Spectrum of the open QCD flux tube and its effective string description

Bastian Brandt

ITP Goethe-Universität Frankfurt

02.06.2018

Contents

1. Introduction
   Confinement, flux tubes and strings
2. String tension and KKN prediction
3. EST analysis without massive modes
4. Testing the presence of massive modes
5. Conclusions

1. Introduction

Confinement, flux tubes and strings

Static $q\bar{q}$-potentials

static $q\bar{q}$-potential:

energy of static $q\bar{q}$ pair at distance R

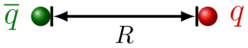
for states with excited gluons configurations:

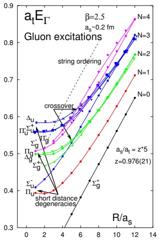
hybrid $q\bar{q}$-potentials

Physical relevance:

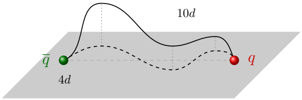
- linearly rising potential $\Leftrightarrow$ confinement
- input for model calculations (hybrid mesons, ...)
  $\Rightarrow$ analytic description is wanted
- can be used to make contact to AdS/CFT duals of pure gauge theory

For the latter: effective string theory



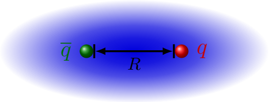


[Juge, Kuti, Morningstar, PRL90 (2003)]



Confinement and flux tubes

Heuristic confinement mechanism:



- $q\bar{q}$ pair connected by region of strong chromo-electromagnetic flux
- pulling the quarks appart:
  flux gets squeezed into a narrow region

  $\Rightarrow$ **flux tube**

  - squeezing due to dual Meissner effect
  - here all quarks are static
    (no string breaking)
- for such a tube:
  expect constant energy density

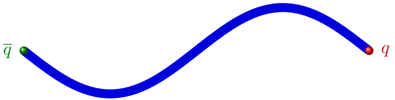
  $\Rightarrow$ linearly rising potential $V(R) = \sigma R$

  σ : string tension

Flux tubes and string theory

at large R: flux tube looks like a thin energy string

excitation spectrum will be dominated by stringy excitations!



$\Rightarrow$ formulation of effective string theories (EST) for the flux tube.

[Nambu, PLB 80, 372 (1979); Lüscher, Symanzik, Weisz, NPB 173, 365 (1980); Polyakov, NPB 164, 171 (1980)]

since then: formalism has been developed and action is known up to $O(R^{-5})$

[Lüscher, Weisz, . . . , Polchinski, Strominger, . . . , Casselle, . . . , Aharony, . . . , Dubovsky, Flauger, Gorbenko . . .]

for details and references see review [BB, Meineri, IJMP A31 (2016)]

Historically:

- idea also motivated by Regge trajectories [Regge, NC14 (1959)]
- origin of first string theories [Goddard *et al*, NPB65 (1963); Goto, PTP46 (1971)]

EST spectrum (open strings)

[Aharony, Klinghoffer, JHEP1012 (2010)]

$$E_{n,l}^{\mathrm{EST}}(R) = \sigma R \sqrt{1 + \frac{2\pi}{\sigma R^2}\left(n - \frac{1}{24}(d-2)\right)} - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3} R^4}\left(B_n^l + \frac{d-2}{60}\right) - \frac{\pi^3(d-26)}{48\sigma^2 R^5} C_n^l + \mathcal{O}(R^{-\xi})$$

LC spectrum (or NG) [J.F. Arvis, PLB127 (1983)] boundary term

$\bar{b}_2$: dimensionless non-universal boundary coefficient $\bar{b}_2 = \sqrt{\sigma^3} b_2$

B_n^l, C_n^l: dimensionless, depend on representation of $SO(d-2)$

$\|n,l\rangle$	$SO(d-2)$ representation		B_n^l	C_n^l
$\|0\rangle$	$1\|0\rangle$	scalar	0	0
$\|1\rangle$	$\alpha_{-1}^i\|0\rangle$	vector	4	$d-3$
$\|2,1\rangle$	$\alpha_{-1}^i\alpha_{-1}^i\|0\rangle$	scalar	8	0
$\|2,2\rangle$	$\alpha_{-2}^i\|0\rangle$	vector	32	$16(d-3)$
$\|2,3\rangle$	$\left(\alpha_{-1}^i\alpha_{-1}^j - \frac{\delta^{ij}}{d-2}\alpha_{-1}^i\alpha_{-1}^i\right)\|0\rangle$	sym. tracel. tensor	8	$4(d-2)$

EST spectrum (open strings)

[Aharony, Klinghoffer, JHEP1012 (2010)]

$$E_{n,l}^{\mathrm{EST}}(R) = \sigma R \sqrt{1 + \frac{2\pi}{\sigma R^2} \left(n - \frac{1}{24}(d-2)\right)}$$
$$-\bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3} R^4}\left(B_n^l + \frac{d-2}{60}\right) - \frac{\pi^3(d-26)}{48\sigma^2 R^5} C_n^l + \mathcal{O}(R^{-\xi})$$

LC spectrum (or NG) [J.F. Arvis, PLB127 (1983)] boundary term

$\bar{b}_2$: dimensionless non-universal boundary coefficient $\bar{b}_2 = \sqrt{\sigma^3} b_2$

B_n^l, C_n^l: dimensionless, depend on representation of $SO(d-2)$

$\|n,l\rangle$	$SO(d-2)$ representation		B_n^l	C_n^l
$\|0\rangle$	$1\|0\rangle$	scalar	0	0
$\|1\rangle$	$\alpha_{-1}^i\|0\rangle$	vector	4	$d-3$
$\|2,1\rangle$	$\alpha_{-1}^i\alpha_{-1}^i\|0\rangle$	scalar	8	0
$\|2,2\rangle$	$\alpha_{-2}^i\|0\rangle$	vector	32	$16(d-3)$
$\|2,3\rangle$	$\left(\alpha_{-1}^i\alpha_{-1}^j - \frac{\delta^{ij}}{d-2}\alpha_{-1}^i\alpha_{-1}^i\right)\|0\rangle$	sym. tracel. tensor	8	$4(d-2)$

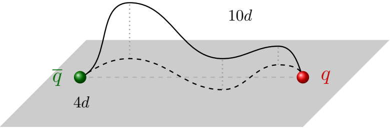
AdS/CFT correspondence and the holographic string

large-N QCD: supposed to have dual 10d AdS superstring description

question: which is the associated holographic string background?

