# Influence of quark masses and strangeness degrees of freedom on inhomogeneous chiral phases

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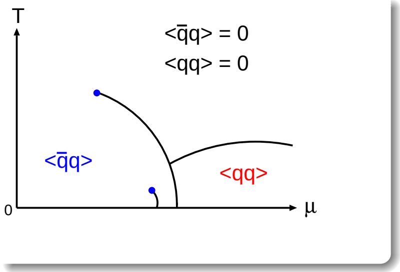
XIIIth Quark Confinement and the Hadron Spectrum,
Maynooth University, Ireland, August 1-6, 2018

# Introduction



- QCD phase diagram (standard picture):



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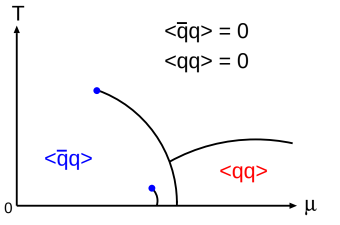


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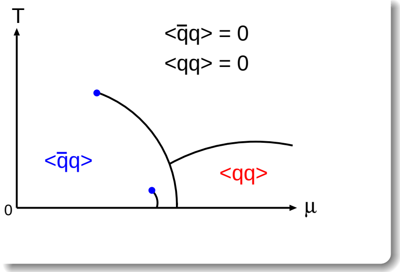
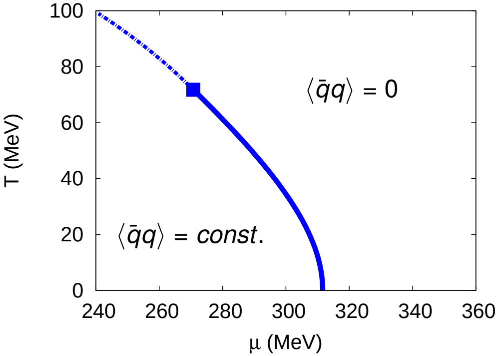



- assumption: $\langle \bar{q}q \rangle$, $\langle qq \rangle$ constant in space
- How about non-uniform phases ?

# Introduction



NJL model, homogeneous phases only




[D. Nickel, PRD (2009)]

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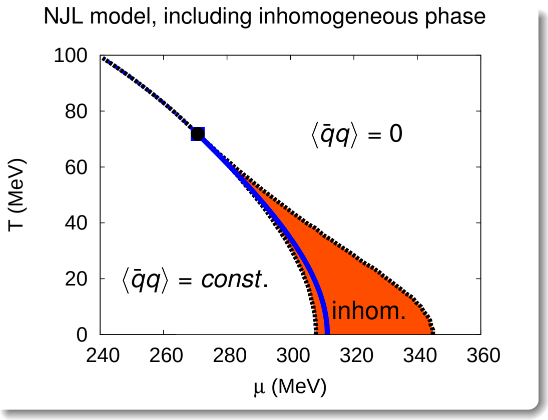


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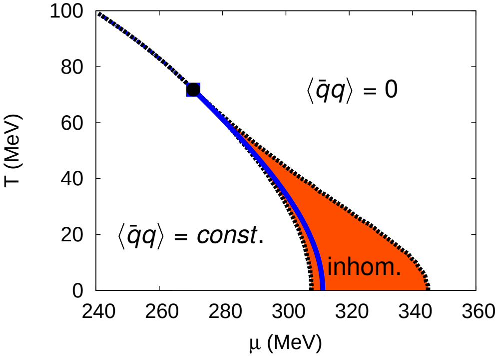


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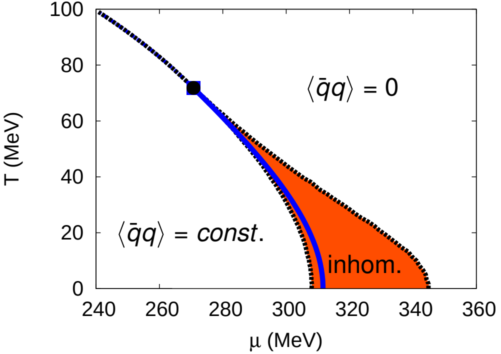


