Ghost Sector in
Minimal Linear Covariant Gauge

Attilio Cucchieri

Instituto de Física de São Carlos – USP (Brazil)

in collaboration with David Dudal, Tereza Mendes,
Orlando Oliveira, Martin Roelfs and Paulo Silva
Abstract

We discuss possible definitions of the Faddeev-Popov matrix for the minimal linear covariant gauge on the lattice and present preliminary results for the ghost propagator.
Abstract

We discuss possible definitions of the Faddeev-Popov matrix for the minimal linear covariant gauge on the lattice and present preliminary results for the ghost propagator.

Why study the linear covariant gauge?
Abstract

We discuss possible definitions of the Faddeev-Popov matrix for the minimal linear covariant gauge on the lattice and present preliminary results for the ghost propagator.

Why study the linear covariant gauge?

- Study Green’s functions in the IR limit of Yang-Mills theories in order to understand low-energy properties of the theory.
- Since they depend on the gauge, consider different gauges (Landau gauge, Coulomb gauge, \(\lambda\)-gauge, MAG, etc.).
- Extend the Gribov-Zwanziger approach to the linear covariant gauge.
- Linear covariant gauge, very popular in continuum studies, proved quite hostile to the lattice approach.
Some Analytic Results

What do we expect for linear covariant gauge?
Some Analytic Results

What do we expect for linear covariant gauge?

- Transverse component of the gluon propagator is similar to the Landau case, with $D(0)$ decreasing when $\xi$ increases (F. Siringo, PRD90 2014, variational method; M.A.L. Capri et al., EPJC75 2015, Gribov-Zwanziger setup; A.C. Aguilar et al., PRD91 2015, SDE of the ghost prop. + Nielsen identitiies).
Some Analytic Results

What do we expect for linear covariant gauge?

- **Transverse component** of the gluon propagator is similar to the Landau case, with $D(0)$ decreasing when $\xi$ increases (F. Siringo, PRD90 2014, variational method; M.A.L. Capri et al., EPJC75 2015, Gribov-Zwanziger setup; A.C. Aguilar et al., PRD91 2015, SDE of the ghost prop. + Nielsen identitiies).

- **Ghost dressing function** is flat in the IR limit (F. Siringo, PRD90 2014, variational method).

- **Ghost dressing function** in the IR limit is decreasing as $\xi$ increases (M. Huber, PRD91 2015, coupled system of DSEs).

We want to impose the gauge condition $\partial_\mu A_\mu^b(x) = \Lambda^b(x)$, for real-valued functions $\Lambda^b(x)$, generated using a Gaussian distribution with width $\sqrt{\xi}$. 
We want to impose the gauge condition $\partial_\mu A_\mu^b(x) = \Lambda^b(x)$, for real-valued functions $\Lambda^b(x)$, generated using a Gaussian distribution with width $\sqrt{\xi}$.

Landau gauge $[\Lambda^b(x) = 0]$ is obtained on the lattice by minimizing the functional

$$\mathcal{E}_{LG}[U^g] = -\Re \text{Tr} \sum_{\mu,x} g(x) U_\mu(x) g^\dagger(x + e_\mu).$$

The set of local minima defines the first Gribov region $\Omega$. 
Linear Covariant Gauge on the Lattice

We want to impose the gauge condition \( \partial_\mu A^b_\mu(x) = \Lambda^b(x) \), for real-valued functions \( \Lambda^b(x) \), generated using a Gaussian distribution with width \( \sqrt{\xi} \).

Landau gauge \([\Lambda^b(x) = 0]\) is obtained on the lattice by minimizing the functional

\[
\mathcal{E}_{LG}[U^g] = - \Re \text{Tr} \sum_{\mu,x} g(x) U_\mu(x) g^\dagger(x + e_\mu).
\]

The set of local minima defines the first Gribov region \( \Omega \).

