Lattice QCD_2 effective action with Bogoliubov transformations

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(in preparation)

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XIIIth Quark Confinement and the Hadron Spectrum

Keypoints and outline

Summary

- Presentation of the 't Hooft model [tH74, BG78, Li86]:
  Wilson lattice formulation, Coulomb gauge, transfer matrix.
- From canonical Bogoliubov formalism [Bar88, KN02]
  to quasiparticle functional representation [CLP07, CPV09].
- Effective action for colourless mesons.

Main results

- Full lattice formulation of a mesonic effective theory,
  from quarks and gluons to complete bosonization.
- Natural interpretation of the hypothesis of boson dominance
  as a projection on colourless mesonic states.

Strategy

Adopt tools and intuition from many-body theories in a relativistic setting.

QCD_2 on the lattice ($T = 0$, $\mu = 0$): motivation

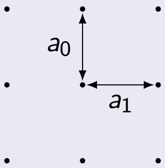
QCD in 2 space-time dimensions and $SU(N_c)$ gauge symmetry.

Toy model for strong interaction

- Quite easy (gauge fixing).
- Solvable for $N_c \to \infty$ (planar limit, 't Hooft '74).
- Non trivial phase space (confinement, chiral symmetry breaking).

Why on the lattice?

- Usual reasons: regularized from the start, non perturbative.
- The particles space of states can be built explicitly, via the formalism of second quantization and transfer matrix.



$\Longrightarrow$ **It's an ideal benchmark for our method!**

Fermionic action

Dirac-Wilson action

$$S_F = a_0 a_1 \sum_{x \in (a_0\mathbb{Z})\times(a_1\mathbb{Z})} \Bigg\{ \left(m + \frac{r_0}{a_0} + \frac{r_1}{a_1}\right) \bar{\psi}(x)\psi(x)$$

$$- \left[\bar{\psi}(x) \frac{r_1 - \gamma^1}{2a_1} U_1(x)\psi(x + a_1\hat{1}) + \bar{\psi}(x + a_1\hat{1}) \frac{r_1 + \gamma^1}{2a_1} U_1^\dagger(x)\psi(x) \right]$$

$$- \left[\bar{\psi}(x) \frac{r_0 - \gamma^0}{2a_0} U_0(x)\psi(x + a_0\hat{0}) + \bar{\psi}(x + a_0\hat{0}) \frac{r_0 + \gamma^0}{2a_0} U_0^\dagger(x)\psi(x) \right] \Bigg\}$$

- ψ, $\bar{\psi}$ fermionic fields (2-component spinors in SU(N_c) fundamental representation);
- γ-matrices terms: lattice discrete derivatives;
- terms with r, Wilson parameter: avoid fermion doubling.

Pure gauge action and weak coupling expansion

U_μ parallel transporter in direction μ. Exponential map:

$$U_\mu(x) = e^{igaA_\mu(x)}$$

Wilson action

$$S_G = \frac{1}{a_0 a_1} \sum_P \frac{1}{g^2} \left[2N_c - \mathrm{Tr}\left(U_P + U_P^\dagger\right)\right]$$

with U_P plaquette variable. Coulomb gauge in 2 dimensions:

$$U_1(x) = 1 \iff A_1(x) = 0$$

't Hooft limit: large N_c $\iff$ weak coupling

$$N_c \to \infty, \qquad g \to 0, \qquad g^2 N_c \text{ fixed}$$

Free gluon propagator at lowest order in g: (→ see Zwanziger talk!)

$$\langle A_0^l(x) A_0^m(y)\rangle = \delta_{lm} \frac{\delta_{x^0,y^0}}{a_0} \int_{-\pi/a_1}^{\pi/a_1} \frac{\mathrm{d}p}{2\pi} \frac{e^{ip\left(x^1-y^1\right)}}{\hat{p}^2}$$

Two equivalent representation for the partition function?

Functional representation

$$\mathcal{Z} = \int \mathcal{D}U\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\; e^{-S[U,\psi,\bar{\psi}]}$$

with $S[U,\psi,\bar{\psi}] = S_G[U] + S_F[U,\psi,\bar{\psi}]$ classical action.

$\Longrightarrow$ correlation functions.

Statistical/canonical representation

$$\mathcal{Z} = \lim_{\beta\to\infty} \int \mathcal{D}U\, e^{-S_G[U]}\, \mathrm{Tr}^F\, e^{-\beta \hat{H}_F} = \int \mathcal{D}U\, e^{-S_G[U]}\, \mathrm{Tr}^F \prod_t \hat{\mathcal{T}}_{t,t+1}$$

with $\hat{H}_F$ fermions Hamiltonian, $\hat{\mathcal{T}}_{t,t+1}$ transfer matrix.