EST string:
4d projection of 10d superstring

for particular backgrounds:
can derive EST action



several suitable backgrounds are known

[Witten, ATMP2 (1998); Klebanov, Strassler, JHEP0008 (2000); Maldacena, Nunez, PRL86 (2001)]

all have the same LO action, consistent with EST [Aharony, Karzbrun, JHEP0906 (2009)]

non-universal coefficients relate to properties of 10d string theory

e.g. $b_2 = -\frac{1}{64\sigma}\sum_\xi \frac{(-1)^{\mathrm{BC}(\xi)}}{m_\xi^b} + b_2^f + \ldots$ [Aharony, Field, JHEP1101 (2011)]

$\Rightarrow$ extraction of non-universal parameters can provide information on AdS side

Rigidity and massive modes

- so far ignored in EST: extrinsic curvature term

  formally higher order; can give contributions under quantisation

  [Billo *et al*, 1205 (2012); Ambjorn *et al*, PRD89 (2014); Caselle *et al*, JHEP1501 (2015)]

  correction term for potential:

$$V_{\mathrm{ext}}(R) = -\frac{m}{2\pi}\sum_{n=1}^{\infty}\frac{K_1(2nmR)}{n} - \frac{(d-2)(d-10)\pi^2}{3840 m\sigma R^4}$$

  K_1: Modified Bessel function of first kind
  m: free parameter with dimension of mass

  $\Rightarrow$ mixes with the boundary term (can change value of $\bar{b}_2$)

Rigidity and massive modes

- so far ignored in EST: extrinsic curvature term

  formally higher order; can give contributions under quantisation

  [Billo *et al*, 1205 (2012); Ambjorn *et al*, PRD89 (2014); Caselle *et al*, JHEP1501 (2015)]

  correction term for potential:

$$V_{\rm ext}(R) = -\frac{m}{2\pi}\sum_{n=1}^{\infty}\frac{K_1(2nmR)}{n} - \frac{(d-2)(d-10)\pi^2}{3840 m\sigma R^4}$$

  K_1: Modified Bessel function of first kind
  m: free parameter with dimension of mass

  $\Rightarrow$ mixes with the boundary term (can change value of $\bar{b}_2$)

Rigidity and massive modes

- so far ignored in EST: extrinsic curvature term

  formally higher order; can give contributions under quantisation

  [Billo *et al*, 1205 (2012); Ambjorn *et al*, PRD89 (2014); Caselle *et al*, JHEP1501 (2015)]

  correction term for potential:

$$V_{\mathrm{ext}}(R) = -\frac{m}{2\pi}\sum_{n=1}^{\infty}\frac{K_1(2nmR)}{n} - \frac{(d-2)(d-10)\pi^2}{3840 m\sigma R^4}$$

  K_1: Modified Bessel function of first kind
  m: free parameter with dimension of mass

  $\Rightarrow$ mixes with the boundary term (can change value of $\bar{b}_2$)

- other possible contribution: massive modes

  found to be important to describe 4d spectrum

  [Dubovsky, Flauger, Gorbenko, PRL111 (2013); JETP120 (2015)]

  however: 4d coupling term not allowed in 3d

  can only couple indirectly via the induced metric

  $\Rightarrow$ formally similar contribution to rigidity term!

Current status of lattice simulations

A large number of lattice studies in the past 35 years:

- static potential and excited states:

  good agreement with LC spectrum

  3d: small deviations can be fitted to $\bar{b}_2$ correction

  [BB, JHEP1102 (2011); Billo *et al*, 1205 (2012)]

  4d: behaviour of excited states points to presence of massive modes

  [Juge, Kuti, Morningstar, PRL90 (2003)]

  3d Z_2 and $U(1)$: presence/importance of rigidity term has been observed

  [Billo *et al*, 1205 (2012); Caselle *et al*, JHEP1501 (2015)]

- closed flux tubes:

  3d: good agreement with EST

  4d: massive modes found to be important to describe 4d spectrum

  [Dubovsky, Flauger, Gorbenko, PRL111 (2013); JETP120 (2015)]

- can also study: flux tube width, finite temperature, ...

for a review and more references see [BB, Meineri, IJMP A31 (2016)]

Goals and setup of this study

first goal: extract EST parameters at finite N and in the $N \to \infty$ limit

in particular: use pure gauge lattice simulation in 3d

- extract $\sqrt{\sigma} r_0$ and $\bar{b}_2$ in 3d $SU(N = 2, 3, 4, 5, 6)$ from $V(R)$
  - multiple lattice spacings $a \approx 0.11$, 0.08, 0.06 fm with $V \gtrsim 5$ fm
  - error reduction: LW algorithm
    (2000 total meas; 20 000 sub. updates; $t_s = 2, 4, 6$)
- extrapolate to continuum $a \to 0$ and subsequently $N \to \infty$.
- check the consistency of results with the excited states

  here: use old $SU(2)$ data from [BB, JHEP1102 (2011)]

second goal: test consistency with massive modes/string rigidity

- extract mass m and investigate impact on $\bar{b}_2$
- once more: compare continuum results for different values of N

  $\Rightarrow$ extrapolate $N \to \infty$?

