Mass-deformed Yang-Mills theory in the covariant gauge and its gauge-invariant extension through the gauge-independent BEH mechanism

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and a paper in preparation.

§ Introduction

In this talk we consider

⊙ Mass-deformed Yang-Mills theory in the covariant Lorenz gauge described by the ordinary massless Yang-Mills theory in the (manifestly Lorentz) covariant Lorenz gauge with an additional naive mass term,

$$
\begin{aligned}
\mathscr{L}_{\rm mYM} =& \mathscr{L}_{\rm YM}^{\rm tot} + \mathscr{L}_{\rm m}, \quad \mathscr{L}_{\rm m} = \frac{1}{2} M^2 \mathscr{A}^{\mu A} \mathscr{A}_\mu^A, \\
& \mathscr{L}_{\rm YM}^{\rm tot} = \mathscr{L}_{\rm YM} + \mathscr{L}_{\rm GF} + \mathscr{L}_{\rm FP}, \\
& \mathscr{L}_{\rm YM} = -\frac{1}{4} \mathscr{F}^{\mu\nu A} \mathscr{F}_{\mu\nu}^A, \\
& \mathscr{L}_{\rm GF} = \mathscr{N}^A \partial^\mu \mathscr{A}_\mu^A + \frac{\alpha}{2} \mathscr{N}^A \mathscr{N}^A \to -\frac{1}{2} \alpha^{-1} (\partial^\mu \mathscr{A}_\mu^A)^2 \quad (\alpha \to 0), \\
& \mathscr{L}_{\rm FP} = i \bar{\mathscr{C}}^A \partial^\mu \mathscr{D}_\mu[\mathscr{A}]^{AB} \mathscr{C}^B = i \bar{\mathscr{C}}^A \partial^\mu (\partial_\mu \mathscr{C}^A + g f_{ABC} \mathscr{A}_\mu^B \mathscr{C}^C),
\end{aligned} \tag{1}
$$

with parameters $g, M, \alpha \to 0$. We call this model the massive Yang-Mills theory for simplicity.

• The massive Yang-Mills theory is a non-gauge theory. The gauge symmetry is lost.

• The massive Yang-Mills theory well reproduces propagators and vertices of the decoupling solution in the covariant Landau gauge in the confining phase of the Yang-Mills theory [Tissier, Wschebor, Serreau, Reinosa, ... (2010–), Kondo (2015), ...]
e.g., confinement-deconfinement temperature, $T_c = 0.36M$ for SU(3), $T_c = 0.34M$ for SU(2),

⊙ The massive Yang-Mills theory is a special case of the Curci-Ferrari model [Curci-Ferrari, 1976] and the extension [Baulieu,1985][Baulieu and Thiery-Mieg, 1982]

We can consider a massive extension of the massless Yang-Mills theory in the most general renormalizable gauge having both BRST and anti-BRST symmetries given by [Baulieu (1985)]

$$\mathscr{L}_{m\mathrm{YM}}^{\mathrm{tot}} = \mathscr{L}_{\mathrm{YM}} + \mathscr{L}_{\mathrm{GF+FP}} + \mathscr{L}_m, \tag{2a}$$

$$\mathscr{L}_{\mathrm{YM}} = -\frac{1}{4}\mathscr{F}_{\mu\nu}\cdot\mathscr{F}^{\mu\nu}, \tag{2b}$$

$$\begin{aligned}\mathscr{L}_{\mathrm{GF+FP}} =& \frac{\alpha}{2}\mathscr{N}\cdot\mathscr{N} + \frac{\beta}{2}\mathscr{N}\cdot\mathscr{N} + \mathscr{N}\cdot\partial^\mu\mathscr{A}_\mu - \frac{\beta}{2}g\mathscr{N}\cdot(i\bar{\mathscr{C}}\times\mathscr{C}) \\ &+ i\bar{\mathscr{C}}\cdot\partial^\mu\mathscr{D}_\mu[\mathscr{A}]\mathscr{C} + \frac{\beta}{4}g^2(i\bar{\mathscr{C}}\times\mathscr{C})\cdot(i\bar{\mathscr{C}}\times\mathscr{C}) \\ =& \mathscr{N}\cdot\partial^\mu\mathscr{A}_\mu + i\bar{\mathscr{C}}\cdot\partial^\mu\mathscr{D}_\mu[\mathscr{A}]\mathscr{C} + \frac{\beta}{4}(\bar{\mathscr{N}}\cdot\bar{\mathscr{N}} + \mathscr{N}\cdot\mathscr{N}) + \frac{\alpha}{2}\mathscr{N}\cdot\mathscr{N},\end{aligned} \tag{2c}$$