From the second variation of \( \mathcal{E}_{LG}[U^g] \) we define the symmetric, semi-positive definite Faddeev-Popov operator

\[
\mathcal{M}^{bc}(x, y) \equiv \sum_{\mu=1}^d \left\{ \Gamma^{bc}_\mu(x) \left[ \delta_{x, y} - \delta_{x+e_\mu, y} \right] + \Gamma^{bc}_\mu(x - e_\mu) \left[ \delta_{x, y} - \delta_{x-e_\mu, y} \right] \\
- \sum_{e=1}^{N_c^2-1} f^{bec} \left[ A^e_\mu(x - e_\mu/2) \delta_{x-e_\mu, y} - A^e_\mu(x + e_\mu/2) \delta_{x+e_\mu, y} \right] \right\}.
\]
The New Minimizing Functional

The lattice linear covariant gauge condition can be obtained by minimizing the functional (A.C., T. Mendes and E.M.S. Santos, PRL103 2009)

\[
\mathcal{E}_{LCG}[U^g, g, \Lambda] = \mathcal{E}_{LG}[U^g] + \Re \text{Tr} \sum_x i g(x) \Lambda(x).
\]
The New Minimizing Functional

The lattice linear covariant gauge condition can be obtained by minimizing the functional (A.C., T. Mendes and E.M.S. Santos, PRL103 2009)

\[ \mathcal{E}_{LCG}[U^g, g, \Lambda] = \mathcal{E}_{LG}[U^g] + \Re \text{Tr} \sum_x i g(x) \Lambda(x). \]

One can interpret the Landau-gauge functional \( \mathcal{E}_{LG}[U^g] \) as a spin-glass Hamiltonian for the spin variables \( g(x) \) with a random interaction given by \( U_{\mu}(x) \). Then, the extra term corresponds to a random external magnetic field \( \Lambda(x) \). Note: the functional \( \mathcal{E}_{LCG}[U^g, g, \Lambda] \) is linear in the gauge transformation \( \{g(x)\} \).
The New Minimizing Functional

The lattice linear covariant gauge condition can be obtained by minimizing the functional (A.C., T. Mendes and E.M.S. Santos, PRL103 2009)

\[ \mathcal{E}_{LCG}[U^g, g, \Lambda] = \mathcal{E}_{LG}[U^g] + \Re \text{Tr} \sum_x i g(x) \Lambda(x). \]

One can interpret the Landau-gauge functional \( \mathcal{E}_{LG}[U^g] \) as a spin-glass Hamiltonian for the spin variables \( g(x) \) with a random interaction given by \( U_\mu(x) \). Then, the extra term corresponds to a random external magnetic field \( \Lambda(x) \). Note: the functional \( \mathcal{E}_{LCG}[U^g, g, \Lambda] \) is linear in the gauge transformation \( \{g(x)\} \).

By considering a one-parameter subgroup, it is easy to check that the stationarity condition implies the lattice linear covariant gauge condition

\[ \nabla \cdot A^b(x) = \sum_\mu A^b_\mu(x + e_\mu/2) - A^b_\mu(x - e_\mu/2) = \Lambda^b(x). \]
Numerical Gauge Fixing

Conceptual problem: using the standard compact discretization, the gluon field is bounded while the four-divergence of the gluon field satisfies a Gaussian distribution, i.e. it is unbounded. This can give rise to convergence problems when a numerical implementation of the linear covariant gauge is attempted (A.C. et al., PRL103 2009, PoS QCD-TNT09, PoS FACESQCD 2010, AIP Conf.Proc.1354 2011; P. Bicudo et al., PRD92 2015, PoS LATTICE2015).
Numerical Gauge Fixing

Conceptual problem: using the standard compact discretization, the gluon field is bounded while the four-divergence of the gluon field satisfies a Gaussian distribution, i.e. it is unbounded. This can give rise to convergence problems when a numerical implementation of the linear covariant gauge is attempted (A.C. et al., PRL103 2009, PoS QCD-TNT09, PoS FACESQCD 2010, AIP Conf.Proc.1354 2011; P. Bicudo et al., PRD92 2015, PoS LATITCE2015).

Moreover, the dimensionless gauge-fixing condition is given by $a^2 g_0 \partial_\mu A^b_\mu (x) = a^2 g_0 \Lambda^b (x)$, in a generic $d$-dimensional space. Since $\beta = 2N_c/(a^4 - d g_0^2)$ in the SU($N_c$) case, we have that

$$\frac{\beta/(2N_c)}{2\xi} \sum_{x,b} \left[a^2 g_0 \Lambda^b (x)\right]^2 \to \frac{1}{2\xi} \int d^d x \sum_b \left[\Lambda^b (x)\right]^2$$

in formal continuum limit $a \to 0$. 
We have checked that

$$p^2 D_l(p^2) = \xi$$

as predicted by perturbation theory. In the SU(2) case, for $V = 16^4$, $\beta = 4$ and $\xi = 0.5$ a fit $a/p^b$ for $D_l(p^2)$ gives $a = 0.502(5)$ and $b = 2.01(1)$ with a $\chi^2/dof = 1.1$.