2. String tension and KKN prediction

Extraction of the string tension

First step: Extract string tension σ (defined by $R \to \infty$ behaviour)

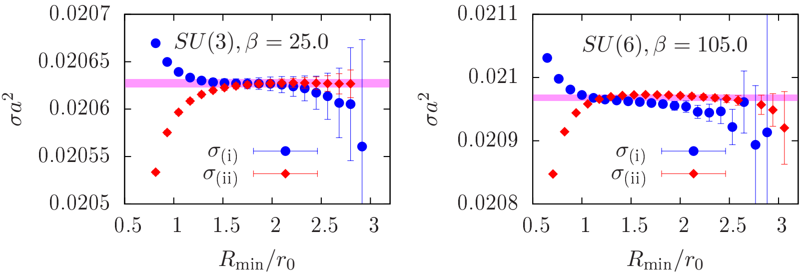
reliable computation: demands extraction of $R \to \infty$ behaviour

strategy: perform two different fits including different $1/R$ corrections

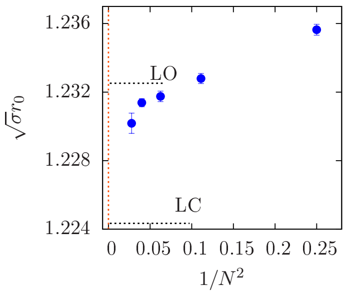
(i) fit to LO force (ii) fit to LC potential

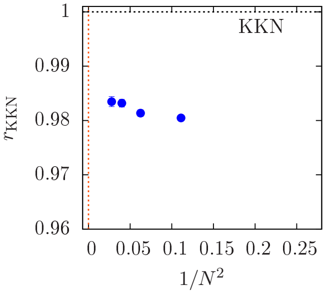
compare $R_{\min}$-dependence of σ from these methods

$\Rightarrow$ extraction of σ is reliable where results agree!



Large-N extrapolations and KKN prediction

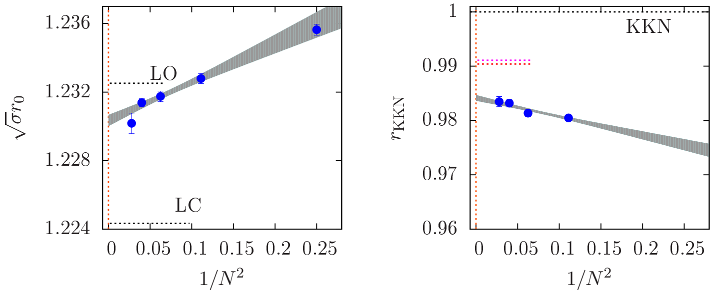




Karabili-Kim-Nair prediction: $\frac{\sqrt{\sigma}}{g^2_{\mathrm{MF}}} = \sqrt{\frac{N^2-1}{8\pi}}$ [Karabili, Kim, Nair, PLB434 (1998)]

$$r_{\mathrm{KKN}} = \frac{\left(\sqrt{\sigma} r_0 / g^2 r_0\right)_{\mathrm{lat}}}{\left(\sqrt{\sigma} r_0 / g^2 r_0\right)_{\mathrm{KKN}}}$$

Large-N extrapolations and KKN prediction



[Teper, Lucini, PRD66 (2002)]
[Teper, Bringoltz, PoS LAT2006 (2006)]]

Karabili-Kim-Nair prediction: $\frac{\sqrt{\sigma}}{g^2_{\mathrm{MF}}} = \sqrt{\frac{N^2-1}{8\pi}}$ [Karabili, Kim, Nair, PLB434 (1998)]

final 3d large-N results:

$$\sqrt{\sigma} r_0 = 1.2304(4)(3) \qquad r_{\mathrm{KKN}} = \frac{\left(\sqrt{\sigma} r_0 / g^2 r_0\right)_{\mathrm{lat}}}{\left(\sqrt{\sigma} r_0 / g^2 r_0\right)_{\mathrm{KKN}}} = 0.9842(6)(14)$$

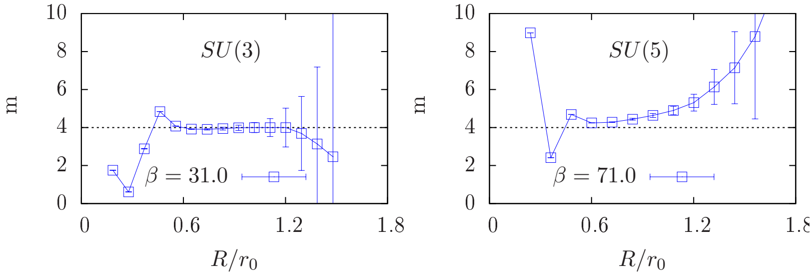
3. EST analysis without massive modes

Order of the leading order correction

first: check consistency of correction to LC potential with R^{-4}

fit $V(R)$ to form: $$V(R) = E_0^{\mathrm{LC}}(R) + \frac{\eta}{(\sqrt{\sigma}R)^m}$$

look at $R_{\min}$ dependence of m:



Extraction strategy

next step: extract the boundary coefficient!

fit data to:

$$V(R) = E_0^{\mathrm{LC}}(R) - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3} R^4} \left(C_n^i + \frac{d-2}{60} \right) + \frac{\gamma_0^{(1)}}{\sqrt{\sigma^5} R^6} + \frac{\gamma_0^{(2)}}{\sigma^3 R^7} + V_0$$