$$\mathscr{L}_m = \frac{1}{2}M^2\mathscr{A}_\mu\cdot\mathscr{A}^\mu + \beta M^2 i\bar{\mathscr{C}}\cdot\mathscr{C}, \tag{2d}$$

where α and β are parameters corresponding to the gauge-fixing parameters in the $M \to 0$ limit, $\mathscr{D}_\mu[\mathscr{A}]\mathscr{C}(x) := \partial_\mu\mathscr{C}(x) + g\mathscr{A}(x)\times\mathscr{C}(x)$, and $\bar{\mathscr{N}} := -\mathscr{N} + gi\bar{\mathscr{C}}\times\mathscr{C}$.

• The Curci-Ferrari model (1976) with the coupling constant g, the mass parameter M and the parameter β is the $\alpha = 0$ case of this model.

• The massive Yang-Mills theory is regarded as a $\beta = 0$ case of the Curci-Ferrari model.

⊙ How is this model meaningful from the field theoretical point of view?

• renormalizable to all orders of perturbation theory [Kugo,text][Sorella et al.]

• existence of the modified BRST symmetry δ' [Ojima, ...]
This theory does not have the usual BRST symmetry. Instead, it has the modified BRST symmetry.

• lack of physical unitarity [Ferrari and Quadri,2004][Kondo,2013]
the Curci-Ferrari model lacks the physical unitarity at least in the perturbation theory.

• The modified BRST symmetry has the unusual nilpotency of the modified BRST symmetry $\delta'\delta'\delta' = 0$, in disagreement with the ordinary nilpotency $\delta\delta = 0$.

• Gribov problem
Gribov copies exist.

⊙ The massive Yang-Mills theory in the covariant gauge has the gauge-invariant extension.

• [Theory 1] massive Yang-Mills theory in the covariant Landau gauge has no longer gauge symmetry, (although it has the modified BRST symmetry).

However, [Theory 1] has a gauge-invariant extension [Theory 2]. [Theory 2] is the gauge-scalar model

$$\mathscr{L}_{\rm RF} = -\frac{1}{2}{\rm tr}[\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}] + (D_\mu[\mathscr{A}]\Phi)^\dagger \cdot (D^\mu[\mathscr{A}]\Phi), \tag{3}$$

with a single scalar field in the fundamental rep. of the gauge group G subject to the radially-fixed constraint,

$$f(\Phi(x)) := \Phi(x)^\dagger \cdot \Phi(x) - \frac{1}{2}v^2 = 0, \ (v > 0) \text{ for } G = SU(2), \tag{4}$$

if an appropriate constraint which we call the reduction condition is imposed (off shell),

$$\chi(x) := \mathscr{D}_\mu[\mathscr{A}]\mathscr{W}^\mu(x) = 0, \quad \mathscr{W}^\mu = \mathscr{W}^\mu[\mathscr{A}, \Phi]. \tag{5}$$

Here $\mathscr{W}^\mu(x) = \mathscr{W}^\mu[\mathscr{A}(x), \Phi(x)]$ is the massive vector field mode defined shortly in terms of $\mathscr{A}$ and Φ, which follows from the gauge-independent Brout-Englert-Higgs (BEH) mechanism $M = gv/2$.
In other words, if we take the covariant Landau gauge and eliminate the scalar field, [Theory 2] reduces to [Theory 1],

$$\mathscr{L}_{\rm RF} + \mathscr{L}_{\rm GF} + \mathscr{L}_{\rm FP} \to \mathscr{L}_{m\rm YM} = \mathscr{L}_{YM} + \mathscr{L}_m + \mathscr{L}_{GF} + \mathscr{L}_{FP} \tag{6}$$

• Based on this correspondence, [Theory 1] can describe also the Higgs phase by choosing parameters g, M or g, v.
This is regarded as a continuum realization of the Fradkin-Shenker continuity shown on the lattice.