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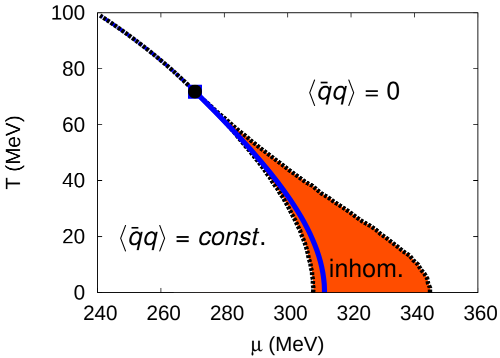


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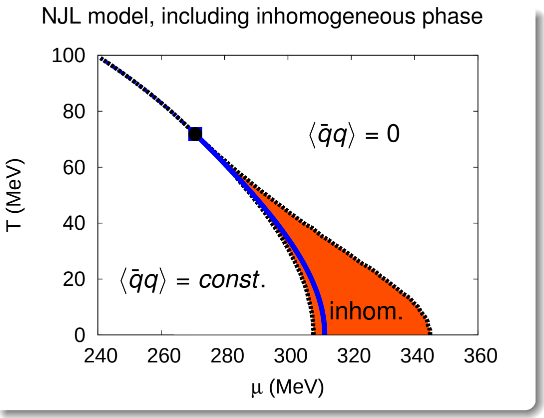


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# Digression: Localized quark matter

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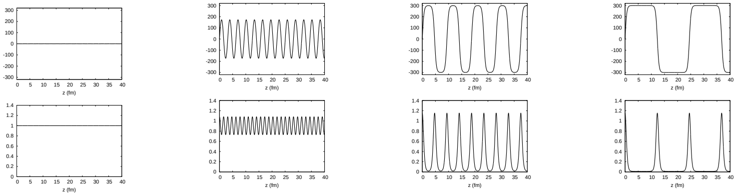
- Particular 1D modulation (most favored solution known so far):

$$\langle \bar{q}q \rangle(z) \propto \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu) \quad \rightarrow \quad \begin{cases} \sqrt{\nu}\Delta \sin(\Delta z) & \text{for} \quad \nu \rightarrow 0 \\ \Delta \tanh(\Delta z) & \text{for} \quad \nu \rightarrow 1 \end{cases}$$

$\mu$ = 345 MeV | 320 MeV | 308 MeV | 307.5 MeV

condensate:

density:




- If it was 3D (but it isn't yet):
  Smooth transition from uniform quark matter to localized "baryons"!
- Revisit chiral solitons ! [Alkofer, Reinhardt, Weigel; Goeke et al.; Ripka; ... ]

# Including strange quarks

# Motivation

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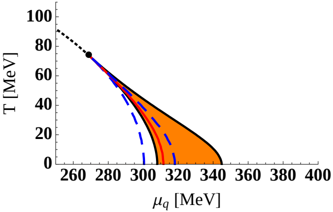
- 2-flavor NJL: CP $\rightarrow$ LP

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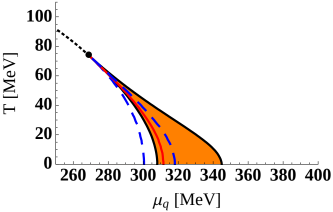


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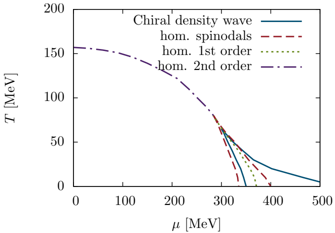
- No proof yet, but similar picture from QCD Dyson-Schwinger studies

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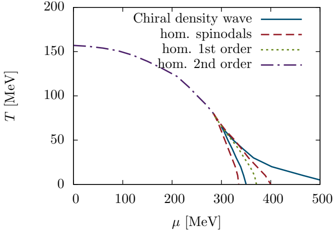


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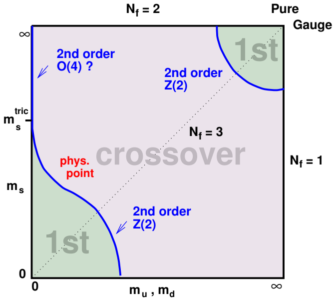
  - CP reaches $T$-axis
  - $\stackrel{?}{\Rightarrow}$ LP reaches $T$-axis
  - chance to be studied on the lattice!