To quantify systematic errors: perform different fits

Extraction strategy

next step: extract the boundary coefficient!

fit data to:

$$V(R) = E_0^{\mathrm{LC}}(R) - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3} R^4} \left(C_n^i + \frac{d-2}{60} \right) + \frac{\gamma_0^{(1)}}{\sqrt{\sigma^5} R^6} + \frac{\gamma_0^{(2)}}{\sigma^3 R^7} + V_0$$

To quantify systematic errors: perform different fits

A: use σ, V_0 from above – use $\bar{b}_2$, $\gamma_0^{(1)}$, $\gamma_0^{(2)}$ as free params

Extraction strategy

next step: extract the boundary coefficient!

fit data to:

$$V(R) = E_0^{\mathrm{LC}}(R) - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3} R^4} \left(C_n^i + \frac{d-2}{60} \right) \qquad + V_0$$

To quantify systematic errors: perform different fits

A: use σ, V_0 from above – use $\bar{b}_2$, $\gamma_0^{(1)}$, $\gamma_0^{(2)}$ as free params

B: use σ, V_0, $\bar{b}_2$ as free params. – set $\gamma_0^{(1)} = \gamma_0^{(2)} = 0$

Extraction strategy

next step: extract the boundary coefficient!

fit data to:

$$V(R) = E_0^{\mathrm{LC}}(R) - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3} R^4}\left(C_n^i + \frac{d-2}{60}\right) + \frac{\gamma_0^{(1)}}{\sqrt{\sigma^5} R^6} \qquad + V_0$$

To quantify systematic errors: perform different fits

A: use σ, V_0 from above – use $\bar{b}_2$, $\gamma_0^{(1)}$, $\gamma_0^{(2)}$ as free params

B: use σ, V_0, $\bar{b}_2$ as free params. – set $\gamma_0^{(1)} = \gamma_0^{(2)} = 0$

C: Use σ, V_0, $\bar{b}_2$, $\gamma_0^{(1)}$ as free params. – set $\gamma_0^{(2)} = 0$

Extraction strategy

next step: extract the boundary coefficient!

fit data to:

$$V(R) = E_0^{\mathrm{LC}}(R) - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3} R^4}\left(C_n^i + \frac{d-2}{60}\right) + \frac{\gamma_0^{(2)}}{\sigma^3 R^7} + V_0$$

To quantify systematic errors: perform different fits

A: use σ, V_0 from above – use $\bar{b}_2$, $\gamma_0^{(1)}$, $\gamma_0^{(2)}$ as free params

B: use σ, V_0, $\bar{b}_2$ as free params. – set $\gamma_0^{(1)} = \gamma_0^{(2)} = 0$

C: Use σ, V_0, $\bar{b}_2$, $\gamma_0^{(1)}$ as free params. – set $\gamma_0^{(2)} = 0$

D: Use σ, V_0, $\bar{b}_2$, $\gamma_0^{(2)}$ as free params. – set $\gamma_0^{(1)} = 0$

Extraction strategy

next step: extract the boundary coefficient!

fit data to:

$$V(R) = E_0^{\mathrm{LC}}(R) + \frac{\gamma_0^{(1)}}{\sqrt{\sigma^5} R^6} + \frac{\gamma_0^{(2)}}{\sigma^3 R^7} + V_0$$

To quantify systematic errors: perform different fits

A: use σ, V_0 from above – use $\bar{b}_2$, $\gamma_0^{(1)}$, $\gamma_0^{(2)}$ as free params

B: use σ, V_0, $\bar{b}_2$ as free params. – set $\gamma_0^{(1)} = \gamma_0^{(2)} = 0$

C: Use σ, V_0, $\bar{b}_2$, $\gamma_0^{(1)}$ as free params. – set $\gamma_0^{(2)} = 0$

D: Use σ, V_0, $\bar{b}_2$, $\gamma_0^{(2)}$ as free params. – set $\gamma_0^{(1)} = 0$

E: Use σ, V_0, $\gamma_0^{(1)}$, $\gamma_0^{(2)}$ as free params. – set $\bar{b}_2 = 0$

Extraction strategy

next step: extract the boundary coefficient!

fit data to:

$$V(R) = E_0^{\mathrm{LC}}(R) - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3} R^4}\left(C_n^i + \frac{d-2}{60}\right) + \frac{\gamma_0^{(1)}}{\sqrt{\sigma^5} R^6} + \frac{\gamma_0^{(2)}}{\sigma^3 R^7} + V_0$$

To quantify systematic errors: perform different fits

A: use σ, V_0 from above – use $\bar{b}_2$, $\gamma_0^{(1)}$, $\gamma_0^{(2)}$ as free params

B: use σ, V_0, $\bar{b}_2$ as free params. – set $\gamma_0^{(1)} = \gamma_0^{(2)} = 0$

C: Use σ, V_0, $\bar{b}_2$, $\gamma_0^{(1)}$ as free params. – set $\gamma_0^{(2)} = 0$

D: Use σ, V_0, $\bar{b}_2$, $\gamma_0^{(2)}$ as free params. – set $\gamma_0^{(1)} = 0$

E: Use σ, V_0, $\gamma_0^{(1)}$, $\gamma_0^{(2)}$ as free params. – set $\bar{b}_2 = 0$

fits **A** and **E** are checks whether $\bar{b}_2 \neq 0$

fits **B–D** are used in the final analysis

Extraction, limites and estimation of systematic errors

- higher order terms:

  final result: average over fits **B–D**
  estimate for uncertainty: largest deviation from final result

Extraction, limites and estimation of systematic errors

- higher order terms:

  final result: average over fits **B–D**
  estimate for uncertainty: largest deviation from final result

