# Temperature dependence of $\eta/s$: uncertainties from the equation of state

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# Equations of state

Hadron resonance gas at low temperatures, fit to lattice at high $T$

- **s95p**: $M < 2$ GeV HG particles from 2005 PDG summary tables
  P. Huovinen and P. Petreczky, NPA 837, 26 (2010)
  Lattice data used for fitting: Bazavov et al., PRD 80, 014504 (2009); Cheng et al., PRD 77, 014511 (2008)
- s83z: Hadron gas particles from 2016 PDG summary tables
  Lattice data: Borsanyi et al., PLB 730, 99 (2014)
- s87r: $M < 2$ GeV HG particles from 2005 PDG summary tables
  Lattice data: Bazavov et al., PRD 90, 094503 (2014); Bazavov et al., PRD 97, 014510 (2018)
- s88s: HG particles from 2016 PDG summary tables
  Lattice data: Bazavov et al., PRD 90, 094503 (2014); Bazavov et al., PRD 97, 014510 (2018)

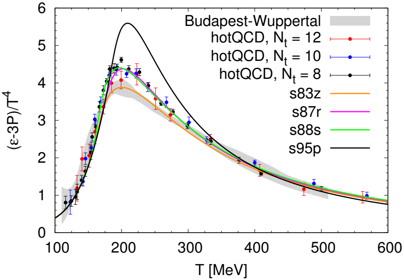



Figure: P. Huovinen

# EKRT+hydrodynamics model

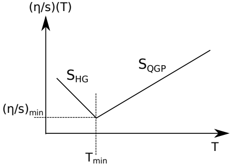
- Initial energy density from the EKRT minijet saturation model

Paatelainen et al., PRC 87, 044904 (2013); PLB 731, 126 (2014)

$$e(\vec{r}_T, \tau_s(\vec{r}_T)) = \frac{K_{sat}}{\pi}[p_{sat}(\vec{r}_T, K_{sat})]^4;\ \tau_s(\vec{r}_T) = 1/p_{sat}(\vec{r}_T, K_{sat})$$

- 2+1D viscous hydrodynamics with linear temperature dependence on shear viscosity coefficent $\eta/s$ Niemi et al., PRC 93, 024907 (2016)

$$\eta/s(T) = S_{HG}(T_{min} - T) + (\eta/s)_{min},\ T < T_{min}$$
$$\eta/s(T) = S_{QGP}(T - T_{min}) + (\eta/s)_{min},\ T > T_{min}$$




Kinetic decoupling temperature $T_{dec} = 120$ MeV

# Bayesian analysis

Model parameters (input): $\vec{x} = (x_1, ..., x_n)$

$(K_{sat},\ T_{min},\ (\eta/s)_{min},\ S_{HG},\ S_{QGP})$

$\Downarrow$

Model output $\vec{y} = (y_1, ..., y_m) \Leftrightarrow$ Experimental values $\vec{y}^{\,\mathrm{exp}}$

$(N_{\mathrm{ch}}(\sqrt{s_{NN}}, \mathrm{centrality}),\ v_2(\sqrt{s_{NN}}, \mathrm{centrality}))$

Bayes' theorem:

$$\boxed{\text{Posterior probability} \propto \text{Likelihood} \cdot \text{Prior knowledge}}$$

# Bayesian analysis

$$\boxed{\text{Posterior probability} \propto \text{Likelihood} \cdot \text{Prior knowledge}}$$

- Prior knowledge: Range of input parameter values to investigate (uniform hypercube)
- Likelihood: $\mathcal{L}(\vec{x}) \propto \exp\left(-\frac{1}{2}(\vec{y}(\vec{x}) - \vec{y}^{\,\exp})\Sigma^{-1}(\vec{y}(\vec{x}) - \vec{y}^{\,\exp})^T\right)$, where $\Sigma$ is the covariance matrix

Use Markov chain Monte Carlo to sample the posterior probability
"emcee" ensemble sampler by D. Foreman-Mackey: `http://dfm.io/emcee/`

- Random walk in input parameter space, guided by the likelihood, constrained by the prior
- Running the full hydrodynamics model during random walk to compute likelihood infeasible timewise
  $\Rightarrow$ use Gaussian process (GP) emulator to estimate model output for likelihood computations `http://scikit-learn.org/stable/modules/gaussian_process.html`

# Analysis procedure

Produce training data for emulator conditioning

- run simulations with $\mathcal{O}(100)$ different parameter combinations
- rule of thumb: $\sim 20$ points per parameter

