Making Sense of Divergent Series: Resummation of Logarithms in Nonrelativistic Expansions of Light-Cone Amplitudes

G. T. Bodwin, H. S. Chung, J.-H. Ee and J. Lee (BCEL)

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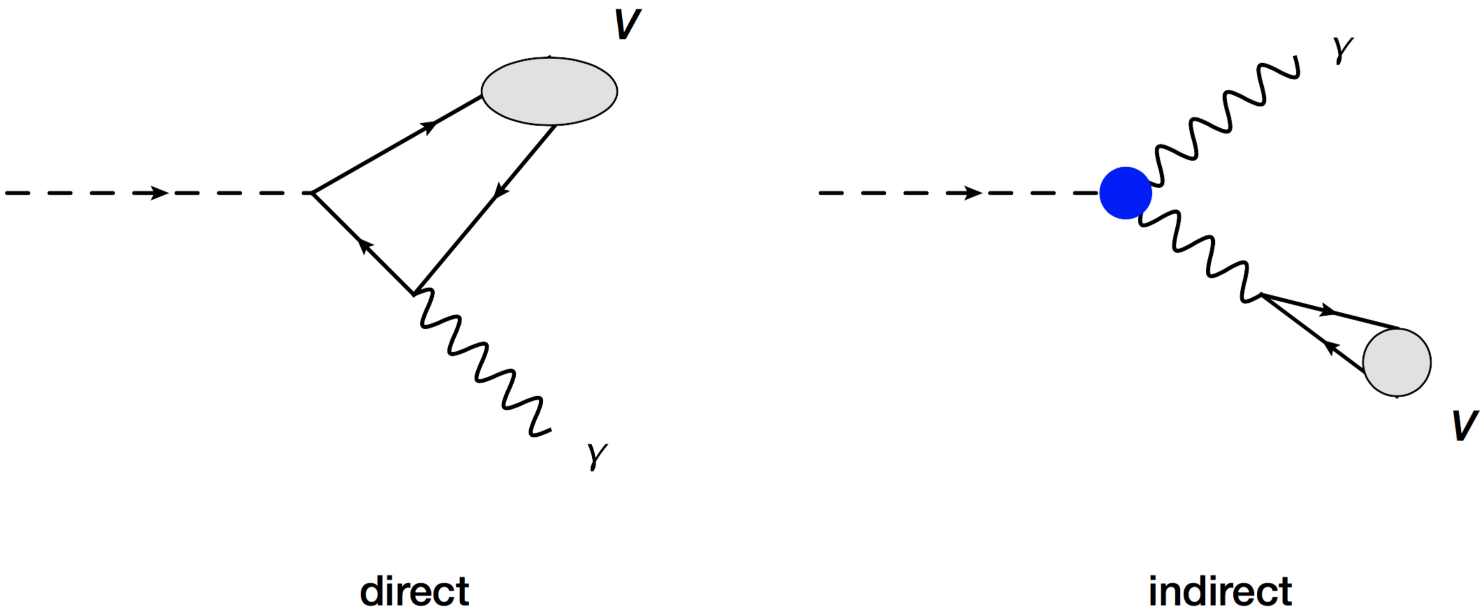
The Decays $H, Z \to V + \gamma$

- The rare Higgs decay $H \to J/\psi + \gamma$ can be used to measure the $Hc\bar{c}$ coupling at a high-luminosity LHC [Bodwin, Petriello, Stoynev, Velasco (2013)].
- The rare Higgs decays $H \to \Upsilon(nS) + \gamma$ are very sensitive to deviations from the standard model $Hb\bar{b}$ coupling.
- The decay of the Z boson to a vector quarkonium V and a photon is interesting in its own right:
  - as a test of the standard model
  - as a test of our understanding of quarkonium production.
- It is also important as a calibration for experimental measurements of the final state $V + \gamma$.

These processes proceed through a direct amplitude and an indirect amplitude:



- In the indirect amplitude, H and Z couple to $\gamma\gamma^*$ through t and W triangle diagrams.
- The indirect amplitude can be determined very precisely from knowledge of the $V\gamma$ coupling (from V decays).
- The direct amplitude can be calculated in nonrelativistic QCD (NRQCD).
- We will focus on the direct amplitude.

The Light-Cone Formalism

- In calculating the direct amplitude, it is convenient to use the light-cone formalism for exclusive processes.
  [Brodsky and Lepage (1980); Chernyak and Zhitnitsky (1984)]
  - Expansion in powers of m_V^2/m_H^2 or m_V^2/m_Z^2.
  - We work at leading order in the expansion.
  - Greatly simplifies the calculation at fixed-order in α_s.
  - A natural framework within which to resum large logs of m_H^2/m_Q^2 or m_Z^2/m_Q^2.
- The NRQCD expansion of the light-cone distribution amplitude (LCDA) leads to distributions (generalized functions):
  Dirac δ-function, its derivatives, $+$ and $++$ distributions, ...
- Problem: The generalized functions cause the standard expansion of the LCDA in eigenfunctions of the evolution operator to diverge.