⊙ Fradkin and Shenker Continuity, Osterwalder and Seiler theorem 温故知新

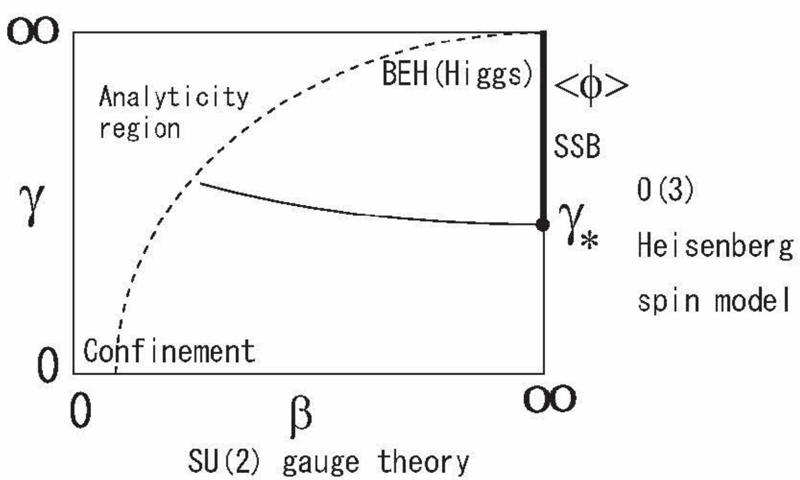
gauge-scalar model in the lattice gauge theory (at zero temperature)

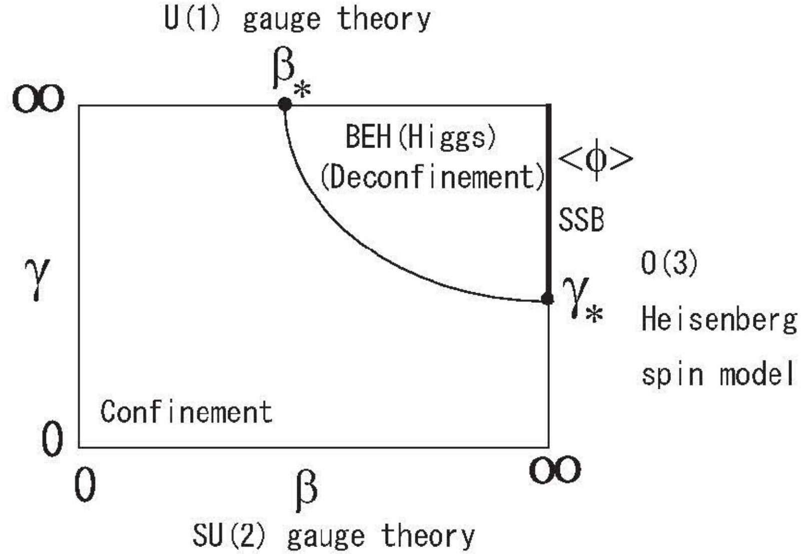
$$S = \beta S_{\text{gauge}}[U] + \gamma S_{\text{scalar}}[\phi, U] \text{ with a gauge group } G \text{ in } D \text{ spacetime dimensions}$$

1. radially fixed scalar field $||\phi(x)||^2 \equiv v^2$ [Fradkin and Shenker,1979]

   Fundamental scalar vs. Adjoint scalar

   $(G = SU(2),\ D = 4)$ $(G = SU(2),\ D = 4)$





In the gauge-scalar model with the fundamental scalar field, Higgs phase and Confinement phase are analytically continued in the phase diagram and are not separated by the phase transition. ("complementarity" between Higgs and Confinement)
This holds for any compact group (continuous and discrete) $G = SU(N), U(1), Z(N)$.