One GP $\leftrightarrow$ One observable
Reduce the dimension of the output space with principal component analysis (PCA)

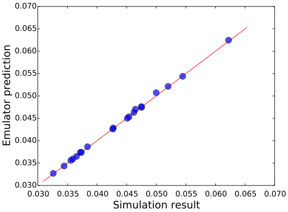
http://scikit-learn.org/stable/modules/decomposition.html

$\Downarrow$

Condition the emulators on training data; check prediction quality

$\Downarrow$

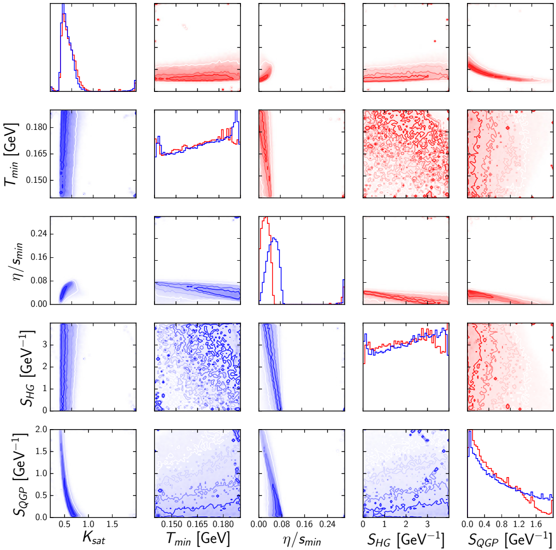
Calibrate on experimental data by running MCMC
Walker density in parameter space $\rightarrow$ Posterior probability distribution as number of steps $\rightarrow \infty$




Emulator verification for $v_2$ at $\sqrt{s_{NN}} = 200$ GeV at (10-20)% centrality, using s95p EoS

# Posterior probability distributions

s83z



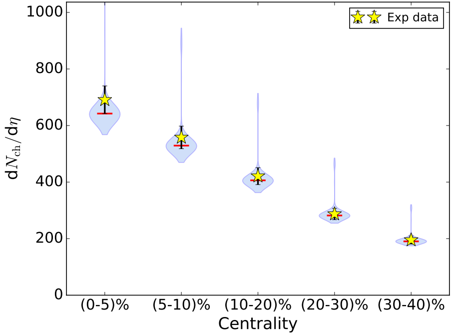

s95p

# Observables at $\sqrt{s_{NN}} = 200$ GeV

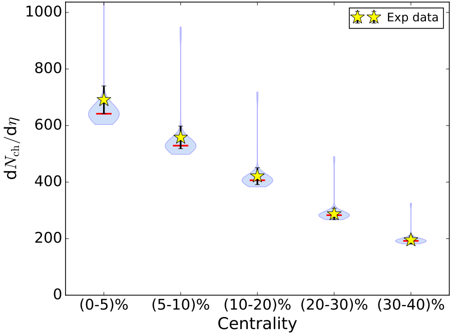
Emulator estimates for samples from the posterior distribution

Charged particle yield vs. centrality

s83z




s95p



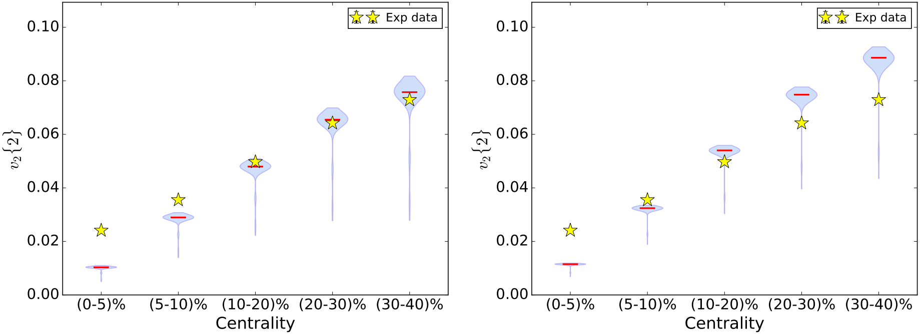

Experimental data: STAR, PRC 79, 034909 (2009)