The Light-Cone Amplitude

- For $H, Z \to V + \gamma$ the leading-twist light-cone direct amplitude is proportional to

$$\mathcal{A} = \int_0^1 dx\, T_H(x,\mu)\phi_V(x,\mu).$$

- x is the light-cone momentum fraction.
- $T_H(x,\mu)$ is the hard-scattering kernel at the renormalization scale μ.
  - $T_H(x,\mu)$ can be calculated in QCD perturbation theory.
  - μ is chosen to be of order m_H or m_Z in order to avoid large logs of m_H^2/μ^2 or m_Z^2/μ^2.
- $\phi_V(x,\mu)$ is the quarkonium light-cone distribution amplitude (LCDA).

NRQCD Expansion of the LCDA

- At the scale $\mu_0 \sim m_Q$, $\phi_V(x,\mu_0)$ has an NRQCD (nonrelativistic) expansion. [Yu Jia, Deshan Yang (2008)]

$$\phi_V(x,\mu_0) = \phi_V^{(0)}(x,\mu_0) + \langle v^2\rangle_V \phi_V^{(v^2)}(x,\mu_0) + \frac{\alpha_s(\mu_0)}{4\pi}\phi_V^{(1)}(x,\mu_0) + O(\alpha_s^2, \alpha_s v^2, v^4).$$

- $\langle v^2\rangle_V$ is the ratio of the order-v^2 LDME to the order-v^0 LDME:

$$\langle v^2\rangle_V = \frac{1}{m_Q^2}\frac{\langle V(\boldsymbol{\epsilon}_V)|\psi^\dagger(-\frac{i}{2}\overleftrightarrow{\boldsymbol{\nabla}})^2\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_V\chi|0\rangle}{\langle V(\boldsymbol{\epsilon}_V)|\psi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_V\chi|0\rangle}.$$

- The LO LCDA is

$$\phi_V^{(0)}(x,\mu_0) = \delta(x-\tfrac{1}{2}).$$

  $\delta(x-\frac{1}{2})$ is the Dirac delta function.

- The order-v^2 contribution to the LCDA is proportional to

$$\phi_V^{(v^2)}(x,\mu_0) = \frac{1}{24}\delta^{(2)}(x-\tfrac{1}{2}).$$

  $\delta^{(n)}(x-\frac{1}{2})$ is the nth derivative of the Dirac delta function.

Evolution of the LCDA

- We need to evolve the LCDA from $\mu_0 \sim m_Q$ to $\mu \sim m_H, m_Z$.
  - The evolution takes into account logs of m_H^2/m_Q^2 or m_Z^2/m_Q^2 to all orders in perturbation theory.
  - In practice, we work to NLL accuracy.
- The LCDA satisfies the Efremov-Radyushkin-Brodsky-Lepage (ERBL) evolution equation:

$$\mu^2 \frac{\partial}{\partial \mu^2} \phi_V(x,\mu) = C_F \frac{\alpha_s(\mu)}{2\pi} \int_0^1 dy\, V_T(x,y) \phi_V(y,\mu),$$

$V_T(x,y)$ is the evolution kernel.

Standard Method of Solution of the ERBL Equation

- Decompose ϕ_V into eigenfunctions $|n, x\rangle$ of the LO evolution kernel.
- The LO evolution kernel is diagonalized by Gegenbauer polynomials of order $3/2$:

$$
\begin{aligned}
|n, x\rangle &= N_n C_n^{(3/2)}(2x-1), \\
\langle n, x| &= N_n w(x) C_n^{(3/2)}(2x-1).
\end{aligned}
$$

  (Sometimes suppress the argument x in $|n, x\rangle$ and $\langle n, x|$.)

  - $N_n = \frac{4(2n+3)}{(n+1)(n+2)}$ is the normalization factor.
  - $w(x) = x(1-x)$ is the weight factor.
- Orthonormality: $\langle n|m\rangle = \delta_{nm}$.
  (The inner product denotes integration over x.)
- Completeness: $\sum_n |n, x'\rangle\langle n, x| = \delta(x'-x)$.

- The evolution equation for the LCDA can be solved in closed form for each eigenstate:

$$|\phi_V(\mu)\rangle = \sum_{m,n} |m\rangle\langle m|U(\mu,\mu_0)|n\rangle\langle n|\phi_V(\mu_0)\rangle.$$

$\langle m|U(\mu,\mu_0)|n\rangle$ is the evolution matrix.

  - Depends on the eigenvalues of the evolution operator.
  - Diagonal at LL order.