[from de Forcrand et al., POSLAT 2007]

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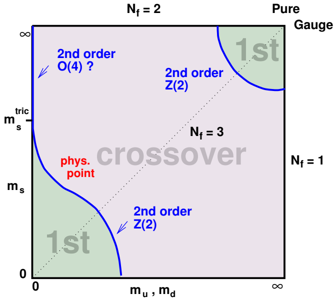


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- Here: Ginzburg-Landau study of CP and LP for 3-flavor NJL

# Ginzburg-Landau analysis

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- Expansion of the thermodynamic potential:

$$\Omega[\Delta] = \Omega[0] + \frac{1}{V}\int_V d^3x \left\{ a_2 |\Delta(\vec{x})|^2 + a_{4,a}(\vec{x})|\Delta|^4 + a_{4,b}|\vec{\nabla}\Delta(\vec{x})|^2 + ... \right\}$$

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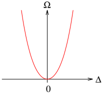

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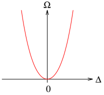


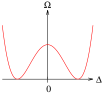

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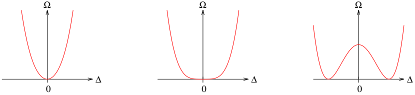

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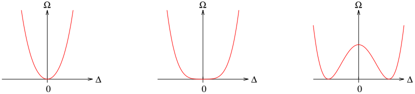


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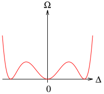

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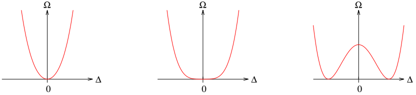


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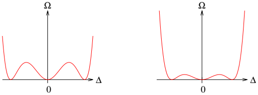

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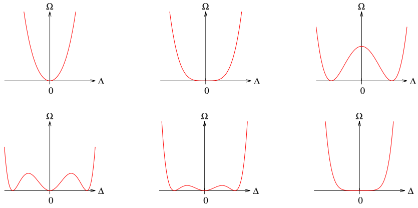


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Lifshitz point (CP): $a_2 = a_{4,b} = 0$

- 2-flavor NJL: $a_{4,a} = a_{4,b}$ $\Rightarrow$ CP = LP ! [Nickel, PRL (2009)]

# 3-flavor NJL model

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- Lagrangian: $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - \hat{m})\psi + \mathcal{L}_4 + \mathcal{L}_6$
  - fields and bare masses: $\psi = (u, d, s)^T$, $\hat{m} = \mathrm{diag}_f(0, 0, m_s)$
  - 4-point interaction: $\mathcal{L}_4 = G\sum_{a=0}^{8}\left[(\bar{\psi}\tau_a\psi)^2 + (\bar{\psi}i\gamma_5\tau_a\psi)^2\right]$
  - 6-point ('t Hooft) interaction: $\mathcal{L}_6 = -K\left[\det{}_f\bar{\psi}(1+\gamma_5)\psi + \det{}_f\bar{\psi}(1-\gamma_5)\psi\right]$

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- Mean fields:
  - light sector: $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle \equiv \frac{S}{2}$, $\langle \bar{u}i\gamma_5 u \rangle = -\langle \bar{d}i\gamma_5 d \rangle \equiv \frac{P}{2}$

    ($\Rightarrow$ $\langle \bar{\psi}_\ell \psi_\ell \rangle \equiv \langle \bar{u}u \rangle + \langle \bar{d}d \rangle = S$, $\langle \bar{\psi}_\ell i\gamma_5\tau_3\psi_\ell \rangle \equiv \langle \bar{u}i\gamma_5 u \rangle - \langle \bar{d}i\gamma_5 d \rangle = P$)
  - strange sector: $\langle \bar{s}s \rangle \equiv S_s$, $\langle \bar{s}i\gamma_5 s \rangle = 0$
  - no flavor-nondiagonal mean fields
  - allow for inhomogeneities: $S = S(\vec{x})$, $P = P(\vec{x})$, $S_s = S_s(\vec{x})$

# Mean-field Thermodynamic Potential

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- $\Omega_{MF}(T,\mu) = -\frac{T}{V}\mathrm{Tr}\,\log\left(i\partial\!\!\!/ + \mu\gamma^0 - \hat{M}\right) + \frac{1}{V}\int d^3x\,\mathcal{V}(\vec{x})$
  - dressed "masses": $\hat{M}_{u,d}(\vec{x}) = -(2G - KS_s(\vec{x}))\left(S(\vec{x}) \pm i\gamma_5 P(\vec{x})\right)$