# Observables at $\sqrt{s_{NN}} = 200$ GeV

## Elliptic flow $v_2\{2\}$

s83z

s95p

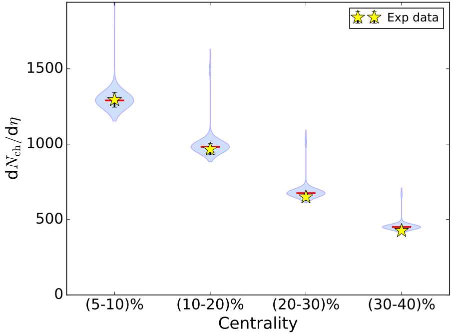



Experimental data: STAR, PRC 72, 014904 (2005)

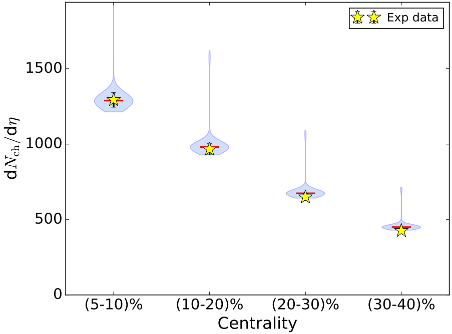
# Observables at $\sqrt{s_{NN}} = 2.76$ TeV

Charged particle yield vs. centrality

s83z




s95p



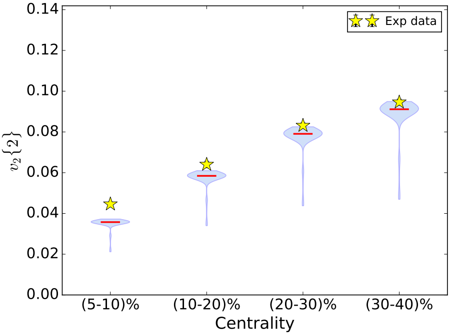

Experimental data: ALICE PRL 106, 032301 (2011)

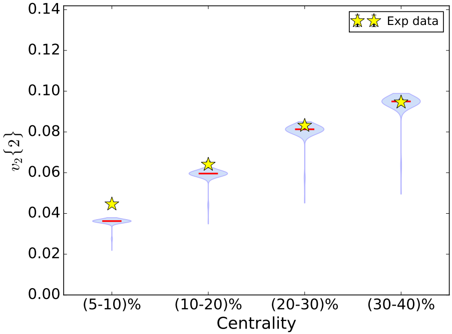
# Observables at $\sqrt{s_{NN}} = 2.76$ TeV

## Elliptic flow $v_2\{2\}$

s83z

s95p

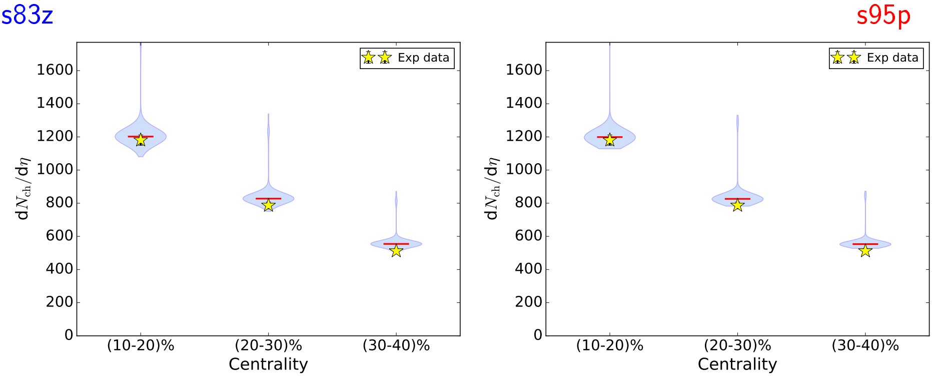



Experimental data: ALICE PRL 116, 132302 (2016)

# Observables at $\sqrt{s_{NN}} = 5.02$ TeV

Charged particle yield vs. centrality

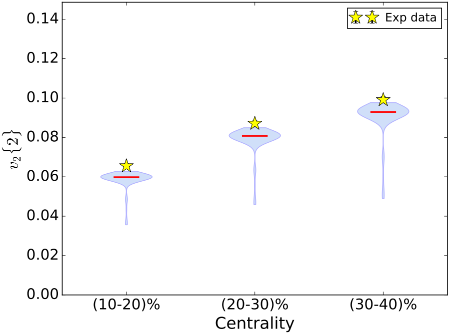



Experimental data: ALICE PRL 116, 222302 (2016)