- The light-cone amplitude is now

$$\mathcal{A} = \sum_{m,n} \underbrace{\langle T_H(\mu)|m\rangle}_{T_m(\mu)} \underbrace{\langle m|U(\mu,\mu_0)|n\rangle}_{U_{mn}(\mu,\mu_0)} \underbrace{\langle n|\phi_V(\mu_0)\rangle}_{\phi_n(\mu_0)}.$$

- Charge conjugation symmetry: ϕ_n is nonzero only for n even.

Problem: The eigenfunction series sometimes diverges.

- Example from H decays [Bodwin, Chung, Ee, Lee, Petriello (2014)]:

  For $T_H = T_H^{(0)} = \frac{1}{x(1-x)}$ and $\phi_V = \delta^{(2k)}(x - \frac{1}{2})$,
  $T_n\phi_n \sim (-1)^{(n/2-k)}\, n^{(2k-1/2)}$.

  – In particular, for $\phi_V = \delta^{(2)}(x - \frac{1}{2})$ (the order-v^2 correction), the series is divergent:

n	0	2	4	6	8	10
$T_n\phi_n$	0	17.5	−38.5	63.3	−91.4	122.6

- $\langle m|U(\mu,\mu_0)|n\rangle$ improves the convergence, but the series doesn't converge until μ is much greater than m_H.
- The essence of the problem: Generalized functions, such as $\delta^{(k)}(x - \frac{1}{2})$, unlike ordinary functions, are not guaranteed to have convergent eigenfunction expansions.
- Also a problem for the order-α_s correction to ϕ_V ($+$ and $++$ distributions).

Solution of the Problem of Diverging Eigenfunction Expansions

Abel Summation

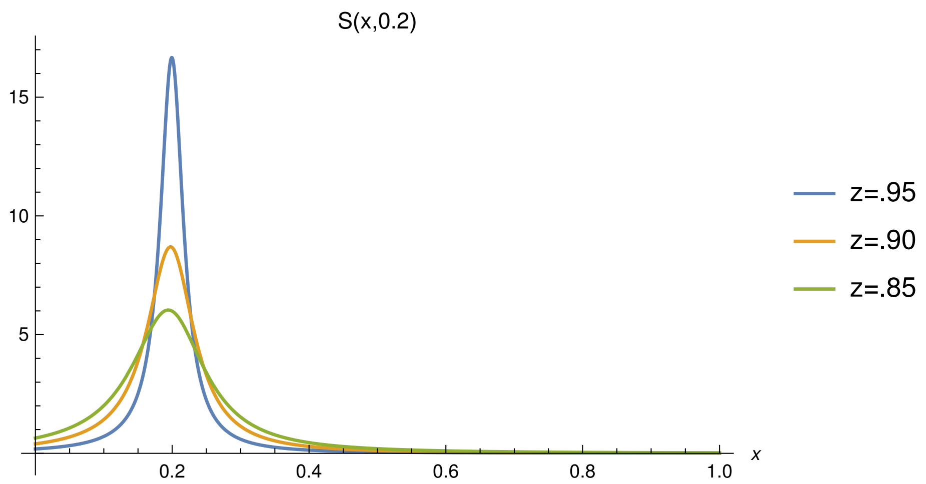
- A general way to assign a value to a divergent series.
  - Multiply the nth term in the series by z^n.
  - z is a complex number with $|z| < 1$.
  - Take the limit $z \to 1^-$.
- In our case, we have

$$\mathcal{A} = \lim_{z \to 1^-} \sum_{m,n} \langle T_H(\mu)|m\rangle\langle m|U(\mu,\mu_0)|n\rangle z^n \langle n|\phi_V(\mu_0)\rangle.$$

- How do we know that Abel summation gives the correct answer?
- Interpretation:

$$S(x,x',z) = \sum_n |n,x'\rangle z^n \langle n,x|$$

gives a representation of a Dirac δ-function as a sequence of ordinary functions.



- $S(x,x',z)$ becomes more and more peaked around $x = x'$ as $z \to 1$.
- The area under $S(x,x',z)$ goes to 1 as $z \to 1$.

- $$\sum_n |n,x'\rangle z^n \langle n,x|\phi_V(\mu_0)\rangle = \int dx'\, S(x,x',z)\phi_V(x')$$

  smears any generalized functions in ϕ_V, turning them into ordinary functions.

- The Abel summation defines generalized functions in ϕ_V as a limit of a sequence of ordinary functions.