Transverse gluon propagator $D_t(p^2)$ [using the stereographic projection in the SU(2) case] as a function of the momentum $p$ (both in physical units) for the lattice volume $V = 16^4$, $\beta = 2.3$ and $\xi = 0 (+), 0.05 (\times), 0.1 (*)$ (A.C. et al., PRL103 2009, PoS QCD-TNT09).

$D_t(0)$ decreases as $\xi$ increases (in agreement with L. Giusti et al. NP Proc. Supp. 94 2001 and P. Bicudo et al., PRD92 2015, PoS LATTICE2015).
Transverse gluon propagator $D_t(p^2)$ [using the stereographic projection in the SU(2) case] as a function of the momentum $p$ (both in physical units) for $\xi = 0.05$, $\beta = 2.3$, and the lattice volumes $V = 8^4(\times)$, $16^4(\times)$, $24^4(\times)$ (A.C. et al., PRL103 2009, PoS QCD-TNT09).

$D_t(0)$ decreases as $V$ increases (as in Landau gauge).
Can we extend to the linear covariant gauge the lattice Landau-gauge approach? Can we define the first Gribov region $\Omega$?
Can we extend to the linear covariant gauge the lattice Landau-gauge approach? Can we define the first Gribov region $\Omega$?

In the continuum we have three possible setups:

1) complex ghost fields $\bar{c} = c^\dagger$: the FP matrix $-\partial \cdot D^{ab}$ and the Lagrangian density are not Hermitian;

2) complex ghost fields $\bar{c} = c^\dagger$: a symmetric FP matrix $-(\partial \cdot D^{ab} + D^{ab} \cdot \partial)/2$, plus a quartic ghost self-interaction term;

3) real independent ghost/anti-ghost fields $u, iv$: the “effective” FP matrix

$$\frac{i}{2} \begin{pmatrix} 0 & -\partial \cdot D^{bc} \\ D^{bc} \cdot \partial & 0 \end{pmatrix}$$

is Hermitian.
Second Variation of $\mathcal{E}_{LCG}[U^g, g, \Lambda]$

The second variation of the term $i g(x) \Lambda(x)$ is purely imaginary and it does not contribute to the Faddeev-Popov matrix.
Second Variation of $\mathcal{E}_{LCG}[U^g, g, \Lambda]$

The second variation of the term $i g(x) \Lambda(x)$ is purely imaginary and it does not contribute to the Faddeev-Popov matrix.

The second variation of the (Landau-gauge) term $\mathcal{E}_{LG}[U^g]$ can only give the symmetric Landau FP matrix $\mathcal{M}$!
Second Variation of $\mathcal{E}_{LCG}[U^g, g, \Lambda]$

The second variation of the term $i g(x) \Lambda(x)$ is purely imaginary and it does not contribute to the Faddeev-Popov matrix.

The second variation of the (Landau-gauge) term $\mathcal{E}_{LG}[U^g]$ can only give the symmetric Landau FP matrix $\mathcal{M}$!

One can write the lattice Landau FP matrix as

$$\mathcal{M} = -\frac{1}{2} \left[ \nabla^{(-)}_\mu D_\mu + D^{T}_\mu \left( \nabla^{(-)}_\mu \right)^T \right].$$

This would correspond, in the continuum, to case 2).
Second Variation of $\mathcal{E}_{LCG}[U^g, g, \Lambda]$

The second variation of the term $i g(x) \Lambda(x)$ is purely imaginary and it does not contribute to the Faddeev-Popov matrix.

The second variation of the (Landau-gauge) term $\mathcal{E}_{LG}[U^g]$ can only give the symmetric Landau FP matrix $\mathcal{M}$!

One can write the lattice Landau FP matrix as

$$\mathcal{M} = -\frac{1}{2} \left[ \nabla_{\mu}^{(-)} D_{\mu} + D_{\mu}^{T} \left( \nabla_{\mu}^{(-)} \right)^{T} \right].$$

This would correspond, in the continuum, to case 2).

