Entanglement and Entropy production at High Energy.

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A.K. and M. Lublinsky, Phys.Rev. D92 (2015) no.3, 034016 ; A.K., M. Lublinsky and M. Serino arXiv:1806.01089

Motivation

“Collectivity” in small systems”

p-A and even p-p systems not many particles are produced, but collectivity signatures are observed.

Is it “thermalization” or “hydro”?

Or is it something inherited from the proton wave function?

Quantum Entanglement?

Proton wave function is a complicated quantum object.
We only observe part of the degrees of freedom. In the proton they are entangled with the rest.

Could the rest mimic “thermal bath”? After scattering this might produce thermal like signals, or signs of “collectivity”.

A bunch of ideas:
Kharzeev-Levin Phys.Rev. D95 (2017) no.11, 114008, Baker-Kharzeev arXiv:1712.04558 : scattering is a quench that releases the entanglement entropy of observed modes into the final state.

Somewhat different idea: Eigenstate Thermalization Hypothesis.
Scattering produces highly energetic state, which behaves like thermal for certain set of observables.
AMO experiments seem to confirm ETH: find the entanglement entropy of a subsystem grows after quench and attains thermal value.

Learn by Doing

Not entirely clear whether it makes sense, but sounds like fun.

Color Glass Condensate: model of the entangled proton wave function.

There soft modes are strongly entangled with the valence modes.

What entropy does this entanglement carry, and how is this entropy reflected in the particle production in high energy scattering?

The CGC

Color Glass Condensate wave fuction: energetic “valence” partons carry Coulomb color field.

Boosted it becomes Weizsacker-Williams equivalent soft gluons.

$$|\psi\rangle = |s\rangle \otimes |v\rangle \,,$$

$$|s\rangle = \Omega|0\rangle = \exp\left\{ i \int_{q^+<\Lambda} \tilde{b}_a^i(q) \, \left[a_i^a(q^+, \mathbf{q}) + a_i^{\dagger a}(q^+, -\mathbf{q}) \right] \right\} |0\rangle \,,$$

The Weizsacker-Williams field :

$$b_i^a(\mathbf{x}) = \frac{g}{2\pi} \int d^2\mathbf{y} \frac{(\mathbf{x}-\mathbf{y})_i}{(\mathbf{x}-\mathbf{y})^2} \, \rho^a(\mathbf{y}) \,, \qquad b_i^a(\mathbf{q}) = -\frac{i\, g\, \rho^a(\mathbf{q})\, \mathbf{q}_i}{\mathbf{q}^2} \,,$$

where $\rho^a(\mathbf{x})$ - **valence** color charge density in the projectile wave function.

Entangled state

This is typical Born-Oppenheimer situation: slow (small k^-) valence modes determine the state of the faster (higher k^-) soft guons.

Note: the state is strongly entangled. The soft state depends on valence ρ, while ρ is distributed in some way in the valence state $|v\rangle$.

Need nonperturbative information for $|v\rangle$.

Will use the McLerran-Venugopalan (MV) model.

$$\langle \rho^a(\mathbf{x})|v\rangle\langle v|\rho^a(\mathbf{x})\rangle = N \exp\left\{-\frac{1}{2}\int d^2\mathbf{x}\, d^2\mathbf{y}\, \rho^a(\mathbf{x})\mu^{-2}(\mathbf{x}-\mathbf{y})\rho^a(\mathbf{y})\right\} ,$$

Eikonal scattering

When scattering on a target, the partons of the projectile undergo eikonal scattering

$$a^a(\mathbf{x}, p^+) \to S^{ab}(\mathbf{x}) a^b(\mathbf{x}, p^+); \qquad \rho^a(\mathbf{x}) \to S^{ab}(\mathbf{x}) \rho^b(\mathbf{x}) \,.$$

The eikonally scattered CGC soft ground state is

$$\hat{S}\,\Omega\,|0\rangle \otimes |v\rangle = \exp\left\{ i \int_{q^+<\Lambda} \int_{\mathbf{x}} \tilde{b}_i'^a(q^+, \mathbf{x})\, S^{ab}(x) \phi^b(q^+, \mathbf{x}) \right\} |0\rangle \otimes \hat{S}|v\rangle \,,$$

where the Weizsäcker-Williams field of the eikonally rotated charge is

$$b_i'^a(\mathbf{x}) = \frac{g}{2\pi} \int d^2\mathbf{y} \frac{(\mathbf{x}-\mathbf{y})_i}{(\mathbf{x}-\mathbf{y})^2}\, \bar{\rho}^a(\mathbf{y}) \,,$$

and

$$\bar{\rho}^a(\mathbf{y}) = S^{ab}(\mathbf{y})\, \rho^a(\mathbf{y}) \,.$$

Between the scattering time and the observation time it evolves with the "free" QCD Hamiltonian, so that at any time t we have

$$\Psi_{out} = U(0,t)\, \hat{S}\,\Omega\,|0\rangle \otimes |v\rangle; \qquad U(0,t) = \exp\{-iHt\} \,,$$

Goals

We have an explicit model of the projectile wave function before scattering and the scattered state.