Padé Approximants

- Problem: The Abel-summation series converges very slowly for z near 1.
  In order to obtain percent level accuracy, it is necessary to retain hundreds of terms.
- Padé approximants replace the Nth partial sum of a series with a ratio of polynomials:
$$[i/j](z) = \frac{a_0 + a_1 z^1 + a_2 z^2 + \ldots + a_i z^i}{1 + b_1 z^1 + b_2 z^2 + \ldots + b_i z^j}.$$
  The a's and b's are chosen so that the series expansion of $[i/j](z)$ reproduces the partial sum though Nth order.
- The Padé approximant gives an approximate analytic continuation that is valid beyond the radius of convergence of the series.
- Simple example: $1/(1+z)$ has a series expansion with partial sums
$$S_N = 1 - z + z^2 + \ldots + (-1)^N z^N.$$
  The series has a radius of convergence 1 because of the singularity at $z = -1$.
- Every Padé approximant of every partial sum is $1/(1+z)$.
  We can evaluate the Padé approximant at $z = 1$, even though the series does not converge there.

- The evolved light-cone amplitude exists because the RHS of the ERBL equation is nonsingular.
- Therefore, the point $z = 1$ is nonsingular.
  We can evaluate the Padé approximant at $z = 1$, instead of taking $\lim z \to 1^-$.
- For $T_H = T_H^{(0)} = \frac{1}{x(1-x)}$ and $\phi_V = \delta^{(2)}(x - \frac{1}{2})$ (no evolution) the Abel-Padé method converges amazingly rapidly to the analytic answer:

N	$\sum_{n=0}^{N} T_n \phi_n$
4	5.468750000
8	3.988747921
12	4.000358243
16	3.999983194
20	4.000000036

Using $[(N/2)/(N/2)](z)$ Padé approximants.

- We have tested the Abel-Padé method against known analytic results in a number of cases: $\phi(x, \mu_0)$ (no evolution) at LO and NLO in α_s, $\phi(x) = \delta^{(2k)}(x - \frac{1}{2})$ up to $k = 5$, fixed-order-in-α_s evolution of ϕ.
  In every case, Abel-Padé method converges rapidly to the correct answer.

Comparison with a Model LCDA

- König and Neubert KN (2015):
  Approximate the order-v^2 and order-α_s corrections to the LCDA with a model LCDA:

$$\phi_V^M(x,\mu_0) = N_\sigma \frac{4x(1-x)}{\sqrt{2\pi}\sigma_V(\mu_0)} \exp\left[-\frac{(x-\frac{1}{2})^2}{2\sigma_V^2(\mu_0)}\right].$$

  – N_σ is chosen so that

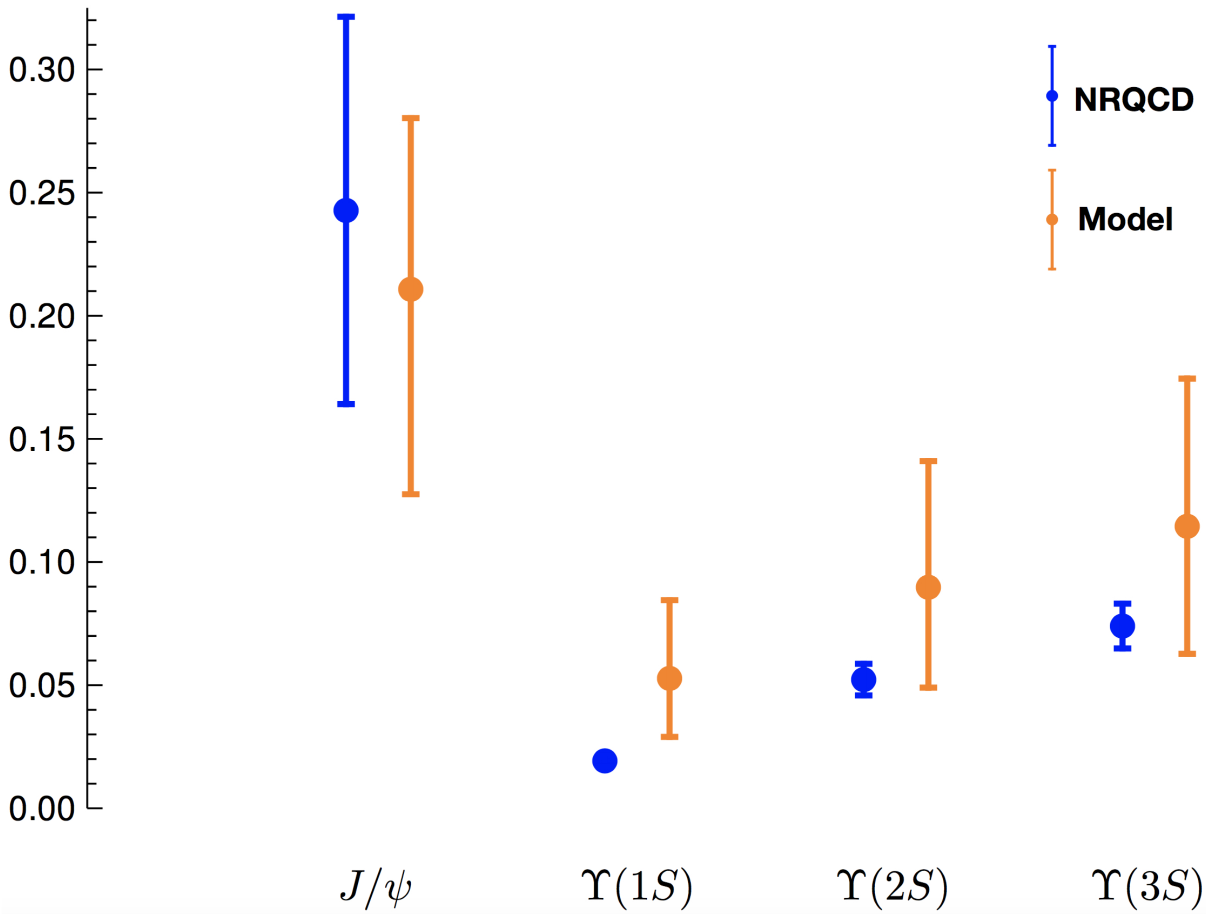
$$\int_0^1 dx\, \phi_V^M(x,\mu_0) = 1.$$