$\Longrightarrow$ thermodynamics.

Which connection?

Building up the fermionc space of states: Fock space

Defined at each time-slice.
Canonical creation and annihilation operators:

$$\left\{\hat{u}_J^\dagger, \hat{u}_K\right\} = \left\{\hat{v}_J^\dagger, \hat{v}_K\right\} = \delta_{JK}, \quad \left\{\hat{u}_J, \hat{u}_K\right\} = \left\{\hat{v}_J, \hat{v}_K\right\} = \cdots = 0$$

with J, K multi-indices: internal (colour, flavour), Dirac and spatial.
Vacuum state:

$$|0\rangle = \bigotimes_K |0\rangle_K \qquad \hat{u}_K |0\rangle_K = 0, \quad \hat{v}_K |0\rangle_K = 0$$

Basis of coherent states:

$$|\rho, \sigma\rangle = \exp\left(-\sum_K \rho_K \hat{u}_K^\dagger - \sum_K \sigma_K \hat{v}_K^\dagger\right) |0\rangle$$

with ρ_K, σ_K anticommuting symbols (Grassmann).

Operatorial $\Longleftrightarrow$ functional representation of $\mathcal{Z}_F$

Resolution of unity:

$$\hat{\mathbb{I}} = \int \prod_K \mathrm{d}\rho_K^\dagger \mathrm{d}\rho_K \mathrm{d}\sigma_K^\dagger \mathrm{d}\sigma_K \, e^{-\sum_J \rho_J^\dagger \rho_J - \sum_J \sigma_J^\dagger \sigma_J} |\rho, \sigma\rangle\langle\rho, \sigma|$$

In the partition function:

$$\begin{aligned}
\mathcal{Z}_F &= \mathrm{Tr}^F \prod_t \hat{\mathcal{T}}_{t,t+1} \\
&= \mathrm{Tr}^F \prod_t \hat{\mathbb{I}}_t \hat{\mathcal{T}}_{t,t+1} \\
&= \mathrm{Tr}^F \prod_t \int \left[\mathrm{d}\rho_t^\dagger \mathrm{d}\rho_t \mathrm{d}\sigma_t^\dagger \mathrm{d}\sigma_t\right] \, e^{-\rho_t^\dagger \rho_t - \sigma_t^\dagger \sigma_t} \langle \rho_t, \sigma_t | \hat{\mathcal{T}}_{t,t+1} | \rho_{t+1}, \sigma_{t+1} \rangle \\
&= \int \prod_t \left[\mathrm{d}\rho_t^\dagger \mathrm{d}\rho_t \mathrm{d}\sigma_t^\dagger \mathrm{d}\sigma_t\right] e^{-S_F[\rho,\sigma]}
\end{aligned}$$

A theory of quasiparticles? Bogoliubov transformations

Unitary transformations of the algebra of canonical operators

Quasiparticle operators

$$\hat{a}_J = \mathcal{R}^{1/2}_{JK}\left(\hat{u}_K - \mathcal{F}^\dagger_{KI}\hat{v}^\dagger_I\right) \qquad \hat{b}_J = \left(\hat{v}_K + \hat{u}^\dagger_I \mathcal{F}^\dagger_{IK}\right)\mathring{\mathcal{R}}^{1/2}_{KJ}$$

$$\hat{a}^\dagger_J = \left(\hat{u}^\dagger_K - \hat{v}_I \mathcal{F}_{IK}\right)\mathcal{R}^{1/2}_{KJ} \qquad \hat{b}^\dagger_J = \mathring{\mathcal{R}}^{1/2}_{JK}\left(\hat{v}^\dagger_K + \mathcal{F}_{KI}\hat{u}_I\right)$$

with

$$\mathcal{R} = \left(1 + \mathcal{F}^\dagger \mathcal{F}\right)^{-1} \qquad \mathring{\mathcal{R}} = \left(1 + \mathcal{F}\mathcal{F}^\dagger\right)^{-1}$$

Mixing of creation and annihilation operators $\implies$ new vacuum state:

$$|\mathcal{F}_t\rangle = \exp\left(\hat{u}^\dagger \mathcal{F}^\dagger_t \hat{v}^\dagger\right)|0\rangle$$

(→ see Reinhardt and Campagnari talks!)