But, how do we simulate the quartic ghost self-interaction term?
Can we obtain the continuum case 1)? The matrix

\[ \mathcal{M}^{bc}(x, y) \equiv \mathcal{M}^{bc}(x, y) + \sum_{e=1}^{N_c^2 - 1} f^{bec} \Lambda^e(x) \delta_{x, y} \]

is a lattice discretization of the continuum operator \(- \partial \cdot D^{bc}\).
Can we obtain the continuum case 1)? The matrix

\[ \mathcal{M}_{+}^{bc}(x, y) \equiv \mathcal{M}^{bc}(x, y) + \sum_{e=1}^{N_c^2-1} f^{bec} \Lambda^e(x) \delta x, y \]

is a lattice discretization of the continuum operator \(- \partial \cdot D^{bc}\).

The extra term is skew-symmetric, under the simultaneous exchanges \(b \leftrightarrow c\) and \(x \leftrightarrow y\), and it cannot be obtained from a second variation!
Second Variation of $\mathcal{E}_{LCG}[U^g, g, \Lambda]$

Can we obtain the continuum case 1)? The matrix

$$\mathcal{M}^{bc}_{\pm}(x, y) \equiv \mathcal{M}^{bc}(x, y) + \sum_{e=1}^{N_c^2-1} f^{bec} \Lambda^e(x) \delta_{x, y}$$

is a lattice discretization of the continuum operator $- \partial \cdot D^{bc}$.

The extra term is skew-symmetric, under the simultaneous exchanges $b \leftrightarrow c$ and $x \leftrightarrow y$, and it cannot be obtained from a second variation! It should be added by hand!
Second Variation of $\mathcal{E}_{\text{LCG}}[U^g, g, \Lambda]$

Can we obtain the continuum case 1)? The matrix

$$M_{bc}^{\pm}(x, y) \equiv M_{bc}(x, y) + \sum_{e=1}^{N_c^2-1} f^{bec} \Lambda^e(x) \delta_{x, y}$$

is a lattice discretization of the continuum operator $- \partial \cdot D^{bc}$.

The extra term is skew-symmetric, under the simultaneous exchanges $b \leftrightarrow c$ and $x \leftrightarrow y$, and it cannot be obtained from a second variation! It should be added by hand!

Equivalently, we could add to the minimizing functional $\mathcal{E}_{\text{LCG}}[U; \Lambda; h]$, the null term $-\Re \text{Tr} \sum_x i \left[ g(x), \Lambda(x) \right] g(x)^\dagger$. Indeed, by expanding to second order the above expression, we find (by a convenient re-ordering of the null terms) the term $f^{bec} \Lambda^e(x) \delta_{x, y}$. 
Second Variation of $\mathcal{E}_{LCG}[U^g, g, \Lambda]$

Can we obtain the continuum case \ref{eq:cont}? The “effective” FP matrix (without the factor $i$) \[
\begin{pmatrix}
0 & -\partial \cdot D^{bc} \\
D^{bc} \cdot \partial & 0
\end{pmatrix}
\] is skew-symmetric and it cannot be obtained from a second variation!
Second Variation of $\mathcal{E}_{LCG}[U^g, g, \Lambda]$

Can we obtain the continuum case 3)? The “effective” FP matrix (without the factor $i$) \[ \frac{1}{2} \begin{pmatrix} 0 & -\partial \cdot D^{bc} \\ D^{bc} \cdot \partial & 0 \end{pmatrix} \] is skew-symmetric and it cannot be obtained from a second variation!

Should we just consider a proper lattice discretization of this “effective” FP matrix?
Can we obtain the continuum case \(3\)? The “effective” FP matrix (without the factor \(i\)) 
\[
\frac{1}{2} \begin{pmatrix}
0 & -\partial \cdot D^{bc} \\
D^{bc} \cdot \partial & 0
\end{pmatrix}
\]
is skew-symmetric and it cannot be obtained from a second variation!

Should we just consider a proper lattice discretization of this “effective” FP matrix?

We can also consider the bilinear form 
\[
\sum_{b,c,x,y} \gamma^b_1(x) \mathcal{M}^{bc}(x, y) \gamma^c_2(y)
\]
and extend it to the complex case \([\gamma^b_1(x), \gamma^b_2(x) \in \mathbb{C}]\). Then, the corresponding sesquilinear form is a positive semi-definite Hermitian form. Moreover, its imaginary part is skew-symmetric and gives us a natural way of obtaining the above FP matrix.
The three real FP matrices considered have a rather different spectrum.
Spectrum of the FP Matrices

The three real FP matrices considered have a rather different spectrum.