  – $\sigma_V(\mu_0)$ is chosen so that $\phi_V^M(x,\mu_0)$ yields the second moment of $\phi_V(x,\mu)$ through linear order in v^2 and α_s.

- Model avoids the problem of generalized functions.

- How well does the model reproduce the first-principles result from NRQCD with Abel-Padé resummation?

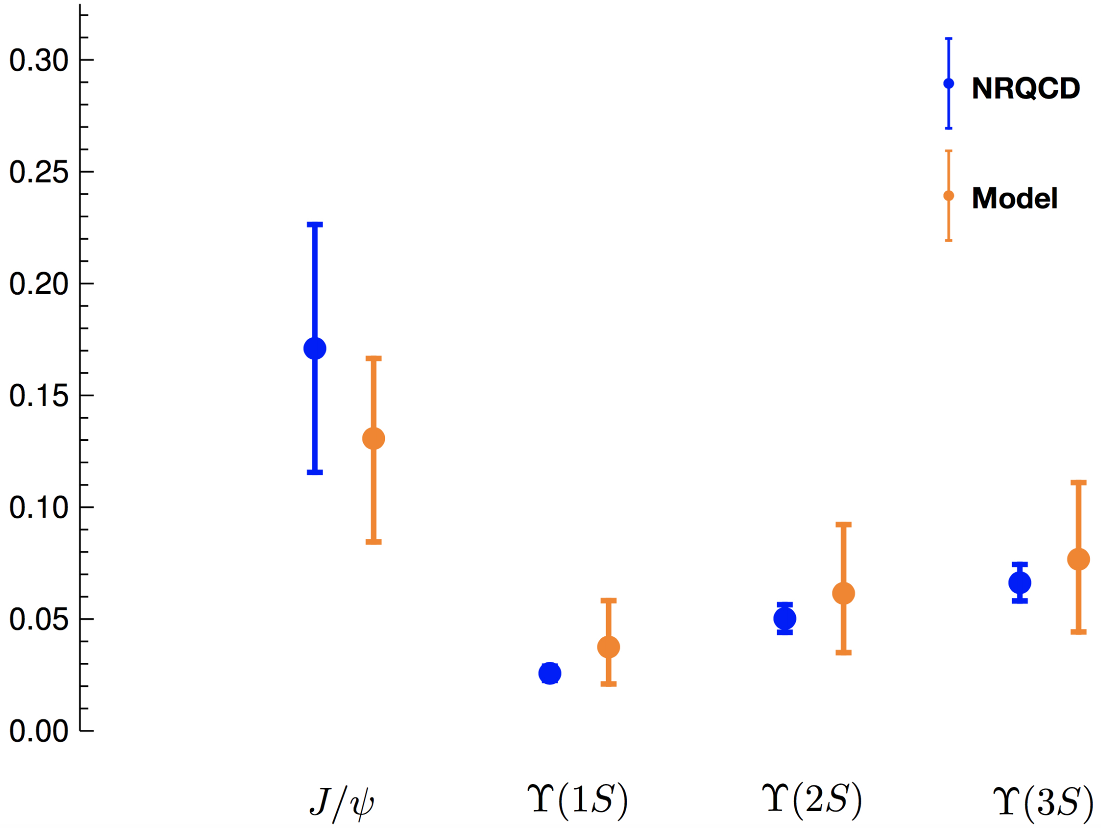
Sum of corrections of NLO in α and v^2 at the scale $\mu_0 = 1$ GeV:



- In units of the LO amplitude at the scale μ_0
- NRQCD errors are estimates of uncalculated terms of higher order in α_s and v^2.
- Model errors are from variation of $\sigma_V(\mu_0)$ by an arbitrary $\pm 25\%$.