- fitrange for $\bar{b}_2$:

  $R_{\rm min}$: defined by the second fit for which χ^2/dof is acceptable.
  estimate for uncertainty: deviation from fits with $R_{\rm min} \pm 1$

Extraction, limites and estimation of systematic errors

- higher order terms:

  final result: average over fits **B–D**
  estimate for uncertainty: largest deviation from final result

- fitrange for $\bar{b}_2$:

  $R_{\min}$: defined by the second fit for which χ^2/dof is acceptable.
  estimate for uncertainty: deviation from fits with $R_{\min} \pm 1$

- continuum extrapolation:

  final result: use a linear continuum extrapolation (in a^2)
  estimate for uncertainty: deviation from fit with only last points

Extraction, limites and estimation of systematic errors

- higher order terms:

  final result: average over fits **B–D**
  estimate for uncertainty: largest deviation from final result

- fitrange for $\bar{b}_2$:

  $R_{\rm min}$: defined by the second fit for which χ^2/dof is acceptable.
  estimate for uncertainty: deviation from fits with $R_{\rm min} \pm 1$

- continuum extrapolation:

  final result: use a linear continuum extrapolation (in a^2)
  estimate for uncertainty: deviation from fit with only last points

- large-N extrapolation:

  final result: obtained from linear large-N extrapolation (in $1/N^2$)
  estimate for uncertainty: deviation from fit with only last points

Extraction, limites and estimation of systematic errors

- higher order terms:

  final result: average over fits **B–D**
  estimate for uncertainty: largest deviation from final result

- fitrange for $\bar{b}_2$:

  $R_{\min}$: defined by the second fit for which χ^2/dof is acceptable.
  estimate for uncertainty: deviation from fits with $R_{\min} \pm 1$

- continuum extrapolation:

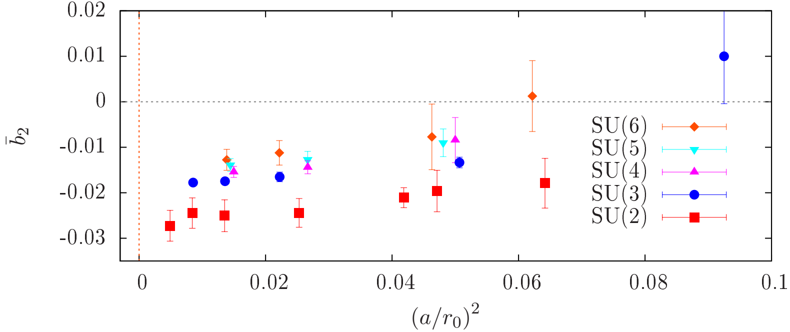
  final result: use a linear continuum extrapolation (in a^2)
  estimate for uncertainty: deviation from fit with only last points

- large-N extrapolation:

  final result: obtained from linear large-N extrapolation (in $1/N^2$)
  estimate for uncertainty: deviation from fit with only last points

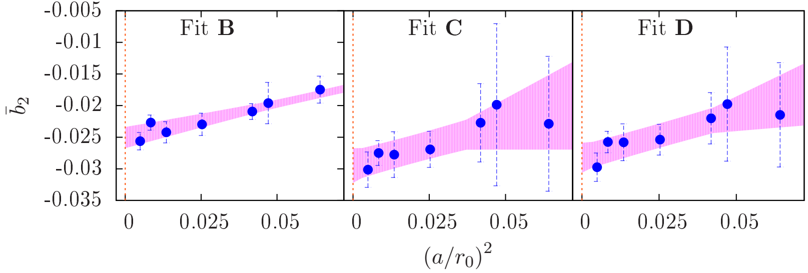
course of analysis: perform all possible combinations of fits