    $\hat{M}_s(\vec{x}) = m_s - 4GS_s(\vec{x}) + \frac{1}{2}K\left(S^2(\vec{x}) + P^2(\vec{x})\right)$
  - "potential field": $\mathcal{V}(\vec{x}) = G\left(S^2(\vec{x}) + P^2(\vec{x}) + 2S_s(\vec{x})\right) - KS_s(\vec{x})\left(S^2(\vec{x}) + P^2(\vec{x})\right)$

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- $K = 0$: light and strange sectors decouple!

  $\hat{M}_{u,d} = -2G\big(S \pm i\gamma_5 P\big), \quad \hat{M}_s(\vec{x}) = m_s - 4GS_s; \qquad \mathcal{V} = G\big(S^2 + P^2\big) + 2GS_s$

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    $\hat{M}_s(\vec{x}) = m_s - 4GS_s(\vec{x}) + \frac{1}{2}K\left(S^2(\vec{x}) + P^2(\vec{x})\right)$
  - "potential field": $\mathcal{V}(\vec{x}) = G\big(S^2(\vec{x}) + P^2(\vec{x}) + 2S_s(\vec{x})\big) - KS_s(\vec{x})\big(S^2(\vec{x}) + P^2(\vec{x})\big)$
- $K = 0$: light and strange sectors decouple!

  $\hat{M}_{u,d} = -2G\big(S \pm i\gamma_5 P\big), \quad \hat{M}_s(\vec{x}) = m_s - 4GS_s; \qquad \mathcal{V} = G\big(S^2 + P^2\big) + 2GS_s$
- Chiral density wave ansatz for the light sector:

  $S(\vec{x}) = \phi_0\cos(\vec{q}\cdot\vec{x}), \quad P(\vec{x}) = \phi_0\sin(\vec{q}\cdot\vec{x}), \quad S_s = \phi_s = const.$

  $\Rightarrow \quad \hat{M}_{u,d} = \Delta\, e^{\pm i\gamma_5\vec{q}\cdot\vec{x}}, \quad \Delta \equiv -(2G - K\phi_s)\phi_0,$

  $M_s = const., \quad \mathcal{V} = const.$

  consistent with the literature [Moreira et al., PRD (2014)]

# Ginzburg-Landau expansion



- Difficulty at $m_s \neq 0$: No $SU(3)_L \times SU(3)_R$ restored solution
- $m_u = m_d = 0$

  $\Rightarrow$ Expand about two-flavor restored solution $S = P = 0$:

$$\Omega_{MF}[S, P, S_s] = \Omega_{MF}[0, 0, S_s^{(0)}] + \frac{1}{V}\int d^3x\, \Omega_{GL}[S(\vec{x}), P(\vec{x}), X(\vec{x})]$$

  - strange condensate: $S_s(\vec{x}) = S_S^{(0)} + X(\vec{x})$
  - $S_S^{(0)}$: homogeneous solution of the gap equation for $S = P = 0$ at given $T$ and $\mu$
  - Expand $\Omega_{GL}$ in $S$, $P$ and $X$, and their gradients.

# Ginzburg-Landau potential



- Define: $\Delta_\ell = -2G(S + iP), \quad \Delta_s = -4GX$

  $[\Delta_i] = (\text{mass}) \quad \rightarrow \quad$ counting scheme: $\mathcal{O}(\vec{\nabla}) = \mathcal{O}(\Delta_i)$

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- Resulting structure:

$$\begin{aligned}\Omega_{GL} = {} & a_2|\Delta_\ell|^2 + a_{4,a}|\Delta_\ell|^4 + a_{4,b}|\vec{\nabla}\Delta_\ell|^2 \\ & + b_1\Delta_s + b_2\Delta_s^2 + b_3\Delta_s^3 + b_{4,a}\Delta_s^4 + b_{4,b}(\vec{\nabla}\Delta_s)^2 \\ & + c_3|\Delta_\ell|^2\Delta_s + c_4|\vec{\nabla}\Delta_\ell|^2(\vec{\nabla}\Delta_s)^2 \qquad + \ \mathcal{O}(\Delta_i^5)\end{aligned}$$