1) The FP matrix $M_+$ has complex-conjugate eigenvalues (and eigenvectors) with a non-negative real part.
The three real FP matrices considered have a rather different spectrum.

1) The FP matrix $\mathcal{M}_+$ has complex-conjugate eigenvalues (and eigenvectors) with a non-negative real part.

2) The FP matrix $\mathcal{M}$ has real non-negative eigenvalues and real eigenvectors.
Spectrum of the FP Matrices

The three real FP matrices considered have a rather different spectrum.

1) The FP matrix $\mathcal{M}_+$ has complex-conjugate eigenvalues (and eigenvectors) with a non-negative real part.

2) The FP matrix $\mathcal{M}$ has real non-negative eigenvalues and real eigenvectors.

3) Since the FP matrix $\frac{1}{2} \begin{pmatrix} 0 & -\partial \cdot D^{bc} \\ D^{bc} \cdot \partial & 0 \end{pmatrix}$ is skew-symmetric, its eigenvalues are complex-conjugate and purely imaginary, and they are related to the SVD of $\mathcal{M}_+$, i.e. to the eigenvalues of $\mathcal{M}_+^T \mathcal{M}_+$. 
Numerical Simulations: Ghost Propagator

We have done some preliminary tests, evaluating the ghost propagator for the continuum case 1), i.e. with the FP matrix

\[ M^{bc}_{\pm}(x, y) = M^{bc}(x, y) + \sum_e f^{bec} \Lambda^e(x) \delta_x, y. \]
We have done some preliminary tests, evaluating the ghost propagator for the continuum case 1), i.e. with the FP matrix $M^{bc}_{\pm}(x, y) = M^{bc}(x, y) + \sum_e f^{bec} \Lambda^e(x) \delta_{x, y}$.

Since the matrix is real and not symmetric, we cannot use the CG algorithm, as in Landau gauge. We are using the bi-conjugate gradient stabilized algorithm for the numerical inversion [P. Silva is using the generalized conjugate residual for the SU(3) case].
Numerical Simulations: Ghost Propagator

We have done some preliminary tests, evaluating the ghost propagator for the continuum case 1), i.e. with the FP matrix

\[ M^{bc}_{\pm}(x, y) = M^{bc}(x, y) + \sum_e f^{bec} \Lambda^e(x) \delta_x, y. \]

Since the matrix is real and not symmetric, we cannot use the CG algorithm, as in Landau gauge. We are using the bi-conjugate gradient stabilized algorithm for the numerical inversion [P. Silva is using the generalized conjugate residual for the SU(3) case].

For the moment, \( SU(2) \) gauge group, \( \beta = 2.4469 \), corresponding to \( a \approx 0.1 \text{ fm} \) [and \( \beta = 6.0 \) in the SU(3) case], with \( \xi \approx 0.163472 \), corresponding to \( \xi = 0.1 \) in the continuum, using a point source for the inversion.
Ghost propagator $G(p^2)$, in the SU(2) case, as a function of the square of the momentum $p^2$ (both in lattice units) for the lattice volume $V = 24^4$, $\beta = 2.4469$ and $\xi \approx 0.163472$, using 60 configurations and 20 sets of $\{\Lambda(x)\}$ for each configuration: comparison of Landau gauge (+) with Linear Covariant gauge (*).

Similar results for SU(3) by P. Silva.

Here, $p_{min} \approx 500$ MeV and $p_{max} \approx 7.8$ GeV.
Conclusions
Conclusions

- Numerical evaluation of the ghost propagator in linear covariant gauge seems feasible.
Conclusions

- Numerical evaluation of the ghost propagator in linear covariant gauge seems feasible.

- We should consider case 1) for larger physical volumes and evaluate the ghost propagator for case 3).
Conclusions

- Numerical evaluation of the ghost propagator in linear covariant gauge seems feasible.
- We should consider case 1) for larger physical volumes and evaluate the ghost propagator for case 3).
- Can we simulate case 2)?
Conclusions

- Numerical evaluation of the ghost propagator in linear covariant gauge seems feasible.
- We should consider case 1) for larger physical volumes and evaluate the ghost propagator for case 3).
- Can we simulate case 2)?
- Can we define the first Gribov region $\Omega$?