The questions we are asking:

1. What is the entanglement entropy between the soft gluons and valence partons?

2. What is the entropy of the ensemble of soft gluons **produced** in the scattering process?

3. Can we sensibly talk about entropy produced in a single event?

The reduced density matrix.

This defines the density matrix (operator) on the soft gluon Hilbert space:

$$\hat{\rho} = \mathcal{N} \int D[\rho]\; e^{-\int_k \frac{1}{2\mu^2(k)}\rho_a(k)\rho_a(-k)} e^{i\int_q b^i_b(q)\phi^i_b(-q)} |0\rangle\langle 0|\; e^{-i\int_p b^j_c(p)\phi^j_c(-p)}$$

with

$$\phi^i_a(k) = a^i_a(k) + a^{\dagger i}_a(-k)$$

Easiest in the "ϕ-basis":

$$\langle\phi|0\rangle = N e^{-\frac{\pi}{2}\int_{\mathbf{k}} \phi^i_a(\mathbf{k})\phi^i_a(-\mathbf{k})}$$

$$\begin{aligned}\langle\phi_1|\hat{\rho}|\phi_2\rangle &= \mathcal{N}\int D[\rho]\; e^{-\int_k \frac{1}{2\mu^2(k)}\rho_a(k)\rho_a(-k)} e^{i\int_q b^i_b(q)[\phi^{1i}_b(-q)-\phi^{2i}_b(-q)]} \\ &\times\quad e^{-\frac{\pi}{2}\int_{\mathbf{k}}[\phi^{1i}_a(\mathbf{k})\phi^{1i}_a(-\mathbf{k})+\phi^{2i}_a(\mathbf{k})\phi^{2i}_a(-\mathbf{k})]}\end{aligned}$$

Entanglement Entropy in the Projectile

The gaussian integral over ρ:

$$\langle\phi_1|\hat{\rho}|\phi_2\rangle = e^{-\frac{1}{2}[\phi^1-\phi^2]M[\phi^1-\phi^2]}e^{-\frac{\pi}{2}[(\phi^1)^2+(\phi^2)^2]}$$

With:

$$M_{ij} \equiv g^2\int_{u,v}\mu^2(u,v)\frac{(x-u)_i}{(x-u)^2}\frac{(y-v)_j}{(y-v)^2}\delta^{ab}$$

Entanglement entropy:

$$\sigma^E = -tr[\hat{\rho}\ln\hat{\rho}]$$

How to calculate ln? The standard "replica trick":

$$\ln\hat{\rho} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\hat{\rho}^\epsilon - 1\right)$$

Calculate ρ^N and take $N\to 1$. N copies of the field - replicas, do the job.

The result

$$\sigma^E = \frac{1}{2}tr\left\{\ln\frac{M}{\pi}+\sqrt{1+\frac{4M}{\pi}}\ln\left[1+\frac{\pi}{2M}\left(1+\sqrt{1+\frac{4M}{\pi}}\right)\right]\right\}$$

Calculating σ^E

Translationally invariant limit (and original MV model):

$$M_{ij}^{ab}(p) = g^2\mu^2 \frac{p_i p_j}{p^4}\delta^{ab}$$

For small M, or the UV contribution

$$\sigma_{UV}^E = tr\left[\frac{M}{\pi}\ln\frac{\pi e}{M}\right] = -\frac{N_c^2-1}{\pi} S\int_{p^2>\frac{Q_s^2}{g^2}} \frac{d^2p}{(2\pi)^2}\frac{Q_s^2}{g^2p^2}\ln\frac{Q_s^2}{eg^2\,p^2}$$

where $Q_s^2 = \frac{g^4}{\pi}\mu^2$ In all σ^E is formally UV divergent

$$\sigma_{UV}^E = \frac{Q_s^2}{4\pi g^2}(N_c^2-1)S\left[\ln^2\frac{g^2\Lambda^2}{Q_s^2} + \ln\frac{g^2\Lambda^2}{Q_s^2}\right]$$

The large M, IR contribution is

$$\sigma_{IR}^E \simeq \frac{1}{2}tr[\ln\frac{e^2M}{\pi}] = \frac{N_c^2-1}{2} S\int_{p2<\frac{Q_s^2}{g^2}} \frac{d^2p}{(2\pi)^2}\ln\frac{e^2\,Q_s^2}{g^2p^2} = \frac{3(N_c^2-1)}{8\pi g^2}SQ_s^2$$

Properties of σ^E.

