Spectrum of the open QCD flux tube and its effective string description

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1. Introduction

Confinement, flux tubes and strings
Static $q\bar{q}$-potentials

static $q\bar{q}$-potential:
energy of static $q\bar{q}$ pair at distance $R$

for states with excited gluons configurations:
hybrid $q\bar{q}$-potentials

Physical relevance:
- linearly rising potential $\Leftrightarrow$ confinement
- input for model calculations
  (hybrid mesons, . . .)
  $\Rightarrow$ analytic description is wanted
- can be used to make contact to AdS/CFT duals of pure gauge theory

For the latter: effective string theory
Confinement and flux tubes

Heuristic confinement mechanism:

- $q\bar{q}$ pair connected by region of strong chromo-electromagnetic flux
- pulling the quarks apart: flux gets squeezed into a narrow region
  \[ \Rightarrow \text{flux tube} \]
  - squeezing due to dual Meissner effect
  - here all quarks are static (no string breaking)
- for such a tube:
  expect constant energy density
  \[ \Rightarrow \text{linearly rising potential} \quad V(R) = \sigma R \]
  $\sigma$ : string tension
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Introduction

Flux tubes and string theory

at large $R$: flux tube looks like a thin energy string

excitation spectrum will be dominated by stringy excitations!

⇒ formulation of effective string theories (EST) for the flux tube.
[ Nambu, PLB 80, 372 (1979); Lüscher, Symanzik, Weisz, NPB 173, 365 (1980); Polyakov, NPB 164, 171 (1980) ]

since then: formalism has been developed and action is known up to $O(R^{-5})$
[ Lüscher, Weisz, …, Polchinski, Strominger, …, Casselle, …, Aharony, …, Dubovsky, Flauger, Gorbenko … ]

for details and references see review [ BB, Meineri, IJMP A31 (2016) ]

Historically:

▶ idea also motivated by Regge trajectories [ Regge, NC14 (1959) ]
▶ origin of first string theories [ Goddard et al, NPB65 (1963); Goto, PTP46 (1971) ]
## EST spectrum (open strings) [Aharony, Klinghoffer, JHEP1012 (2010)]

\[
E_{n,l}^{\text{EST}}(R) = \sigma R \sqrt{1 + \frac{2\pi}{\sigma R^2} \left( n - \frac{1}{24} (d - 2) \right)} \\
- \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3 R^4}} \left( B^l_n + \frac{d - 2}{60} \right) - \frac{\pi^3(d - 26)}{48\sigma^2 R^5} C^l_n + \mathcal{O}(R^{-\xi})
\]

### LC spectrum (or NG) [J.F. Arvis, PLB127 (1983)]

**boundary term**

\( \bar{b}_2 \): dimensionless non-universal boundary coefficient \( \bar{b}_2 = \sqrt{\sigma^3 b_2} \)

\( B^l_n, C^l_n \): dimensionless, depend on representation of \( SO(d - 2) \)

<table>
<thead>
<tr>
<th>( n, l )</th>
<th>( SO(d - 2) ) representation</th>
<th>( B^l_n )</th>
<th>( C^l_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>( 1</td>
<td>0\rangle ) scalar</td>
<td>0</td>
</tr>
<tr>
<td>( 1 )</td>
<td>( \alpha^j_{-1}</td>
<td>0\rangle ) vector</td>
<td>4</td>
</tr>
<tr>
<td>( 2, 1 )</td>
<td>( \alpha^j_{-1}\alpha^j_{-1}</td>
<td>0\rangle ) scalar</td>
<td>8</td>
</tr>
<tr>
<td>( 2, 2 )</td>
<td>( \alpha^j_{-2}</td>
<td>0\rangle ) vector</td>
<td>32</td>
</tr>
<tr>
<td>( 2, 3 )</td>
<td>( (\alpha^j_{-1}\alpha^j_{-1} - \frac{\delta^{ij}}{d - 2}\alpha^j_{-1}\alpha^j_{-1})</td>
<td>0\rangle ) sym. tracel. tensor</td>
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</tr>
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**EST spectrum (open strings)**  

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E_{n,l}^{\text{EST}}(R) = \sigma R \sqrt{1 + \frac{2\pi}{\sigma R^2} \left( n - \frac{1}{24} (d - 2) \right)} - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3 R^4}} \left( B'_n + \frac{d - 2}{60} \right) - \frac{\pi^3 (d - 26)}{48\sigma^2 R^5} C'_n + \mathcal{O}(R^{-\xi})
\]