2. Fradkin-Shenker continuity still hold for almost radially fixed scalar field $\lambda \gg 1$ with the potential term $V = \lambda(||\phi(x)||^2 - v^2)^2$ [Osterwalder and Seiler,1978]

• We want to realize the Fradkin-Shenker continuity in quantum field theory on the continuum spacetime. If this is possible, we can understand confinement and gluon mass generation from a different viewpoint based on the BEH mechanism.
[' t Hooft,1979][Fröhlich, Morchio and Strocchi,1980,1981]

However, we immediately encounter the obstructions. Confinement phase respects the gauge symmetry (with no SSB) and confinement is believed to be understood in the gauge-invariant (or gauge-independent) way. The usual description of BEH mechanism is based on spontaneous breaking of the gauge symmetry. How the BEH phase with spontaneously broken gauge symmetry can be continued to the confinement phase with the unbroken gauge symmetry?

We need the gauge-invariant BEH mechanism without relying on SSB.

[Kondo, Phys.Lett. B762, 219-224 (2016), e-Print: arXiv:1606.06194 [hep-th]] $\rightarrow$ partial SSB

[Kondo, Eur.Phys.J. C 78, 577 (2018), e-Print: arXiv:1804.03279 [hep-th]] $\rightarrow$ complete SSB

Contents

§ Massive Yang-Mills theory and decoupling solution

In order to reproduce the decoupling solution, we calculate one-loop quantum corrections to the gluon and ghost propagators in the massive Yang-Mills theory. The Feynman rules are given as follows.

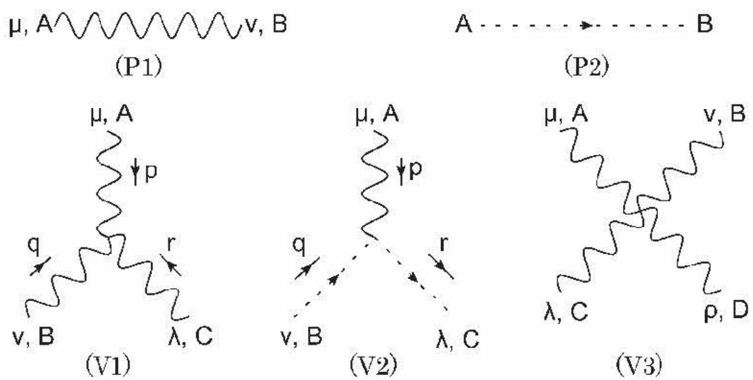


Figure 1: Feynman rules for the covariant gauge massive Yang-Mills theory.

The Feynman rules for the ghost propagator and the vertex functions are the same as those of the usual Yang-Mills theory in the covariant gauge, except for

(P1) gluon propagator $\langle \mathscr{A} \mathscr{A} \rangle$

$$\tilde{D}_{\mu\nu}^{AB}(k) := \frac{-\delta^{AB}}{k^2 - M^2}\left[g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2 - \alpha M^2}\right] \tag{1}$$

$$= \delta^{AB}\left[\frac{-1}{k^2 - M^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2}\right) - \frac{k_\mu k_\nu}{M^2}\frac{1}{k^2 - \alpha M^2}\right] \tag{2}$$

$$= \delta^{AB}\left[\frac{-1}{k^2 - M^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) - \frac{\alpha}{k^2 - \alpha M^2}\frac{k_\mu k_\nu}{k^2}\right], \tag{3}$$

We take into account the one-loop diagrams which contribute to the gluon and ghost propagators.

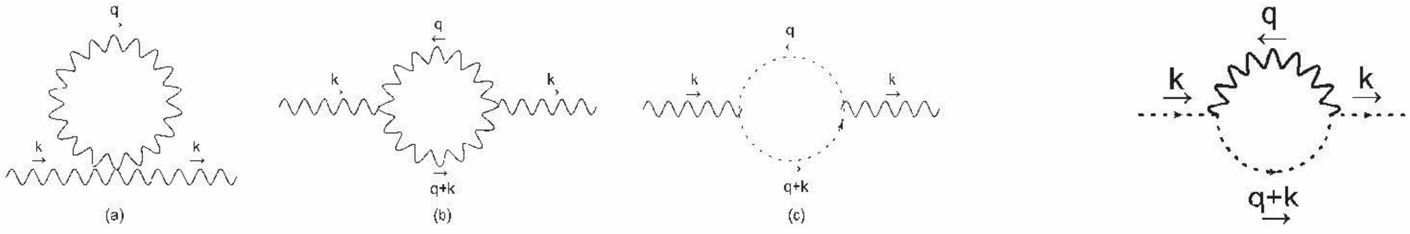


Figure 2: (Left) gluon vacuum polarization diagrams,(Right) ghost self-energy diagram to one loop