Putting it together:

$$\sigma^E \approx \sigma^E_{UV} + \sigma^E_{IR} = \frac{SQ_s^2}{4\pi g^2}(N_c^2 - 1)\left[\ln^2 \frac{g^2\Lambda^2}{Q_s^2} + \ln \frac{g^2\Lambda^2}{Q_s^2} + \frac{3}{2}\right]$$

UV divergent: the divergence is cutoff physically at $\Lambda \sim Me^{Y_0} \gg M$, where eikonal approximation breaks down.

Similar to "topological entropy": insensitive to boundary region between the modes.

But not quite what we would like to know.

Density matrix of the produced system of gluons.

After eikonal scattering:

$$\hat{\rho}_S = \mathcal{N} \int D[\rho]\ e^{-\frac{1}{2}\rho\mu^{-2}\rho} e^{i\int_q b^i_b(S\rho)S\phi}|0\rangle\langle 0|\ e^{-ib(S\rho)S\phi}$$

Not quite the right density matrix: it contains gluons that are part of the Weizsacker-Williams cloud of receding charges. We don't want want those gluons in our Hilbert space.

The correct solution: to "project" onto transformed states (before integrating over ρ:

$$\hat{\rho} \to \Omega^\dagger \hat{\rho} \Omega$$

Long story short: same as the incoming density matrix, but with

$$M \to M^P$$

with

$$M^P \equiv g^2 \int_{u,v} \mu^2(u,v) \frac{(x-u)_i}{(x-u)^2} \frac{(y-v)_j}{(y-v)^2} \left[(S(u)-S(x))(S^\dagger(v)-S^\dagger(y))\right]^{ab}$$

Entropy of the produced gluons

The entropy of the system of produced particles is formally

$$\sigma^P = \frac{1}{2}\langle tr\left\{\ln\frac{M^P}{\pi} + \sqrt{1+\frac{4M^P}{\pi}}\ln\left[1+\frac{\pi}{2M^P}\left(1+\sqrt{1+\frac{4M^P}{\pi}}\right)\right]\right\}\rangle_T$$

Here $\langle ...\rangle_T$ is average over the target.

$$\langle M^P\rangle_T = \delta^{ab}\frac{Q_P^2\pi}{g^2}\int_z \frac{(x-z)_i}{(x-z)^2}\frac{(y-z)_j}{(y-z)^2}[P_A(x,y)+1-P_A(x,z)-P_A(z,y)]$$

P_A - S-matrix of an adjoint dipole on the target
Q_p - saturation momentum of the **projectile**.

Entropy and inclusive gluons production.

Expand σ^P around $\bar{M}$ (dilute projectile limit):

$$\sigma^P = tr\left[\frac{\bar{M}}{\pi}\ln\frac{\pi e}{\bar{M}}\right] - \frac{1}{2\pi}\,tr\left[\left\{\langle(M^P-\bar{M})\,(M^P-\bar{M})\rangle_T\right\}\bar{M}^{-1}\right]\ldots.$$

$\bar{M}$ is *almost* single inclusive gluon.

Second term - *almost* **correlated part** of double inclusive gluon production.

Correlations between gluons decrease entropy of the produced state.

We can naturally define **temperature** through: $T^{-1} = \frac{d\sigma}{dE_T}$

Keeping only mean field term in the entropy: $T = \frac{\pi}{2} < k_T >$.

Entropy event-by-event?

What does our calculation mean?

Remember: ρ are slow valence degrees of freedom. Scattering on an energetic target moving with the speed of light is very fast.

So in each scattering event only a fixed configuration of $\rho^a(x)$ is probed.

Reducing the density matrix over ρ is equivalent to averaging over the ensemble of scattering events!

Can we sensibly speak abut entropy of a single event?

Entropy of a pure state?

Strictly speaking for a fixed $\rho^a(x)$ (and $S(x)$) the soft gluons after scattering are in a pure state.

$$|\Psi\rangle_{out} = \Omega^\dagger[\rho]\hat{S}[S]\Omega[\rho]|0\rangle$$

So the entropy vanishes?

Not what we intuitively feel...