Results for $\bar{b}_2$



Continuum extrapolation of $\bar{b}_2$

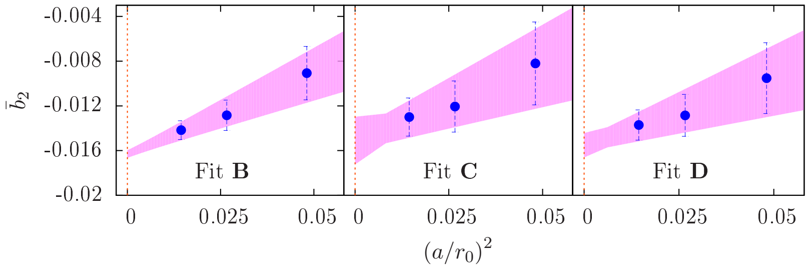
Extrapolation for $SU(2)$



linear continuum extrapolation works well for all N

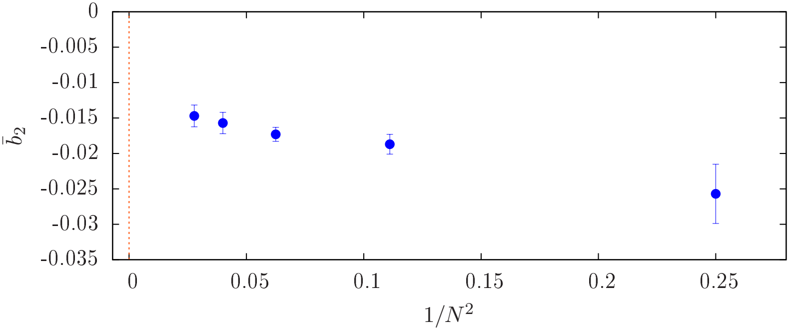
Continuum extrapolation of $\bar{b}_2$

Extrapolation for $SU(5)$

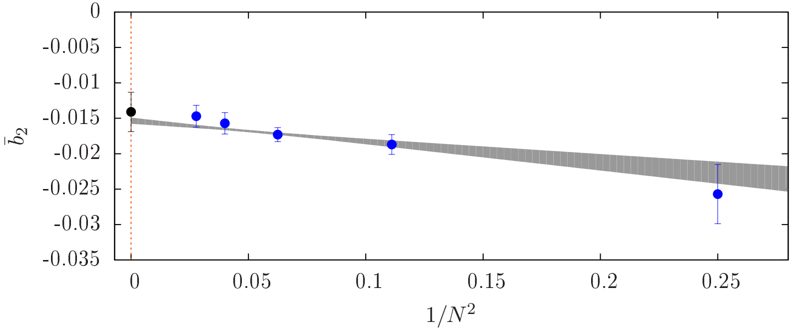


linear continuum extrapolation works well for all N

Final continuum results for $\bar{b}_2$



Large-N extrapolation $\bar{b}_2$

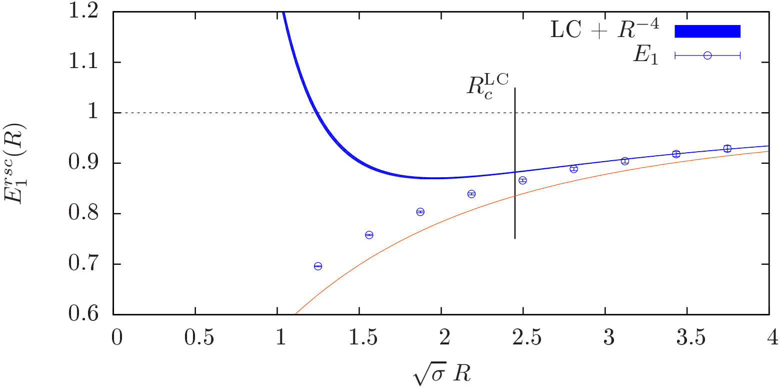


final large-N result: $\bar{b}_2^{N\to\infty} = -0.0141(3)(15)(13)(9)(17)$

errors: statistical, HO corr., $R_{\min}$, cont. extra., large-N extra

Consistency with the excited states

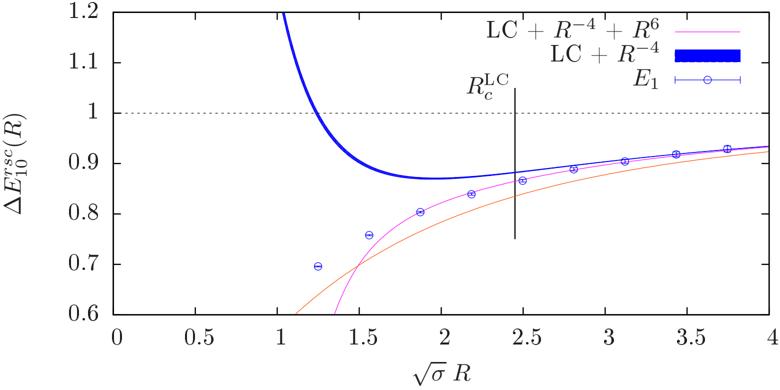
compare results for $\bar{b}_2$ to E_1 in 3d $SU(2)$: ($\beta = 5.0$ data [BB, JHEP1102 (2011)])



Energy levels fully determined by $\bar{b}_2$ up to $O(1/R^{6,7})$.

Consistency with the excited states

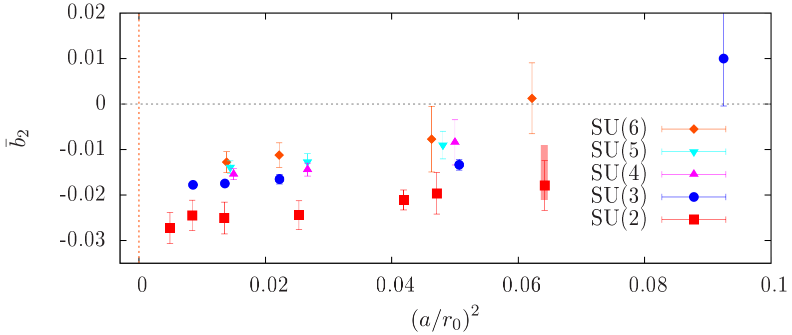
compare results for $\bar{b}_2$ to E_1 in 3d $SU(2)$: ($\beta = 5.0$ data [BB, JHEP1102 (2011)])



Fit the higher order terms: Good description of the data!

Consistency with the excited states

Alternatively: extract $\bar{b}_2$ from fit to excited states [BB, JHEP1102 (2011)]



$\Rightarrow$ excellent agreement with extraction from potential

4. Testing the presence of massive modes

Extraction strategy

up to now: neglected the possible presence of massive modes

to test whether they can be present fit data to:

$$V(R) = E_0^{\mathrm{LC}}(R) - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3} R^4}\left(C_n^i + \frac{d-2}{60}\right) - \frac{(d-2)(d-10)\pi^2}{3840 m \sigma R^4} - \frac{m}{2\pi}\sum_{k=1}^{\infty}\frac{K_1(2kmR)}{k} + \frac{\gamma_0^{(1)}}{\sqrt{\sigma^5} R^6} + \frac{\gamma_0^{(2)}}{\sigma^3 R^7} + V_0$$

perform different fits

Extraction strategy

up to now: neglected the possible presence of massive modes

to test whether they can be present fit data to:

$$V(R) = E_0^{\mathrm{LC}}(R) - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3} R^4}\left(C_n^i + \frac{d-2}{60}\right) - \frac{(d-2)(d-10)\pi^2}{3840 m\sigma R^4} - \frac{m}{2\pi}\sum_{k=1}^{\infty}\frac{K_1(2kmR)}{k} + V_0$$

perform different fits

F use σ, V_0, $\bar{b}_2$ and m as free parameters – set $\gamma_0^{(1)} = \gamma_0^{(2)} = 0$