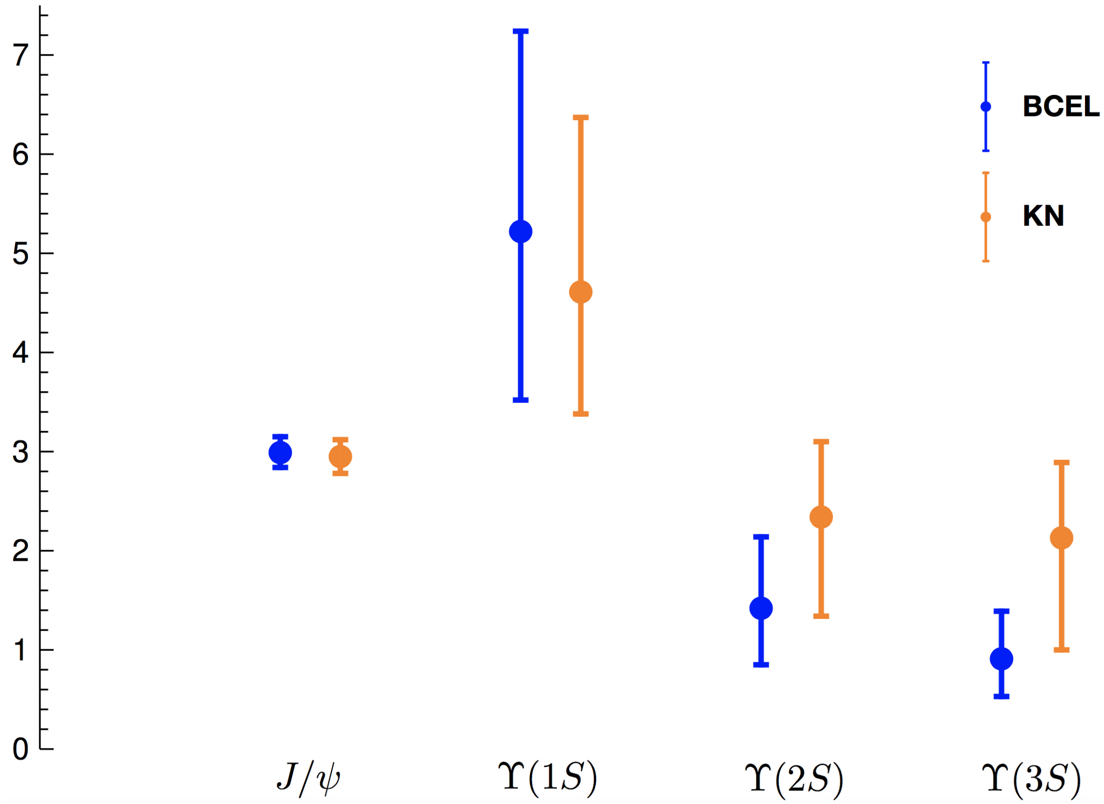
- For the $\Upsilon(nS)$ states, the model errors are much larger than the NRQCD errors.
- For the $\Upsilon(nS)$ states, the model values are shifted by several NRQCD standard deviations from the NRQCD values.
- Shifts of the model values by amounts that are larger than the expected sizes of uncalculated higher-order terms are probably unphysical.

After evolution to the scale $\mu = m_H$, the agreement of the model with NRQCD is better:



There are still substantial shifts of central values for the $\Upsilon(nS)$ states.

