} = i \frac{q}{2\pi} \int_{\mathbb{R}_t} dt \, \phi \, da$$

both variables are angular (2$\pi$ period inherited from compactness)

Hamiltonian=0

quantize: $[\phi, a] = -i \frac{2\pi}{q}$

$$e^{i\phi} \, e^{ia} = e^{i \frac{2\pi}{q}} \, e^{ia} \, e^{i\phi}$$

(+ $e^{iqa} = 1$ $e^{iq\phi} = 1$, trivial as operators)

treating $\phi$ as coordinate

$$e^{i\phi} |P\rangle = |P\rangle e^{i \frac{2\pi}{q}}$$

$$e^{ia} |P\rangle = |P + 1(\text{mod } q)\rangle$$

order parameter of broken 0-form $\mathbb{Z}_q$

“DW” between vacua (not local: spatial charge-1 Wilson loop)

q degenerate states = the q vacua, $|P\rangle$, of the SM

(if $a$ treated as coordinate: topological $\mathbb{Z}_q$ gauge theory)
**Extra technical slide I.IV: Discrete anomalies and anomaly inflow from “the bulk”**

**broken phase long distance theory and anomaly matching:**

\[ i S_{IR} = i \frac{q}{2 \pi} \int \phi \left( da^{(1)} - B^{(2)} \right) \quad \text{: IR action with gauged 1-form center} \]

recalling \( \int B^{(2)} \in \frac{2 \pi Z}{q} \) then: \( \delta_{Z,q} i S_{IR} = i \int_{\mathbb{R}t \times S^1} (da^{(1)} - B^{(2)}) = i \frac{2 \pi}{q} Z \quad \text{anomaly matching!} \)

(the anomaly matching part can be done rather nicely on the lattice, too)

continuum description: a \( Zq \) topological BF theory 0-form \( \phi \) and 1-form \( a^{(1)} \) both compact: \( \phi \equiv \phi + 2 \pi \)

\[ i S_{IR} = i \frac{q}{2 \pi} \int_{\mathbb{R}t \times S^1} \phi \, da^{(1)} \]

\( \quad \begin{align*}
0\text{-form global } Z_q: \quad & \phi \rightarrow \phi + \frac{2 \pi}{q} \\
1\text{-form global } Z_q: \quad & a^{(1)} \rightarrow a^{(1)} + \lambda^{(1)} \\
\end{align*} \]

(\( \text{recall} \delta_{Z,q} B_C^2 = d\lambda^{(1)} \rightarrow d\lambda^{(1)} \in 2\pi Z \))

EOMs imply both fields are flat (topological). On \( \mathbb{R}t \times S^1 \) fix \( a_0=0 \) gauge (Gauss law=constant \( \phi \) on \( S^1 \)). \( a \_x \) only d.o.f. (or rather \( a = \int a_x \, dx \)). Lagrangian for \( \phi, a \) is QM:

\[ i S_{IR} = i \frac{q}{2 \pi} \int_{\mathbb{R}t} dt \, \phi \, \dot{a} \quad \text{both variables are angular (} 2\pi \text{ period inherited from compactness)} \]

quantize: \([\phi, a] = -i \frac{2 \pi}{q} \quad e^{i \phi} e^{ia} = e^{i \frac{2 \pi}{q} a} e^{ia} e^{i \phi} \quad (+ e^{ia} = 1, e^{iq \phi} = 1, \text{trivial as operators}) \)

anomaly matching!

\( q \) degenerate states = the \( q \) vacua, \( |P> \), of the SM
We are now ready, as in \[ \text{di}, \text{k} \]. The Dirac sea states obeying Gauss’ law can be found as was briefly outlined above. The \( i \) is the one where the states of all left moving particles of (gauge non-invariant) momenta \( n, k \). As eigenstates of the total Hamiltonian, however, the same letter, hoping that this does not cause undue confusion.) (Here and elsewhere we take the liberty to denote operators and eigenvalues with the \( n, k \), \( n \) charge vanishes, but the chiral (or axial \( h \)) states are orthogonal; their norm can be defined as unity,

\[ h \text{ matching ensures validity of the Gauss' law} \]

Under large gauge transformations \( \text{large gauge transforms shift} \ cL \text{ by } 2\pi \)

\[ |n\rangle : \text{Dirac sea states obeying Gauss' law (zero charge on circle) explicit construction} \]

\[ Q_5 |n\rangle = |n\rangle \left( 2n - \frac{qcL}{\pi} \right) \] axial \( \text{U(1) charge of Dirac sea depends on holonomy - anomaly!} \]

\[ \tilde{Q}_5 \equiv Q_5 + \frac{qcL}{\pi} \] can define axial \( \text{U(1) charge which does not depend on holonomy but not invariant under large gauge transforms} \)

\[ G : \tilde{Q}_5 \to \tilde{Q}_5 + 2q \] but then, clearly:

\[ X_{2q} \equiv e^{i \frac{2\pi}{2q} \tilde{Q}_5} \] is invariant under \( G \): operator performing \( \mathbb{Z}_{2q} \) chiral transforms

\[ G |n\rangle = |n + q\rangle \] under large gauge transforms \( n \) shifts by \( q \) units

\[ Y_q |n\rangle = |n + 1\rangle \] center symmetry shifts \( n \) by \( 1 \) units
Extra technical slide II.II: Schwinger model discrete-symmetries operator algebra

\[ X_{2q} \equiv e^{i \frac{2\pi}{2q}\tilde{Q}_5} \]

is invariant under large gauge transforms: \( \mathbb{Z}_{2q} \) chiral symmetry

\[ G|n\rangle = |n + q\rangle \]

under large gauge transforms \( n \) shifts by \( q \) units

\[ Y_q|n\rangle = |n + 1\rangle \quad \mathbb{Z}_q \text{ center symmetry } Y_q \text{ shifts } n \text{ by } 1 \text{ units} \]