**LC spectrum (or NG)**  

\[\bar{b}_2: \text{dimensionless non-universal boundary coefficient} \quad \bar{b}_2 = \sqrt{\sigma^3} b_2\]

\[B'_n, C'_n: \text{dimensionless, depend on representation of } SO(d - 2)\]

| \(|n, l\rangle\) | \(SO(d - 2)\) representation | \(B'_n\) | \(C'_n\) |
|-------------|-----------------|--------|--------|
| \(|0\rangle\) | \(1|0\rangle\) | scalar | 0 | 0 |
| \(|1\rangle\) | \(\alpha^i_{-1}|0\rangle\) | vector | 4 | \(d - 3\) |
| \(|2, 1\rangle\) | \(\alpha^i_{-1}\alpha^i_{-1}|0\rangle\) | scalar | 8 | 0 |
| \(|2, 2\rangle\) | \(\alpha^i_{-2}|0\rangle\) | vector | 32 | \(16(d - 3)\) |
| \(|2, 3\rangle\) | \(\left(\alpha^i_{-1}\alpha^j_{-1} - \frac{\delta^{ij}}{d - 2}\alpha^i_{-1}\alpha^i_{-1}\right)|0\rangle\) | sym. tracel. tensor | 8 | \(4(d - 2)\) |
AdS/CFT correspondence and the holographic string

large-N QCD: supposed to have dual 10d AdS superstring description

question: which is the associated holographic string background?

EST string:
  4d projection of 10d superstring

for particular backgrounds:
  can derive EST action

several suitable backgrounds are known
  [ Witten, ATMP2 (1998); Klebanov, Strassler, JHEP0008 (2000); Maldacena, Nunez, PRL86 (2001) ]

all have the same LO action, consistent with EST
  [ Aharony, Karzbrun, JHEP0906 (2009) ]

non-universal coefficients relate to properties of 10d string theory

\[ b_2 = - \frac{1}{64\sigma} \sum_\xi \frac{(-1)^{B C(\xi)}}{m_\xi^b} + b_2^f + \ldots \]
  [ Aharony, Field, JHEP1101 (2011) ]

⇒ extraction of non-universal parameters can provide information on AdS side
Rigidity and massive modes

- so far ignored in EST: extrinsic curvature term

formally higher order; can give contributions under quantisation


correction term for potential:

\[ V_{\text{ext}}(R) = -\frac{m}{2\pi} \sum_{n=1}^{\infty} \frac{K_1(2nmR)}{n} - \frac{(d-2)(d-10)\pi^2}{3840m\sigma R^4} \]

\( K_1 \): Modified Bessel function of first kind
\( m \): free parameter with dimension of mass

⇒ mixes with the boundary term (can change value of \( \bar{b}_2 \))
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Rigidity and massive modes

- so far ignored in EST: extrinsic curvature term
- formally higher order; can give contributions under quantisation
  \[ V_{\text{ext}}(R) = - \frac{m}{2\pi} \sum_{n=1}^{\infty} \frac{K_1(2nmR)}{n} - \frac{(d-2)(d-10)\pi^2}{3840m\sigma R^4} \]
  - \( K_1 \): Modified Bessel function of first kind
  - \( m \): free parameter with dimension of mass
  \Rightarrow mixes with the boundary term (can change value of \( \bar{b}_2 \))
- other possible contribution: massive modes
  found to be important to describe 4d spectrum
  \[ [ \text{Dubovsky, Flauger, Gorbenko, PRL111 (2013); JETP120 (2015)} ] \]
- however: 4d coupling term not allowed in 3d
  can only couple indirectly via the induced metric
  \Rightarrow formally similar contribution to rigidity term!
Current status of lattice simulations

A large number of lattice studies in the past 35 years:

▶ static potential and excited states:
  good agreement with LC spectrum
  3d: small deviations can be fitted to $\tilde{b}_2$ correction
  [ BB, JHEP1102 (2011); Billo et al, 1205 (2012) ]
  4d: behaviour of excited states points to presence of massive modes
  [ Juge, Kuti, Morningstar, PRL90 (2003) ]
  3d $Z_2$ and $U(1)$: presence/importance of rigidity term has been observed