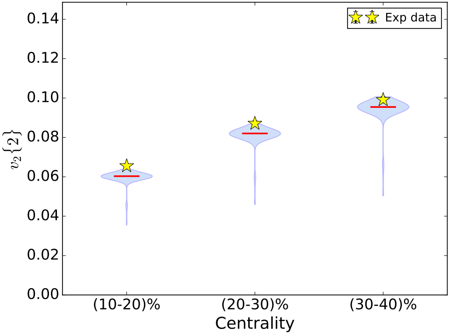
# Observables at $\sqrt{s_{NN}} = 5.02$ TeV

## Elliptic flow $v_2\{2\}$

s83z

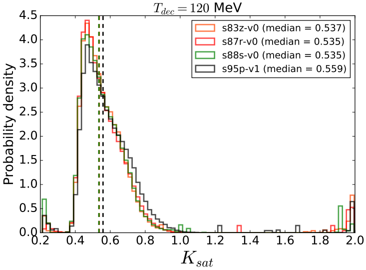



s95p

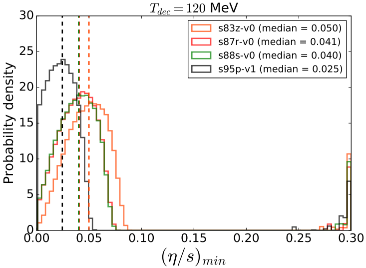


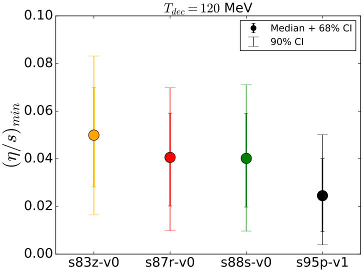

Experimental data: ALICE PRL 116, 132302 (2016)

# Marginal posterior distribution comparisons



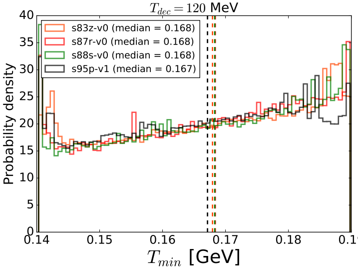

- $K_{sat}$ unaffected by EoS; consistent with the values used in previous studies ($\approx 0.5$ - $0.75$)
- $(\eta/s)_{min}$ peak probability at higher values using newer equations of state compared to **s95p**
- Uncertainties remain large

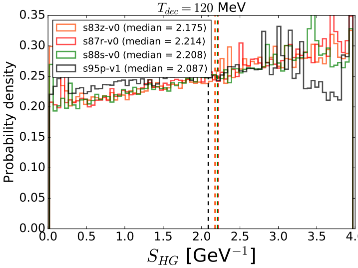


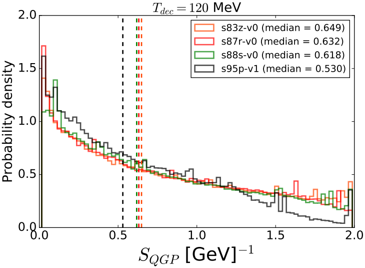



# Marginal posterior distribution comparisons

- $T_{min}, S_{HG}$ not constrained by this analysis; consirerable probability density on both edges ⇒ Prior range too small
- Smaller values of $S_{QGP}$ favored; the skew is more pronounced for **s95p**



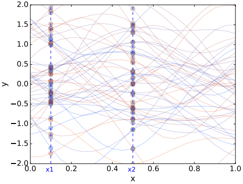
# Summary

- Bayesian analysis is a convenient method for simultaneously determining both the "best-fit" parameter values and their uncertainties.
- The choice of the equation of state has a notable effect on the extracted minimum value of $\eta/s$; significant overlap remain in the posterior probability distributions, however
- Minijet saturation parameter $K_{sat}$ unaffected by EoS
- Smaller slope values for $eta/s(T)$ in QGP favored, weak dependence on EoS
- The place of the minimum (wrt temperature) and the linear slope of $eta/s(T)$ in hadron gas remain unconstrained; prior ranges need to be extended
- Overall, the equations of state based on newer lattice data tend to require higher values of $eta/s(T)$ for the best fit with experimental data

# Extra slides

# Gaussian process

Stochastic process: A parameterized collection of random variables $\{y_t\}_{t\in T}$ ($T$ possibly infinite). E.g. random walk over time.