Extraction strategy

up to now: neglected the possible presence of massive modes

to test whether they can be present fit data to:

$$V(R) = E_0^{\mathrm{LC}}(R) - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3} R^4}\left(C_n^i + \frac{d-2}{60}\right) - \frac{(d-2)(d-10)\pi^2}{3840 m \sigma R^4}$$
$$- \frac{m}{2\pi}\sum_{k=1}^{\infty}\frac{K_1(2kmR)}{k} + \frac{\gamma_0^{(1)}}{\sqrt{\sigma^5} R^6} + V_0$$

perform different fits

F use σ, V_0, $\bar{b}_2$ and m as free parameters – set $\gamma_0^{(1)} = \gamma_0^{(2)} = 0$

G use σ, V_0, $\bar{b}_2$, m and $\gamma_0^{(1)}$ as free parameters – set $\gamma_0^{(2)} = 0$

Extraction strategy

up to now: neglected the possible presence of massive modes

to test whether they can be present fit data to:

$$V(R) = E_0^{\mathrm{LC}}(R) - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3} R^4}\left(C_n^i + \frac{d-2}{60}\right) - \frac{(d-2)(d-10)\pi^2}{3840 m \sigma R^4}$$

$$- \frac{m}{2\pi}\sum_{k=1}^{\infty}\frac{K_1(2kmR)}{k} \qquad + \frac{\gamma_0^{(2)}}{\sigma^3 R^7} + V_0$$

perform different fits

- **F** use σ, V_0, $\bar{b}_2$ and m as free parameters – set $\gamma_0^{(1)} = \gamma_0^{(2)} = 0$
- **G** use σ, V_0, $\bar{b}_2$, m and $\gamma_0^{(1)}$ as free parameters – set $\gamma_0^{(2)} = 0$
- **H** use σ, V_0, $\bar{b}_2$, m and $\gamma_0^{(2)}$ as free parameters – set $\gamma_0^{(1)} = 0$

Extraction strategy

up to now: neglected the possible presence of massive modes

to test whether they can be present fit data to:

$$V(R) = E_0^{\mathrm{LC}}(R) - \frac{(d-2)(d-10)\pi^2}{3840 m\sigma R^4} - \frac{m}{2\pi}\sum_{k=1}^{\infty}\frac{K_1(2kmR)}{k} + V_0$$

perform different fits

- **F** use σ, V_0, $\bar{b}_2$ and m as free parameters – set $\gamma_0^{(1)} = \gamma_0^{(2)} = 0$
- **G** use σ, V_0, $\bar{b}_2$, m and $\gamma_0^{(1)}$ as free parameters – set $\gamma_0^{(2)} = 0$
- **H** use σ, V_0, $\bar{b}_2$, m and $\gamma_0^{(2)}$ as free parameters – set $\gamma_0^{(1)} = 0$
- **J** use σ, V_0 and m as free parameters – set $\gamma_0^{(1)} = \gamma_0^{(2)} = \bar{b}_2 = 0$

Extraction strategy

up to now: neglected the possible presence of massive modes

to test whether they can be present fit data to:

$$V(R) = E_0^{\mathrm{LC}}(R) - \bar{b}_2 \frac{\pi^3}{\sqrt{\sigma^3} R^4}\left(C_n^i + \frac{d-2}{60}\right) - \frac{(d-2)(d-10)\pi^2}{3840 m \sigma R^4}$$
$$- \frac{m}{2\pi}\sum_{k=1}^{\infty}\frac{K_1(2kmR)}{k} + \frac{\gamma_0^{(1)}}{\sqrt{\sigma^5} R^6} + \frac{\gamma_0^{(2)}}{\sigma^3 R^7} + V_0$$

perform different fits

F use σ, V_0, $\bar{b}_2$ and m as free parameters – set $\gamma_0^{(1)} = \gamma_0^{(2)} = 0$

G use σ, V_0, $\bar{b}_2$, m and $\gamma_0^{(1)}$ as free parameters – set $\gamma_0^{(2)} = 0$

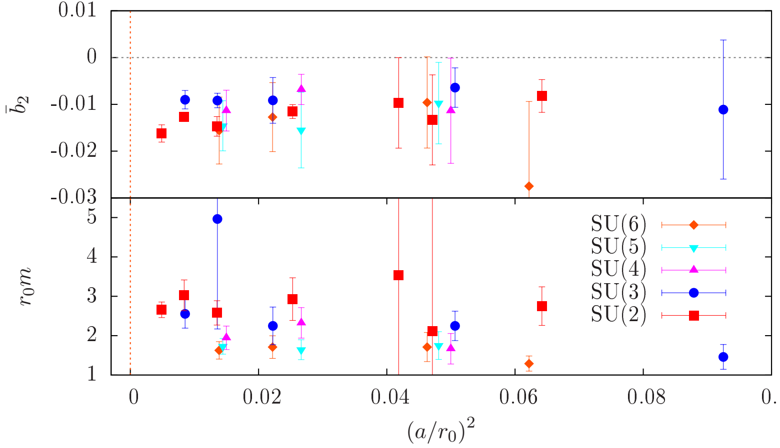
H use σ, V_0, $\bar{b}_2$, m and $\gamma_0^{(2)}$ as free parameters – set $\gamma_0^{(1)} = 0$