Construct \( q \) theta-vacua, eigenstates of \( G \) (theta shown for convenience, not observable with \( m=0 \)):

\[ |\theta, k\rangle \equiv \sum_{n \in \mathbb{Z}} e^{i(k+qn)\theta} |k + qn\rangle, \quad k = 0, 1, \ldots, q - 1 \]

\[ \text{Iso-Murayama '90} \quad \phi(x) \equiv \bar{\psi}_+(x)\psi_-(x). \]

The linear combinations

\[ |P, \theta\rangle \equiv \frac{1}{\sqrt{q}} \sum_{k=0}^{q-1} \omega_q^{kP} |\theta, k\rangle, \quad P = 0, \ldots, q - 1, \]

have diagonal condensate and obey cluster decomposition:

\[ \langle P', \theta|\phi(x)|P, \theta\rangle = e^{-i\theta} \omega_q^{-P} \delta_{P,P'}C' \]

Further, from above, discrete chiral and center act as “shift” and “clock” matrices

\[ X_{2q} |P, \theta\rangle = |P + 1(\text{mod } q), \theta\rangle \]

\[ Y_q |P, \theta\rangle = |P, \theta\rangle \omega_q^{-P} e^{-i\theta} \quad \text{the famous } q\text{-dim representation of the 't Hooft algebra} \]

\[ X_{2q} Y_q = \omega_q Y_q X_{2q} \quad (\omega_q = e^{i \frac{2\pi}{q}}) \]
Extra technical slide II.III: Schwinger model discrete-symmetries operator algebra

\[ X_{2q} |P, \theta\rangle = |P + 1 (\text{mod } q), \theta\rangle \]
\[ Y_q |P, \theta\rangle = |P, \theta\rangle \omega_q^{-P} e^{-i\theta} \]

the famous q-dim representation of the 't Hooft algebra

\[ X_{2q} Y_q = \omega_q Y_q X_{2q} \quad (\omega_q = e^{i \frac{2\pi}{q}}) \]

Discrete chiral symmetry: clearly broken, q-vacua (1st relation above).

Breaking of center symmetry:

perimeter law for q=1 Wilson loop - all charges are screened in the massless Schwinger model due to vacuum polarization.
Screening length at infinite L is \( \sim \frac{1}{e} \)  
Iso-Murayama '90

This establishes all claims in the 2d part of the talk.
Extra technical slide III: further properties of charge-q Schwinger model

\[ X_{2q} |P, \theta\rangle = |P + 1(\mod q), \theta\rangle \]
\[ Y_q |P, \theta\rangle = |P, \theta\rangle \omega_q^{-P} e^{-i\theta} \]

1. Center symmetry and chiral symmetries broken, at any \( T \) in the 2d charge-q model:
   - at \( T >> qe \), condensate still nonzero, exponential \( e^{(-T/qe)} \) falloff
   - at \( T >> qe \), \( A_0 \) component has a center-breaking vev, "GPY" potential

2. Using Shifman/Smilga '94, one can see, in \( eL << 1 \) weak coupling semiclassical limit, quite explicitly
   fractional-1/fractional-1* pairs ("fractons") contributing to bilinear condensate
   essentially: \[ \langle \bar{\psi}_+ \psi_- (x) \bar{\psi}_- \psi_+ (0) \rangle \big|_{x \to \infty} \neq 0 \]
   + clustering: \[ \langle \bar{\psi}_+ \psi_- \rangle \neq 0 \]
   + \( eL << 1 \) weak coupling:
     fracton/anti-fracton (each of charge \( |1/q| \)) saddle point (x-apart)

3. There are no dynamical DWs between the \( |P> \) vacua (these would be new particles in this 1d world);
   a \( q=1 \) Wilson static loop instead serves the purpose (**)
   [see also older work by Hansson, Nielsen, Zahed '94,
    no mention of chiral symmetry or anomalies, but breaking of \( Z_q \) center discussed]

4. The story flows into the theta=\( \pi \) pure YM upon adding mass (with right phase).
   \[ (***) also matches with discussion of Gaiotto et al '17 \]
Extra technical slide IV: nonabelian/multiflavor case - richness...

center breaking vacua for SU(N) QCD(adj) labeled by $w_0=0, w_1, \ldots w_r$ ($r=N-1$, $w_k=k$-th fund. weight)

DWs labeled by: $k, k=1, \ldots, N-1$: $w_0 \rightarrow w_k$

<table>
<thead>
<tr>
<th>matter</th>
<th>gauge group on k-wall</th>
</tr>
</thead>
<tbody>
<tr>
<td>fermion</td>
<td>$SU(N-k)_{gauge}$</td>
</tr>
<tr>
<td>$\lambda_+$</td>
<td>$\square$</td>
</tr>
<tr>
<td>$\lambda_-$</td>
<td>$\square$</td>
</tr>
</tbody>
</table>

There is a $\mathbb{Z}_N$ 1-form center symmetry on the 2d k-wall (min. $U(1)$ charge is $1$)

Again, inherits bulk chiral/center ’t Hooft anomaly…

[…centers of worldvolume gauge groups, + emergent 1-form 2d worldvolume center…]

Presumably saturated on the wall by

$$\langle \lambda_+ \lambda_- \rangle \neq 0$$

(for $nf=1$, as in the low-T 4d bulk)

but remains to be shown…

$nf$ dependence?

[2d-4d relations…?]