For gluons, we introduce the dimensionless vertex function and the propagator as

$$\begin{aligned}\Gamma^{(2)}_{\mathscr{A}}(k) &:= [\mathscr{D}(k)]^{-1} = k^2 + M^2 + \Pi_T(k) + k^2\delta'_Z + M^2\delta'_M \\ &:= M^2[\tilde{\mathscr{D}}(s)]^{-1} = M^2\left[s + 1 + \tilde{\Pi}(s)\right] := M^2\tilde{\Gamma}(s), \quad s := \frac{k^2}{M^2}, \end{aligned} \tag{4}$$

For gluons, we consider the the renormalization conditions.

• [TW] One renormalization condition adopted by [Tissier and Wschebor (2010)] is given in terms of Γ or $\tilde{\Pi}$ as

$$\begin{cases} \Gamma^{(2)}_{\mathscr{A}}(k=0) = M^2 \\ \Gamma^{(2)}_{\mathscr{A}}(k=\mu) = \mu^2 + M^2 \end{cases} \quad \text{or} \quad \begin{cases} \tilde{\Pi}(s=0) = 0 \\ \tilde{\Pi}(s=\nu) = 0 \end{cases} \quad (\text{at} \quad \mu = 1 \text{ GeV}). \tag{TW}$$

• [OS] Another renormalization condition adopted by [Oliveira and Silva (2012)] is given by

$$\begin{cases} \Gamma^{(2)}_{\mathscr{A}}(k=0) = M^2 \\ \Gamma^{(2)}_{\mathscr{A}}(k=\mu) = \mu^2 \end{cases} \quad \text{or} \quad \begin{cases} \tilde{\Pi}(s=0) = 0 \\ \tilde{\Pi}(s=\nu) = -1 \end{cases} \quad (\text{at} \quad \mu = 4 \text{ GeV}). \tag{OS}$$

• The gluon vacuum polarization function in the covariant Landau gauge $\alpha = 0$ has been calculated to one-loop order using the dimensional regularization.
For the renormalization condition [TW], we obtain the renormalized gluon vacuum polarization function,

$$
\begin{aligned}
\tilde{\Pi}_{\mathrm{TW}}(s) = g^2 C_2(G) \frac{1}{48\pi^2}\frac{1}{4} \\
\times s\Bigg[\frac{\mathbf{111}}{\mathbf{2}}\frac{1}{\mathrm{s}} - \frac{1}{s^2} + \left(1 - \frac{s^2}{2}\right)\ln(s) + \left(1+\frac{1}{s}\right)^3 \left(s^2 - 10s + 1\right)\ln(s+1) \\
+ \frac{1}{2}\left(1+\frac{4}{s}\right)^{\frac{3}{2}}\left(s^2 - 20s + 12\right)\ln\left(\frac{\sqrt{4+s}-\sqrt{s}}{\sqrt{4+s}+\sqrt{s}}\right) \boldsymbol{-(s \to \nu)}\Bigg].
\end{aligned} \tag{5}
$$

where constant terms in [...] are canceled by the subtraction: $-(s \to \nu)$.
For another renormalization condition [OS], we have

$$
\hat{\Pi}_{\mathrm{OS}}(s) = \hat{\Pi}_{\mathrm{TW}}(s) - s/\nu \tag{6}
$$

• For ghost, the ghost propagator is related to the ghost self-energy function as

$$
[\Delta_{gh}(k)^{AB}]^{-1} = \delta^{AB}[k^2 + \Pi_{gh}(k) + k^2\delta_C]. \tag{7}
$$

We impose the renormalization condition which is common to both [TW] and [OS]:

$$
\Delta_{gh}(k=\mu) = \frac{1}{\mu^2} \Longleftrightarrow \Gamma^{(2)}_{gh}(k=\mu) = \mu^2 \Longleftrightarrow \tilde{\Pi}_{gh}(s=\nu) = 0. \tag{8}
$$

We use our analytical results (to one-loop order) of the massive Yang-Mills theory to fit the results of numerical simulations on the lattice for the $SU(3)$ Yang-Mills theory in the covariant Landau gauge [A.G. Duarte, O. Oliveira, and P.J. Silva, Phys. Rev. D**94** (2016) 014502.]