▶ closed flux tubes:
  3d: good agreement with EST
  4d: massive modes found to be important to describe 4d spectrum
  [ Dubovsky, Flauger, Gorbenko, PRL111 (2013); JETP120 (2015) ]

▶ can also study: flux tube width, finite temperature, ...

for a review and more references see [ BB, Meineri, IJMP A31 (2016) ]
Goals and setup of this study

first goal: extract EST parameters at finite $N$ and in the $N \to \infty$ limit
in particular: use pure gauge lattice simulation in 3d
  - extract $\sqrt{\sigma r_0}$ and $\bar{b}_2$ in 3d $SU(N = 2, 3, 4, 5, 6)$ from $V(R)$
    - multiple lattice spacings $a \approx 0.11, 0.08, 0.06$ fm with $V \gtrsim 5$ fm
    - error reduction: LW algorithm
      (2000 total meas; 20 000 sub. updates; $t_s = 2, 4, 6$)
  - extrapolate to continuum $a \to 0$ and subsequently $N \to \infty$.
  - check the consistency of results with the excited states
    here: use old $SU(2)$ data from [BB, JHEP1102 (2011)]

second goal: test consistency with massive modes/string rigidity
  - extract mass $m$ and investigate impact on $\bar{b}_2$
  - once more: compare continuum results for different values of $N$
    $\Rightarrow$ extrapolate $N \to \infty$?
2. String tension and KKN prediction
Extraction of the string tension

First step: Extract string tension $\sigma$ (defined by $R \to \infty$ behaviour)

reliable computation: demands extraction of $R \to \infty$ behaviour

strategy: perform two different fits including different $1/R$ corrections

(i) fit to LO force  (ii) fit to LC potential

compare $R_{\text{min}}$-dependence of $\sigma$ from these methods

$\Rightarrow$ extraction of $\sigma$ is reliable where results agree!
Large-N extrapolations and KKN prediction

Karabili-Kim-Nair prediction: \( \frac{\sqrt{\sigma}}{g_{MF}^2} = \sqrt{\frac{N^2 - 1}{8\pi}} \)  
[Karabili, Kim, Nair, PLB434 (1998)]

\[ r_{KKN} = \frac{(\sqrt{\sigma r_0/g^2 r_0})_{\text{lat}}}{(\sqrt{\sigma r_0/g^2 r_0})_{\text{KKN}}} \]
Karabili-Kim-Nair prediction: \[ \sqrt{\sigma} \frac{g^2}{g_{\text{MF}}^2} = \sqrt{\frac{N^2 - 1}{8\pi}} \]

final 3d large-N results:

\[ \sqrt{\sigma} r_0 = 1.2304(4)(3) \quad r_{KKN} = \frac{(\sqrt{\sigma} r_0 / g^2 r_0)_{\text{lat}}}{(\sqrt{\sigma} r_0 / g^2 r_0)_{KKN}} = 0.9842(6)(14) \]
3. EST analysis without massive modes
Order of the leading order correction

first: check consistency of correction to LC potential with $R^{-4}$

fit $V(R)$ to form: $V(R) = E_0^{LC}(R) + \frac{\eta}{(\sqrt{\sigma R})^m}$

look at $R_{\text{min}}$ dependence of $m$:
Extraction strategy

next step: extract the boundary coefficient!

fit data to:

\[ V(R) = E_0^{LC}(R) - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3} R^4} \left( C_i^i + \frac{d - 2}{60} \right) + \frac{\gamma_0^{(1)}}{\sqrt{\sigma^5} R^6} + \frac{\gamma_0^{(2)}}{\sigma^3 R^7} + V_0 \]

To quantify systematic errors: perform different fits
Extraction strategy

next step: extract the boundary coefficient!

fit data to:

\[ V(R) = E_0^{LC}(R) - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3} R^4} \left( C_n^i + \frac{d - 2}{60} \right) + \frac{\gamma_0^{(1)}}{\sqrt{\sigma^5} R^6} + \frac{\gamma_0^{(2)}}{\sigma^3 R^7} + V_0 \]

To quantify systematic errors: perform different fits

A: use \( \sigma, V_0 \) from above – use \( \bar{b}_2, \gamma_0^{(1)}, \gamma_0^{(2)} \) as free params
Extraction strategy

next step: extract the boundary coefficient!