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$$\Rightarrow \quad M_s^{(0)} = m_s - 16N_c\, G\, T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{M_s^{(0)}}{(i\omega_n+\mu)^2 - \vec{p}^{\,2} - M_s^{(0)\,2}}$$

(= gap equation for $M_s^{(0)} \equiv \hat{M}_s|_{S=P=X=0} = m_s - 4GS_S^{(0)}$)

# Eliminating the strange condensate

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- Extremizing $\Omega_{MF}$ w.r.t. $\Delta_s(\vec{x})$

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  CP and LP don't coincide anymore!

# Discussion

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- Relevant GL coefficients (no guarantee yet!):

$$a_2 = \frac{1}{4G}(1 + 2\delta) + (1+\delta)^2\, 4N_c\, \frac{1}{V_4}\sum \frac{1}{p^2} + \frac{K}{2G^2}\, N_c\, \frac{1}{V_4}\sum \frac{M_s^{(0)}}{p^2 - M_s^{(0)\,2}}$$

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  - $m_s \to 0 \;\Rightarrow\; M_s^{(0)}, S_s^{(0)}, \delta \to 0 \;\Rightarrow\;$ LP $\to$ LP(K=0) $\neq$ CP

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  - $m_s \to 0 \;\Rightarrow\; M_s^{(0)}, S_s^{(0)}, \delta \to 0 \;\Rightarrow\;$ LP $\to$ LP(K=0) $\neq$ CP
- Numerical survey of the general case still to be done.

# Finite bare quark masses

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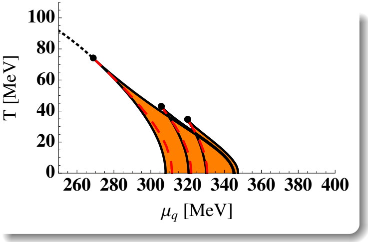

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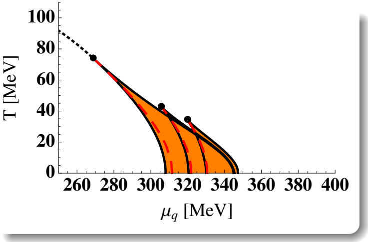


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- Can we investigate this more systematically within GL?

# Ginzburg-Landau analysis with nonzero bare masses

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- No restored phase $\Rightarrow$ Expand about arbitrary homogeneous $\Delta_0$:

  $$\Omega_{GL} = a_1(\Delta - \Delta_0) + a_2(\Delta - \Delta_0)^2 + a_3(\Delta - \Delta_0)^3 + a_{4,a}(\Delta - \Delta_0)^4 + a_{4,b}(\vec{\nabla}\Delta)^2 + \dots$$

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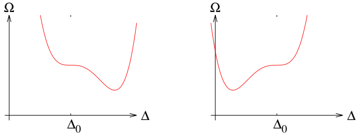

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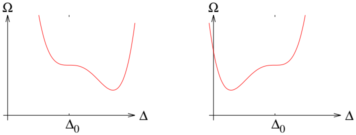


- "Lifshitz point" = upper corner of the inhomogeneous phase?
  - @ CEP: We find $a_{4,b} < 0$ $\Rightarrow$ The CEP is *inside* the inhomogeneous phase.
  - No point with $a_2 = a_{4,b} = 0$ $\Rightarrow$ No point with $\vec{\nabla}\Delta = 0$ at the phase boundary

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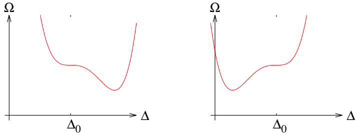


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  Ongoing work: Determine phase boundary via $1 - \Pi_{\sigma,\pi}(\omega = 0, \vec{q}) = 0$

# Conclusions

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- Ginzburg-Landau analysis of the effect of strangeness and bare quark masses on the inhomogeneous chiral phase in NJL
- strange quarks: CP and LP no longer agree
- nonzero $m_{u,d}$ (very preliminary):
  - CEP *inside* the inhomogeneous phase
  - No LP-like point with $\vec{\nabla}\Delta = 0$
- Detailed numerical study to be done.