New coherent states:

$$|\alpha, \beta; \mathcal{F}_t\rangle = \exp\left(-\alpha \hat{a}^\dagger - \beta \hat{b}^\dagger\right)|\mathcal{F}_t\rangle$$

Quasiparticles in functional representation

Original theory $\overset{Bogoliubov}{\Longrightarrow}$ Quasiparticles theory unitarily equivalent:

$$\mathcal{Z} = \int \mathcal{D}U \, e^{-S_G[U]} e^{-S_0[\mathcal{F}]} \int \prod_t \left[\mathrm{d}\alpha_t^\dagger \mathrm{d}\alpha_t \mathrm{d}\beta_t^\dagger \mathrm{d}\beta_t \right] e^{-S_Q[\alpha,\beta;\mathcal{F}]}$$

Quasiparticles action

$$S_Q[\alpha, \beta; \mathcal{F}] = -\sum_t \Big[\beta_t \mathcal{I}_t^{(2,1)} \alpha_t + \alpha_t^\dagger \mathcal{I}_t^{(1,2)} \beta_t^\dagger + \alpha_t^\dagger (\nabla_t - \mathcal{H}_t) \alpha_{t+1} - \beta_{t+1} (\mathring{\nabla}_t - \mathring{\mathcal{H}}_t) \beta_t^\dagger \Big]$$

- $\mathcal{I}^{(2,1)}$, $\mathcal{I}^{(1,2)}$ mixing terms;
- $\mathcal{H}$, $\mathring{\mathcal{H}}$ quasiparticles energies;
- ∇, $\mathring{\nabla}$ covariant derivatives.

Vacuum contribution: variational principle

Vacuum action

$S_0[\mathcal{F}]$:

- does not contains quasiparticles excitations;
- depends on the parameters $\mathcal{F}$;
- depends on the gauge fields U_μ.

$\Longrightarrow$ it is a "vacuum contribution".

The physical vacuum must be the state of minimal energy
$\Longrightarrow$ $\mathcal{F}$ can be fixed via a variational principle
$\Longrightarrow$ **saddle point equations** for $\mathcal{F}$, $\mathcal{F}^\dagger$.

But the equations depend on the gauge fields configuration!

In weak coupling, can be solved after **averaging over gauge fields**:

- expand to second order in A_0,
- use $\langle A_0 \rangle = 0$ and substitute $\langle A_0 A_0 \rangle$ with the free gluon propagator.

Vacuum contribution in continuum limit

Bogoliubov-Valatin angle: $\mathcal{F}(q) = \tan\frac{\theta_q}{2}$.

Vacuum contribution

In continuum limit, we get

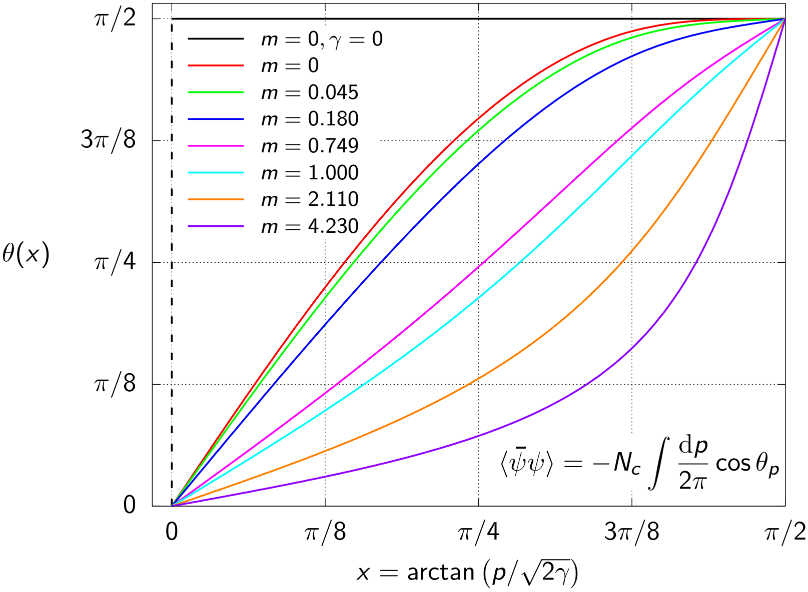
$$S_0[\theta] = -\frac{VN_c}{2\pi}\left[\int \mathrm{d}q\,(m\cos\theta_q + q\sin\theta_q) - \frac{g^2\,(N_c^2-1)}{4\pi N_c}\int \mathrm{d}q\int \mathrm{d}k\,\frac{1}{(q-k)^2}\sin^2\frac{\theta_q-\theta_k}{2}\right]$$

$$\omega_q^0 = m\cos\theta_q + q\sin\theta_q - \gamma\int \mathrm{d}k\,\frac{1}{(q-k)^2}\sin^2\frac{\theta_q-\theta_k}{2}$$