fit data to:

\[ V(R) = E_0^{Lc}(R) - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3 R^4}} \left( C_n + \frac{d-2}{60} \right) + V_0 \]

To quantify systematic errors: perform different fits

**A:** use \( \sigma, V_0 \) from above – use \( \bar{b}_2, \gamma_0^{(1)}, \gamma_0^{(2)} \) as free params

**B:** use \( \sigma, V_0, \bar{b}_2 \) as free params. – set \( \gamma_0^{(1)} = \gamma_0^{(2)} = 0 \)
Extraction strategy

next step: extract the boundary coefficient!

fit data to:

\[ V(R) = E_0^{LC}(R) - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3} R^4} \left( C_i + \frac{d - 2}{60} \right) + \frac{\gamma_0^{(1)}}{\sqrt{\sigma^5} R^6} + V_0 \]

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D: Use \( \sigma, V_0, \bar{b}_2, \gamma_0^{(2)} \) as free params. – set \( \gamma_0^{(1)} = 0 \)
Extraction strategy

next step: extract the boundary coefficient!

fit data to:

\[ V(R) = E_0^{LC}(R) + \frac{\gamma_0^{(1)}}{\sqrt{\sigma^5 R^6}} + \frac{\gamma_0^{(2)}}{\sigma^3 R^7} + V_0 \]

To quantify systematic errors: perform different fits

A: use \( \sigma, V_0 \) from above – use \( \bar{b}_2, \gamma_0^{(1)}, \gamma_0^{(2)} \) as free params

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C: Use \( \sigma, V_0, \bar{b}_2, \gamma_0^{(1)} \) as free params. – set \( \gamma_0^{(2)} = 0 \)

D: Use \( \sigma, V_0, \bar{b}_2, \gamma_0^{(2)} \) as free params. – set \( \gamma_0^{(1)} = 0 \)

E: Use \( \sigma, V_0, \gamma_0^{(1)}, \gamma_0^{(2)} \) as free params. – set \( \bar{b}_2 = 0 \)
Extraction strategy

next step: extract the boundary coefficient!

fit data to:

\[ V(R) = E_0^{LC}(R) - b_2 \frac{\pi^3}{\sqrt{\sigma^3 R^4}} \left( C_i + \frac{d - 2}{60} \right) + \frac{\gamma_0^{(1)}}{\sqrt{\sigma^5 R^6}} + \frac{\gamma_0^{(2)}}{\sigma^3 R^7} + V_0 \]

To quantify systematic errors: perform different fits

A: use \( \sigma, V_0 \) from above – use \( b_2, \gamma_0^{(1)}, \gamma_0^{(2)} \) as free params

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D: Use \( \sigma, V_0, b_2, \gamma_0^{(2)} \) as free params. – set \( \gamma_0^{(1)} = 0 \)

E: Use \( \sigma, V_0, \gamma_0^{(1)}, \gamma_0^{(2)} \) as free params. – set \( b_2 = 0 \)

fits A and E are checks whether \( b_2 \neq 0 \)

fits B–D are used in the final analysis
Extraction, limits and estimation of systematic errors

- **higher order terms:**

  final result: average over fits B–D
  estimate for uncertainty: largest deviation from final result
Extraction, limits and estimation of systematic errors

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  - final result: average over fits B–D
  - estimate for uncertainty: largest deviation from final result

- **fitrange for $\bar{b}_2$:**
  - $R_{\text{min}}$: defined by the second fit for which $\chi^2$/dof is acceptable.
  - estimate for uncertainty: deviation from fits with $R_{\text{min}} \pm 1$
Extraction, limits and estimation of systematic errors

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  final result: average over fits B–D
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- **continuum extrapolation:**
  
  final result: use a linear continuum extrapolation (in \( a^2 \))
  estimate for uncertainty: deviation from fit with only last points
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EST analysis without massive modes

Extraction, limits and estimation of systematic errors

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- **large-N extrapolation:**
  - final result: obtained from linear large-N extrapolation (in \( 1/N^2 \))
  - estimate for uncertainty: deviation from fit with only last points

- course of analysis: perform all possible combinations of fits
Extraction, limits and estimation of systematic errors