“Almost mixed state”

Pure state evolved in time:

$$|\psi(t)\rangle = \sum_n e^{-iE_n t} c_n |\psi_n\rangle \, ,$$

Density matrix:

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)| = \begin{pmatrix} |c_1|^2 & c_1 c_2^* \, e^{i(E_1 - E_2)t} & \dots \\ c_2 c_1^* \, e^{i(E_2 - E_1)t} & |c_2|^2 & \dots \\ \dots & \dots & \dots \end{pmatrix} .$$

Finite time resolution - same as “averaging over time”.
If $T > |E - 1 - E_2|^{-1}$ almost like “reduction”

$$\hat{\rho} \sim \begin{pmatrix} |c_1|^2 & 0 & \dots \\ 0 & |c_2|^2 & \dots \\ \dots & \dots & \dots \end{pmatrix} ,$$

This is mixed and has entropy.

The white noise

Except this is not quite "reduction over part of Hilbert space". Time is not degree of freedom.

But real measurement has time resolution, and provides external degree of freedom coupled to the observed system.

"Calorimeter" degree of freedom ("White noise")

- Should couple to energy;
- Should have a typical time scale;

Couple in the "calorimeter" in the Hilbert space:

$$\hat{\rho}_{P,\xi} = e^{-iH\xi}\,\Omega^\dagger \hat{S}\,\Omega|G\rangle \otimes |0\rangle \otimes |v\rangle\langle v| \otimes \langle 0| \otimes \langle G|\Omega^\dagger\,\hat{S}^\dagger\,\Omega e^{iH\xi}$$

with a (rather arbitrary, but convenient)

$$\langle \xi | G \rangle = e^{-\frac{\xi^2}{2T^2}}$$

The Entropy

For fixed ρ and fixed S - single event

$$\langle \xi_1 | \hat{\rho}'_{P,\xi} | \xi_2 \rangle = \frac{1}{\sqrt{\pi}\, T}\, e^{-\frac{\xi_1^2+\xi_2^2}{2T^2}}\, e^{-iH_0\xi_1}\, e^{i\int_q \Delta\tilde{b}_i^a(q)\,\phi_i^a(q^+,\mathbf{q})} |0\rangle\langle 0| e^{-i\int_{\mathbf{p}} \Delta\tilde{b}_j^b(p)\,\phi_j^b(p^+,\mathbf{p})}\, e^{iH_0\xi_2}\,,$$

with

$$\Delta b_i^a(\mathbf{x}) \equiv \frac{g}{2\pi}\int d^2\mathbf{z}\,\frac{(\mathbf{x}-\mathbf{z})_i}{(\mathbf{x}-\mathbf{z})^2}\Big(S^{ab}(\mathbf{x})-S^{ab}(\mathbf{z})\Big)\,\rho^b(\mathbf{z})\,,$$

$$\Delta b_i^a(\mathbf{q}) = i\,g\int\frac{d^2\mathbf{l}}{(2\pi)^2}\left[\frac{\mathbf{q}_i}{\mathbf{q}^2}-\frac{\mathbf{l}_i}{\mathbf{l}^2}\right]S^{ab}(\mathbf{q}-\mathbf{l})\,\rho^b(\mathbf{l})\,,$$

Now trace over ξ and calculate entropy.

Weak Field Limit

Not completely straightforward, but doable in the weak field limit.

$$\sigma^E_{\Delta b^2 \ll 1} \approx -\left[\int_q \Delta\tilde{b}^2(q)\left(1 - e^{-\frac{E_q^2 T^2}{2}}\right)\right] \log\left[\int_p \Delta\tilde{b}^2(p)\left(1 - e^{-\frac{E_p^2 T^2}{2}}\right)\right] .$$

Very natural interpretation. As time goes by after scattering more and more energy eigenstates "decohere" from each other due to phase oscillations. Those soft gluons are "produced" - or resolved by the apparatus.

Number of particles produced by the time T

$$n(T) = \int_q \Delta\tilde{b}^2(q)\left(1 - e^{-\frac{E_q^2 T^2}{2}}\right),$$

The entropy produced (in a singe event) at time T:

$$\sigma^E(T) = -n(T)\log n(T) .$$

As $T \to \infty$, $n(T) \to \int_q \Delta\tilde{b}^2(q)$ - total number of produced particles in the CGC approximation.

Monogamy of entanglement

An interesting exercise: calculate time dependent entropy (time given by the calorimeter resolution T) of the ensemble of events. That means introduce both the "white noise" ξ, and the reduction over the valence space.

We find:

$$\sigma = -n \log n$$

with

$$n = \langle \int_q \Delta \tilde{b}_i^a(q) \Delta \tilde{b}_i^a(-q) \rangle_{(\rho, S)}$$

No time dependence! (Although it does appear in next order in n.)

Possible reason: "monogamy of entanglement. Two degrees of freedom which are maximally entangled with each other cannot be entangled with anything else.

Summary

None.