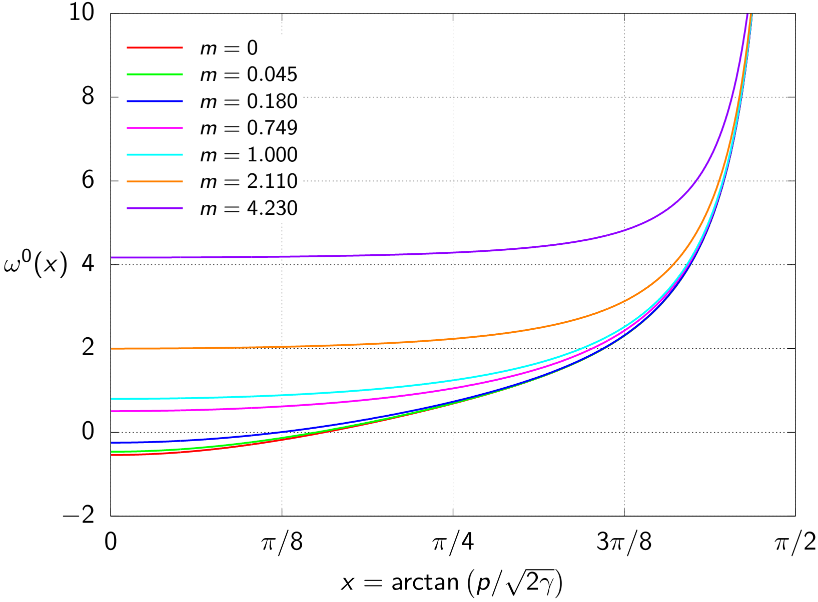
Saddle point equation

$$-m\sin\theta_q + q\cos\theta_q - \frac{\gamma}{2}\int \mathrm{d}k\,\frac{\sin(\theta_q-\theta_k)}{(q-k)^2} = 0$$

θ and chiral symmetry breaking [LWB87, JLLX17]



Vacuum dispersion relation



Quasiparticles on the saddle point

Mixing term

$$\int \frac{\mathrm{d}q}{2\pi} \mathcal{I}_q \left[\beta(q)\alpha(-q) + \alpha^\dagger(q)\beta(-q)\right]$$

$$\mathcal{I}_q = -m\sin\theta_q + q\cos\theta_q - \gamma \int \mathrm{d}k \, \frac{1}{(q-k)^2} \cos\frac{\theta_q - \theta_k}{2} \sin\frac{\theta_q - \theta_k}{2} \Bigg]$$

$\Longrightarrow$ null on the saddle point!

Energy term

$$\int \frac{\mathrm{d}q}{2\pi} \left[\omega_q^\alpha \alpha^\dagger(q)\alpha(-q) + \omega_q^\beta \beta(q)\beta^\dagger(-q)\right]$$

$$\omega_q^{\alpha,\beta} = m\cos\theta_q + q\sin\theta_q + \frac{\gamma}{2} \int \frac{\mathrm{d}k}{2\pi} \frac{\cos(\theta_q - \theta_k)}{(q-k)^2}$$

IR divergent on the saddle point $\Longrightarrow$ quasiparticles confinement.

How to build a theory of mesons?

The average over gauge fields produces a quartic fermionic theory:

$$S_Q \simeq S_Q^{(0)} + gS_Q^{(1)} + g^2 S_Q^{(2)}$$
$$\implies \exp\left(-S_Q\right) \simeq \exp\left(-S_Q^{(0)} - g^2 S_Q^{(2)}\right)\left[1 - gS_Q^{(1)} + \frac{1}{2}g^2\left(S_Q^{(1)}\right)^2\right]$$

Possible interpretation as a quadratic theory for the composite fields

$$\Gamma_n = \sum_{J,K} \Phi_{n;JK}\beta_J\alpha_K \qquad \Gamma_n^\dagger = \sum_{J,K} \Phi^\dagger_{n;JK}\alpha_J^\dagger\beta_K^\dagger$$

Not so easy!

- Bilinears in Grassmann fields are not proper complex fields:

$$(\alpha_J)^2 = 0 \implies (\Gamma_n)^\Omega = 0 \qquad \text{for a certain } \Omega \in \mathbb{N}$$

$\implies$ functional measure $\mathcal{D}\Gamma\,\mathcal{D}\Gamma^\dagger$ not well defined.