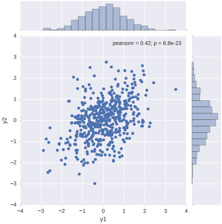



Gaussian process: A stochastic process, in which every finite set $Y = \{y_i\}_{i=1}^{N}$ is a multivariate Gaussian random variable $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \mu(X) = \{\mu(x_1), ..., \mu(x_N)\}$$

is the mean and

$$\boldsymbol{\Sigma} = \sigma(X, X) = \begin{pmatrix} \sigma(\vec{x}_1, \vec{x}_1) & \cdots & \sigma(\vec{x}_1, \vec{x}_N) \\ \vdots & \ddots & \vdots \\ \sigma(\vec{x}_N, \vec{x}_1) & \cdots & \sigma(\vec{x}_N, \vec{x}_N) \end{pmatrix}$$




is the covariance matrix with covariance function $\sigma(\vec{x}, \vec{x}')$.

# Gaussian process

Choice: Squared-exponential covariance function with a noise term

$$\sigma(\vec{x},\vec{x}') = \theta_0 \exp\left(-\sum_{i=1}^{n}\frac{(x_i - x_i')^2}{2\theta_i^2}\right) + \theta_{\text{noise}}\delta_{\vec{x}\vec{x}'}$$

The *hyperparameters* $\vec{\theta} = (\theta_0, \theta_1, ...\theta_n, \theta_{\text{noise}})$ are not known a priori and must be estimated from the given data

$\Rightarrow$ emulator training: Maximise the marginal likelihood (aka "evidence")

$$\log P(Y|X,\vec{\theta}) = \underbrace{-\frac{1}{2}Y^T\mathbf{\Sigma}^{-1}(X,\vec{\theta})Y}_{\text{data fit}} \underbrace{-\frac{1}{2}\log|\mathbf{\Sigma}(X,\vec{\theta})|}_{\text{complexity penalty}} \underbrace{-\frac{N}{2}\log(2\pi)}_{\text{normalization}}$$

# Gaussian process

To predict the model output $y_0$ in an arbitrary point $\vec{x_0}$, we write the joint distribution

$$\begin{pmatrix} y_0 \\ Y \end{pmatrix} = \mathcal{N}\left( \begin{pmatrix} \mu(\vec{x_0}) \\ \mu(X) \end{pmatrix}, \begin{pmatrix} \Sigma_{0,0} & \Sigma_{0,X} \\ \Sigma_{X,0} & \Sigma_{X,X} \end{pmatrix} \right)$$

and calculate the conditional predictive mean

$$\bar{\mu}(\vec{x_0}) = \mu(\vec{x_0}) + \Sigma_{0,X}\Sigma_{X,X}^{-1}(Y - \mu(X)).$$

Typically we define the mean function $\mu(\vec{x}) \equiv 0$ and the prediction is simply

$$\bar{\mu}(\vec{x_0}) = \Sigma_{0,X}\Sigma_{X,X}^{-1} Y$$

with the associated predictive (co)variance

$$\bar{\Sigma} = \Sigma_{0,0} - \Sigma_{0,X}\Sigma_{X,X}^{-1}\Sigma_{X,0}.$$

# Principal component analysis

$m$ observables $\Rightarrow$ $m$ Gaussian processes needed for model emulation

However, $m$ can grow to $\mathcal{O}(100)$ at top RHIC energies and at the LHC!
Number of emulators can be reduced with principal component analysis

First principal component represents the direction of largest variance in output space, second PC the direction of second largest variance, etc.

- Fraction of variance explained by principal component $p_q$:
  $\mathrm{Var}(p_q) = \frac{\lambda_q}{\sum_{i=1}^{m} \lambda_i}$
- Select the number of principal components which together explain desired fraction of total variance; often only a few PCs are needed to explain 99% of the variance

# Likelihood function

The likelihood function used in MCMC:

$$\frac{1}{|2\pi(\Sigma_{\text{exp}} + \Sigma_{GP})|} \exp\left( -\frac{1}{2} (\vec{z}^{*}_{GP} - \vec{z}_{\text{exp}})(\Sigma_{\text{exp}} + \Sigma_{GP})^{-1} (\vec{z}^{*}_{GP} - \vec{z}_{\text{exp}})^{T} \right)$$

- $\vec{z}^{*}_{GP}$ is the emulator prediction at the input parameter point $\vec{x}^{*}$
- $\vec{z}_{\text{exp}}$ is the experimental data transformed to principal component space
- $\Sigma_{GP}$ is the predictive variance (emulator uncertainty)
- $\Sigma_{\text{exp}}$ is the experimental error squared, transformed to PC space