- higher order terms:
  final result: average over fits B–D
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  $R_{\text{min}}$: defined by the second fit for which $\chi^2$/dof is acceptable.
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  final result: obtained from linear large-N extrapolation (in $1/N^2$)
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course of analysis: perform all possible combinations of fits
Results for $\bar{b}_2$
Continuum extrapolation of $\bar{b}_2$

Extrapolation for $SU(2)$

Linear continuum extrapolation works well for all $N$
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**Continuum extrapolation of \( \bar{b}_2 \)**

*Extrapolation for SU(5)*

linear continuum extrapolation works well for all \( N \)
Final continuum results for $\bar{b}_2$
Large-N extrapolation $\bar{b}_2$

Final large-N result: $\bar{b}_2^N \rightarrow \infty = -0.0141(3)(15)(13)(9)(17)$

Errors: statistical, HO corr., $R_{\text{min}}$, cont. extra., large-N extra
Consistency with the excited states

compare results for $\bar{b}_2$ to $E_1$ in 3d $SU(2)$: ($\beta = 5.0$ data [BB, JHEP1102 (2011)])

Energy levels fully determined by $\bar{b}_2$ up to $O(1/R^{6,7})$. 
Consistency with the excited states

compare results for $\tilde{b}_2$ to $E_1$ in 3d $SU(2)$: ($\beta = 5.0$ data [ BB, JHEP1102 (2011) ] )

Fit the higher order terms: Good description of the data!
Consistency with the excited states

Alternatively: extract $\bar{b}_2$ from fit to excited states  

[ BB, JHEP1102 (2011) ]

$\Rightarrow$ excellent agreement with extraction from potential
4. Testing the presence of massive modes
Extraction strategy

up to now: neglected the possible presence of massive modes

to test whether they can be present fit data to:

\[ V(R) = E_0^L C(R) - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3} R^4} \left( C_n^i + \frac{d - 2}{60} \right) - \frac{(d - 2)(d - 10)\pi^2}{3840 m\sigma R^4} \]

\[ - \frac{m}{2\pi} \sum_{k=1}^{\infty} \frac{K_1(2kmR)}{k} + \frac{\gamma_0^{(1)}}{\sqrt{\sigma^5} R^6} + \frac{\gamma_0^{(2)}}{\sigma^3 R^7} + V_0 \]

perform different fits
Extraction strategy

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\[ V(R) = E_0^{LC}(R) - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma}^3 R^4} \left( C_n + \frac{d - 2}{60} \right) - \frac{(d - 2)(d - 10)\pi^2}{3840 m \sigma R^4} \]

\[ - \frac{m}{2\pi} \sum_{k=1}^{\infty} \frac{K_0(2kmR)}{k} + V_0 \]

perform different fits

F use \( \sigma, V_0, \bar{b}_2 \) and \( m \) as free parameters – set \( \gamma_0^{(1)} = \gamma_0^{(2)} = 0 \)
Extraction strategy

up to now: neglected the possible presence of massive modes

to test whether they can be present fit data to:

\[
V(R) = E_0^{LC}(R) - \overline{b}_2 \frac{\pi^3}{\sqrt{\sigma^3} R^4} \left( C_n + \frac{d - 2}{60} \right) - \frac{(d - 2)(d - 10)\pi^2}{3840 m \sigma R^4}
\]

\[
- \frac{m}{2\pi} \sum_{k=1}^{\infty} \frac{K_1(2kmR)}{k} + \frac{\gamma_0^{(1)}}{\sqrt{\sigma^5} R^6} + V_0
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perform different fits

**F** use \(\sigma, V_0, \overline{b}_2\) and \(m\) as free parameters – set \(\gamma_0^{(1)} = \gamma_0^{(2)} = 0\)

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**G** use \( \sigma, V_0, \bar{b}_2, m \) and \( \gamma_0^{(1)} \) as free parameters – set \( \gamma_0^{(2)} = 0 \)

**H** use \( \sigma, V_0, \bar{b}_2, m \) and \( \gamma_0^{(2)} \) as free parameters – set \( \gamma_0^{(1)} = 0 \)
Extraction strategy

up to now: neglected the possible presence of massive modes

to test whether they can be present fit data to:

\[
V(R) = E_0^{LC}(R) - \frac{(d - 2)(d - 10)\pi^2}{3840m\sigma R^4} - \frac{m}{2\pi} \sum_{k=1}^{\infty} \frac{K_1(2kmR)}{k} + V_0
\]

perform different fits

F use \(\sigma, V_0, \overline{b}_2\) and \(m\) as free parameters – set \(\gamma_0^{(1)} = \gamma_0^{(2)} = 0\)