- Energy terms still quadratic, and not quartic, in fermion fields.

Mesons as two-particle states in the Fock space

In canonical formalism: mesons as *quasiparticles condensates*

$$|\Phi_t; \mathcal{F}_t\rangle = \exp\left(\hat{a}^\dagger \Phi_t^\dagger \hat{b}^\dagger\right) |\mathcal{F}_t\rangle$$

Physical assumption: **boson dominance** $\implies$ the partition function is "well approximated" by its **projection on composites subspace**:

$$\begin{aligned}\mathcal{Z}_F &= \mathrm{Tr}^F \prod_t \hat{\mathcal{T}}_{t,t+1} \\ &\simeq \mathrm{Tr}^F \prod_t \hat{\mathcal{P}}_t \hat{\mathcal{T}}_{t,t+1} := \mathcal{Z}_C\end{aligned}$$

Projection operator:

$$\hat{\mathcal{P}}_t[\mathcal{F}_t] = \int \frac{\left[\mathrm{d}\Phi_t^\dagger \, \mathrm{d}\Phi_t\right]}{\langle \Phi_t; \mathcal{F}_t | \Phi_t; \mathcal{F}_t\rangle} |\Phi_t; \mathcal{F}_t\rangle\langle \mathcal{F}_t; \Phi_t|$$

A lattice theory of mesons

After projection

$$\mathcal{Z}_c = \int \mathcal{D}\Phi^\dagger \, \mathcal{D}\Phi \; e^{-S_0[U;\mathcal{F}]-S_M[\Phi,\Phi^\dagger,U;\mathcal{F}]}$$

Meson effective action

$$S_M[\Phi, \Phi^\dagger, U; \mathcal{F}] = \sum_t \mathrm{Tr}\left\{\log\left(1 + \Phi_t^\dagger \Phi_t\right) - \log\left(\mathcal{D}_{t,t+1}[\Phi, \Phi^\dagger]\right)\right\}$$

$\mathcal{D}_{t,t+1}$ is a term linear and quartic in the Φ fields.

The action is still not a polynomial in Φ, $\Phi^\dagger$!

A way out:

- choose Φ to describe colourless mesons;
- take the large N_c limit;
- average over gauge and evaluate the result on the saddle point.

Colourless mesons in the large N_c limit

Structure of a colourless meson

- specialize multi-index $J = (p, i)$: space, colour;
- define a suitable creator operator: $\hat{\Gamma}^\dagger(p,q) = \sum_{i=1}^{N_c} \frac{\hat{a}_i^\dagger(p)\hat{b}_i^\dagger(q)}{\sqrt{N_c}}$
- define suitable structure matrices: $\Phi_t^\dagger(p,q) = \mathbb{I}_{N_c} \frac{\phi_t^\dagger(p,q)}{\sqrt{N_c}}$

Quadratic mesonic action:

$$S_M \underset{N_c\to\infty}{\longrightarrow} \sum_t \underset{\text{space}}{\mathrm{tr}} \Big\{ -\phi_t \left(\phi_{t+1}^\dagger - \phi_t^\dagger\right) + \left(\mathring{\mathcal{H}}_t' \phi_t \phi_{t+1}^\dagger + \mathcal{H}_t' \phi_{t+1}^\dagger \phi_t\right) + \frac{1}{2}\left(-2\phi_{t+1}^\dagger \mathring{\mathcal{H}}_t' \phi_t \mathcal{H}_t' + \phi_t \mathcal{I}_t^{(1,2)} \phi_t \mathcal{I}_t^{(1,2)} + \phi_t^\dagger \mathcal{I}_t^{(2,1)} \phi_t^\dagger \mathcal{I}_t^{(2,1)}\right) \Big\}$$

To put it in an usual form of the type $\phi^\dagger\phi$, diagonalize it with respect to the doublets $(\phi^\dagger, \phi) \implies$ Bars-Green equations for colourless mesons [BG78, KN02].

Conclusions and outlooks

Results

Starting from the fundamental theory of quarks and gluons (QCD_2), we obtained

- an effective theory for mesons on the lattice which reproduces, in the continuum limit, results well known from Hamiltonian canonical approach;
- a remarkable physical insight about boson dominance, interpreted as a projection on composite states.

Future perspectives

- application to more realistic models of strong interaction;
- study of models at finite temperature and chemical potential (projection on diquarks states? colour superconductivity? deconfinement?).

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Thank you for your attention.