J use σ, V_0 and m as free parameters – set $\gamma_0^{(1)} = \gamma_0^{(2)} = \bar{b}_2 = 0$

fit **J**: check whether $\bar{b}_2 \neq 0$

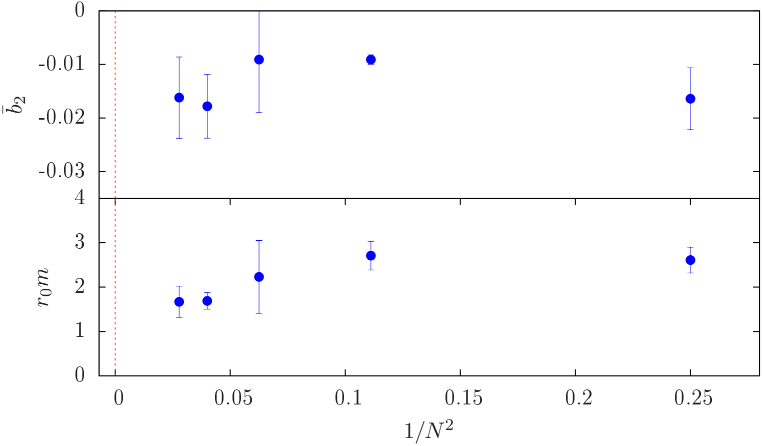
fit **F** used in the final analysis (results of **G** and **H** not accurate enough)

Results for $\bar{b}_2$ and m



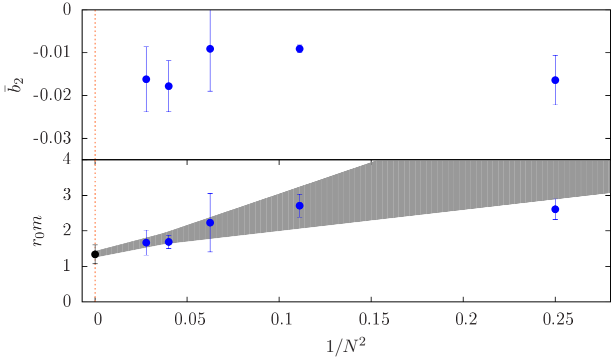
main cause for large uncertainties: R_{min}-dependence of fit

Final continuum results for $\bar{b}_2$ and m



much larger uncertainties $\Rightarrow$ extrapolation for $\bar{b}_2$ unstable

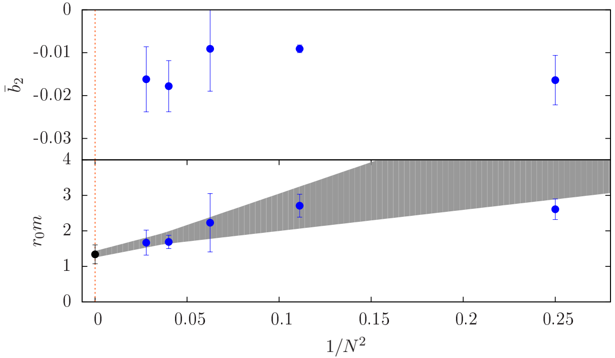
Large-N extrapolation m



final large-N result: $r_0 m^{N\to\infty} = -1.34(4)(8)(25)$

errors: statistical, R_{min}, cont. extra. (HO corr, large-N: uncontrolled)

Large-N extrapolation m



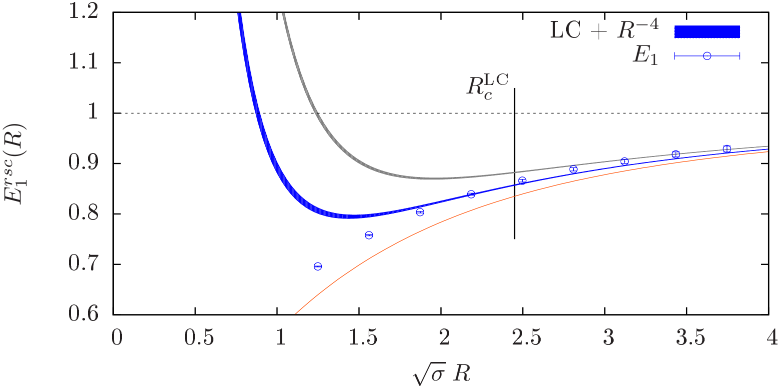
final large-N result: $r_0 m^{N\to\infty} = -1.34(4)(8)(25)$ $\Rightarrow$ $\dfrac{m^{N\to\infty}}{\sqrt{\sigma}} \approx 1.1$

errors: statistical, R_{min}, cont. extra. (HO corr, large-N: uncontrolled)

"worldsheet axion" (4d): $\dfrac{m^{N\to\infty}}{\sqrt{\sigma}} \approx 1.713(4)$ [Athenodorou, Teper, PLB771 (2017)]

Consistency with the excited states

Compare results for $\bar{b}_2$ to state E_1 in 3d $SU(2)$:



$\Rightarrow$ data misses points at large R

Conclusions

Summary:

- computed non-universal EST parameters in continuum and large-N limits
  - KKN prediction for σ: deviation only by 2%
  - $\bar{b}_2$ does not vanish for $N \to \infty$
    (at least in analysis w/o massive modes)
- computed parameters are in good agreement with excited states
- data allows for presence of massive mode/rigidity contributions
  - $\bar{b}_2$ much less precise – cannot reliably extrapolate to large-N
    (appears to remain non-vanishing)
  - m decreases (becomes similar to $\sqrt{\sigma}$ or $\Lambda_Q CD$)

Future prospects:

- include excited states in analysis (more information?)
  (would be good to know contribution from massive modes in EST)
- do the same for 4d theory (extremely difficult)

Thank you for your attention!