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H use \(\sigma, V_0, \overline{b}_2, m\) and \(\gamma_0^{(2)}\) as free parameters – set \(\gamma_0^{(1)} = 0\)

J use \(\sigma, V_0\) and \(m\) as free parameters – set \(\gamma_0^{(1)} = \gamma_0^{(2)} = \overline{b}_2 = 0\)
Extraction strategy

up to now: neglected the possible presence of massive modes

to test whether they can be present fit data to:

\[ V(R) = E_0^{LC}(R) - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3 R^4}} \left( C_n^i + \frac{d - 2}{60} \right) - \frac{(d - 2)(d - 10)\pi^2}{3840 m\sigma R^4} \]

\[ - \frac{m}{2\pi} \sum_{k=1}^{\infty} K_1(2kmR) \frac{\gamma_0^{(1)}}{\sqrt{\sigma^5 R^6}} + \frac{\gamma_0^{(2)}}{\sigma^3 R^7} + V_0 \]

perform different fits

**F** use \( \sigma, V_0, \bar{b}_2 \) and \( m \) as free parameters – set \( \gamma_0^{(1)} = \gamma_0^{(2)} = 0 \)

**G** use \( \sigma, V_0, \bar{b}_2, m \) and \( \gamma_0^{(1)} \) as free parameters – set \( \gamma_0^{(2)} = 0 \)

**H** use \( \sigma, V_0, \bar{b}_2, m \) and \( \gamma_0^{(2)} \) as free parameters – set \( \gamma_0^{(1)} = 0 \)

**J** use \( \sigma, V_0 \) and \( m \) as free parameters – set \( \gamma_0^{(1)} = \gamma_0^{(2)} = \bar{b}_2 = 0 \)

fit **J**: check whether \( \bar{b}_2 \neq 0 \)

fit **F** used in the final analysis (results of **G** and **H** not accurate enough)
Results for $\bar{b}_2$ and $m$

main cause for large uncertainties: $R_{\text{min}}$-dependence of fit
Spectrum of the open QCD flux tube and its effective string description

Testing the presence of massive modes

Final continuum results for $\bar{b}_2$ and $m$

much larger uncertainties $\Rightarrow$ extrapolation for $\bar{b}_2$ unstable
Spectrum of the open QCD flux tube and its effective string description

Testing the presence of massive modes

Large-N extrapolation $m$

![Graph showing the large-N extrapolation of $m$](image)

final large-N result: $r_0 m^{N \to \infty} = -1.34(4)(8)(25)$

errors: statistical, $R_{\text{min}}$, cont. extra. (HO corr, large-N: uncontrolled)
Large-N extrapolation $m$

final large-N result: $r_0 m_{N \to \infty} = -1.34(4)(8)(25) \Rightarrow \frac{m_{N \to \infty}}{\sqrt{\sigma}} \approx 1.1$

errors: statistical, $R_{\text{min}}$, cont. extra. (HO corr, large-N: uncontrolled)

“worldsheet axion” (4d): $\frac{m_{N \to \infty}}{\sqrt{\sigma}} \approx 1.713(4)$

[Athenodorou, Teper, PLB771 (2017)]
Consistency with the excited states

Compare results for $\bar{b}_2$ to state $E_1$ in 3d $SU(2)$:

$$
\text{data misses points at large } R
$$
Conclusions

Summary:

▶ computed non-universal EST parameters in continuum and large-N limits
  ▶ KKN prediction for $\sigma$: deviation only by 2%
  ▶ $\tilde{b}_2$ does not vanish for $N \to \infty$
    (at least in analysis w/o massive modes)
▶ computed parameters are in good agreement with excited states
▶ data allows for presence of massive mode/rigidity contributions
  ▶ $\tilde{b}_2$ much less precise – cannot reliably extrapolate to large-$N$
    (appears to remain non-vanishing)
  ▶ $m$ decreases (becomes similar to $\sqrt{\sigma}$ or $\Lambda_{QCD}$)

Future prospects:

▶ include excited states in analysis (more information?)
  (would be good to know contribution from massive modes in EST)
▶ do the same for 4d theory (extremely difficult)
Thank you for your attention!