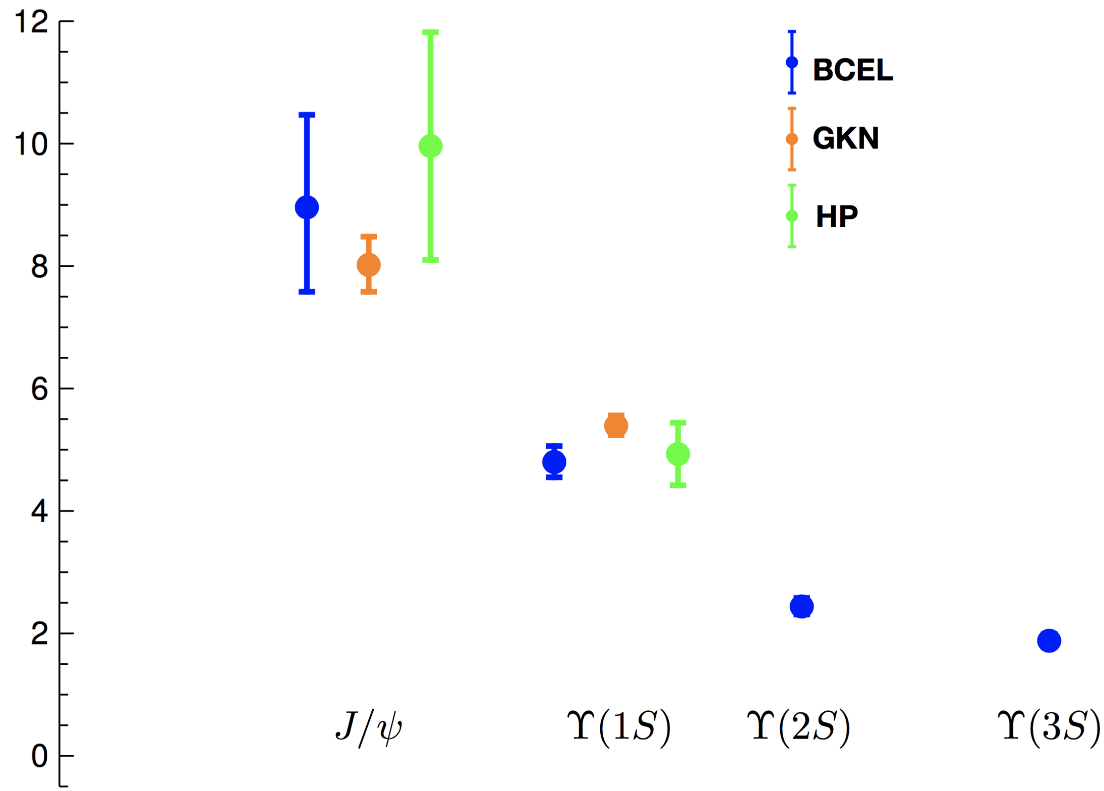
Branching ratios for $H \to V + \gamma$



- $\text{Br}(H \to J/\psi + \gamma)$ in units of 10^{-6}
- $\text{Br}(H \to \Upsilon(nS) + \gamma)$ in units of 10^{-9}
- BCEL (2016, 2018)
- König and Neubert (KN) (2015)

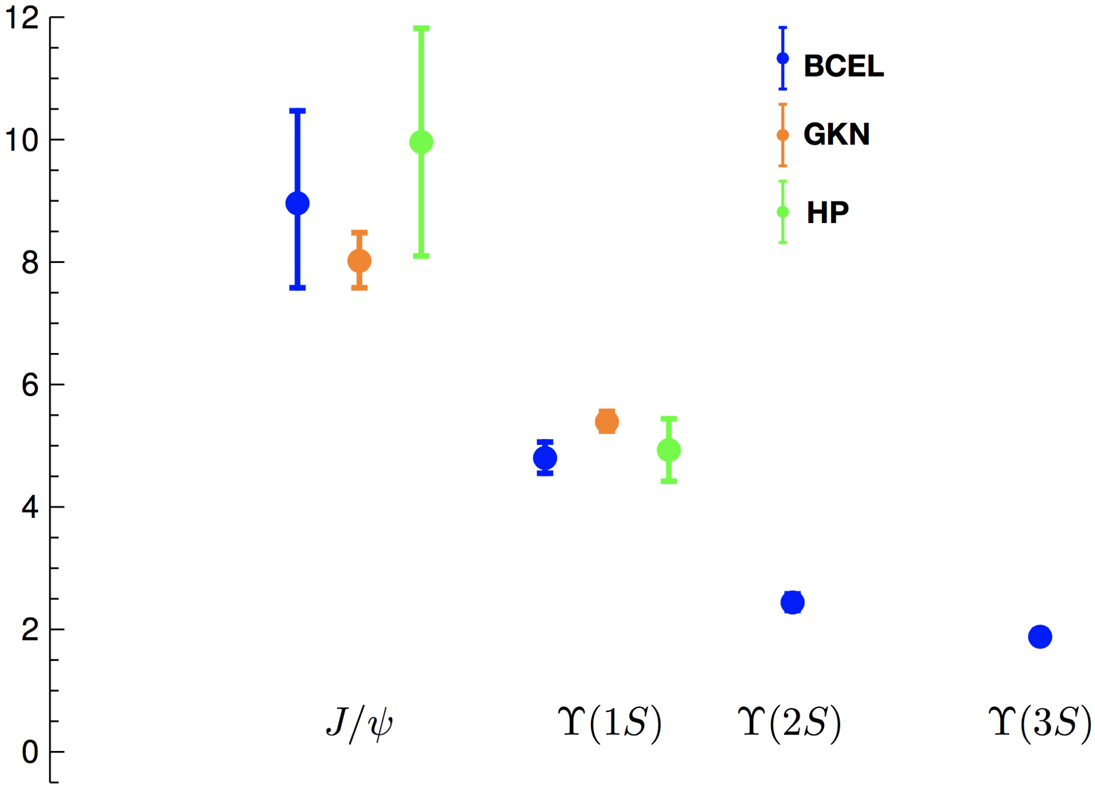
- For the J/ψ, the indirect amplitude is dominant and differences between BCEL and KN are minimized.
- For the $\Upsilon(nS)$ states, the direct amplitude is similar in magnitude to the indirect amplitude. There are significant differences between BCEL and KN.

Branching ratios for $Z \to V + \gamma$



- $\mathrm{Br}(Z \to V + \gamma)$ in units of 10^{-8}
- BCEL (2017)
- Grossman, König, and Neubert (GKN) (2015)
- Huang and Petriello (HP) (2014)

- Compared with BCEL, GKN
  - did not include nonlog α_s corrections to the LCDA,
  - used different values of $\langle v^2 \rangle$,
  - resummed logs at LL accuracy instead of NLL accuracy,
  - computed errors by varying μ instead of estimating uncalculated corrections (did not take into account uncalculated $\alpha_s(\mu_0)$ corrections to the LCDA).



- $\mathrm{Br}(Z \to V + \gamma)$ in units of 10^{-8}
- BCEL (2017)
- Grossman, König, and Neubert (GKN) (2015)
- Huang and Petriello (HP) (2014)

- Compared with BCEL, HP
  - did not resum logs,
  - took the scale of the $\alpha_s(\mu_0)$ corrections to the LCDA to be m_z,
  - took the scale of the wave function at the origin to be m_Q instead of m_V (the extraction scale),
  - made a computational error in the relative sign of the indirect amplitude.
- The first 3 differences are $\sim 30\%$ corrections that tend to cancel.

Summary

- The Abel-Padé method provides a general solution to the problem of the evolution of the NRQCD expansions of quarkonium LCDAs.
- First-principles calculations show that a simple model for the LCDA gives unreliable results.
- We have used the Abel-Padé method to compute the evolution of the order-α_s and order-v^2 corrections to the quarkonium LCDAs for the decays $H \to V + \gamma$ $Z \to V + \gamma$ from first principles.
- Measurements of the decays $Z \to V + \gamma$ will provide new precision tests of quarkonium-production theory.
- Experience with the measurements of $Z \to V + \gamma$ will facilitate measurements of $H \to V + \gamma$.
- Measurements of the decays $H \to V + \gamma$ will constrain the $Hc\bar{c}$ and $Hb\bar{b}$ couplings.

Backup slides

Branching Fractions for $Z \to V + \gamma$

- Using the Abel-Padé method, we computed the branching fractions for $Z \to V + \gamma$.
  1. Including the contribution of NLO in α_s in $\phi_V(x, \mu_0)$. [X.-P. Wang and D. Yang (2017)].
  2. Including the contribution of NLO in α_s in T_H. [X.-P. Wang and D. Yang (2014)].
  3. Including logs of m_Z^2/m_Q^2 resummed to all orders in α_s at NLL accuracy.
  4. Including the indirect amplitude (decay of the Z boson through a fermion loop). Only a 1% effect.
- Compare with Huang and Petriello (HP) (2014) and Grossman, König, and Neubert (GKN) (2015).
  - HP did not include 3.
  - GKN did not include 1 and 4 and did 3 at LL accuracy.
  - GKN used different values for $\langle v^2 \rangle$.
  - We corrected some scale choices in HP. Produced nearly cancelling 30% corrections.
  - We corrected the relative sign of the indirect amplitude in HP.

V	$\mathrm{Br}(Z \to V + \gamma)$	$\mathrm{Br}(Z \to V + \gamma)$ (HP)	$\mathrm{Br}(Z \to V + \gamma)$ (GKN)
J/ψ	$8.96^{+1.51}_{-1.38} \times 10^{-8}$	$(9.96 \pm 1.86) \times 10^{-8}$	$8.02^{+0.46}_{-0.44} \times 10^{-8}$
$\Upsilon(1S)$	$4.80^{+0.26}_{-0.25} \times 10^{-8}$	$(4.93 \pm 0.51) \times 10^{-8}$	$5.39^{+0.17}_{-0.15} \times 10^{-8}$
$\Upsilon(2S)$	$2.44^{+0.14}_{-0.13} \times 10^{-8}$	–	–
$\Upsilon(3S)$	$1.88^{+0.11}_{-0.10} \times 10^{-8}$	–	–

- Our result for $\mathrm{Br}(Z \to J/\psi + \gamma)$ differs from that of HP by -10% and from that of GKN by $+12\%$.
- Our result for $\mathrm{Br}(Z \to \Upsilon(1S) + \gamma)$ differs from that of HP by -3% and from that of GKN by -11%.
- The error bars in GKN seem to be underestimated.
  - Estimated by varying the hard scale μ by a factor two.
  - Does not take into account uncalculated corrections to $\phi_V(x, \mu_0)$ at the heavy-quark scale μ_0.