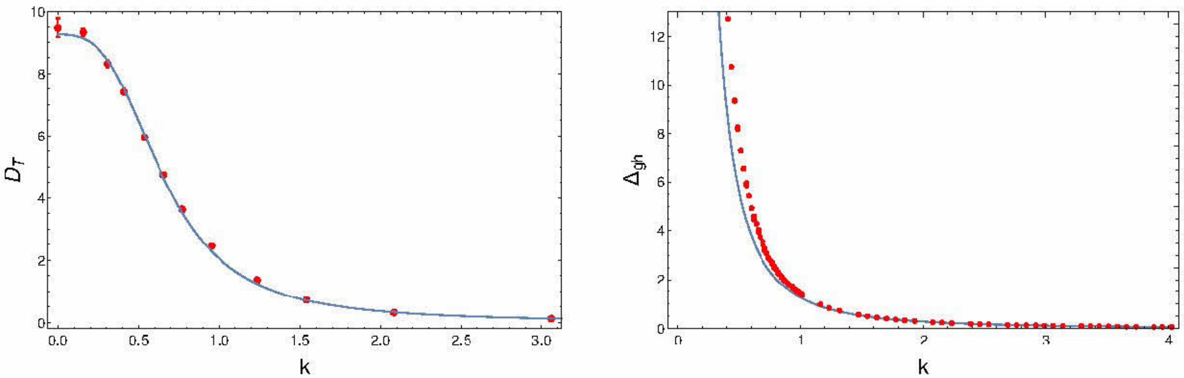


Figure 3: (Left) gluon propagator $\tilde{D}_T(k)$, (Right) ghost propagator $\Delta_{gh}(k)$ for the SU(3) Yang-Mills theory in the covariant Landau gauge. The comparison of the analytical result with numerical simulations [Duarte, Oliveira and Silva (2016)] on the lattice under the renormalization condition [OS] imposed at $\mu = 4$ GeV determines the fitting parameters as $g = 2.4$ and $M = 0.33$.

Both gluon propagator and ghost propagator of the decoupling solution are well reproduced by the same set of parameters (g, M).

$$\text{[OS]:} \quad g = 2.4 \pm 0.4, \quad M = 0.33 \pm 0.05 \text{ GeV}, \tag{9}$$

This should be compared with [M. Tissier and N. Wschebor, Phys.Rev. D**82**, 101701 (2010).]

$$\text{[TW]:} \quad g = 4.9, \quad M = 0.54 \text{ GeV}. \tag{10}$$

§ Gauge-invariant extension

Let $\Phi(x)$ be the $SU(2)$ **doublet** formed from two complex scalar fields $\phi_1(x), \phi_2(x)$,

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}, \ \phi_1(x), \phi_2(x) \in \mathbb{C}. \tag{1}$$

We introduce the **matrix-valued scalar field** Θ by adding another $SU(2)$ doublet

$$\Theta(x) := \begin{pmatrix} \phi_2^*(x) & \phi_1(x) \\ -\phi_1^*(x) & \phi_2(x) \end{pmatrix}, \tag{2}$$

Then the **normalized matrix-valued scalar field** $\hat{\Theta}$ is an element of $SU(2)$:

$$\hat{\Theta}(x) = \Theta(x) / \left(\frac{v}{\sqrt{2}} \right) \in G = SU(2). \tag{3}$$

The massive vector boson field $\mathscr{W}_\mu$ is defined in terms of $\mathscr{A}_\mu$ and $\hat{\Theta} \in G = SU(2)$ [Kondo (2018)] ,

$$\begin{aligned} \mathscr{W}_\mu(x) :=& ig^{-1}(D_\mu[\mathscr{A}]\hat{\Theta}(x))\hat{\Theta}(x)^\dagger = -ig^{-1}\hat{\Theta}(x)(D_\mu[\mathscr{A}]\hat{\Theta}(x))^\dagger \\ =& \frac{1}{2} ig^{-1}[(D_\mu[\mathscr{A}]\hat{\Theta}(x))\hat{\Theta}(x)^\dagger - \hat{\Theta}(x)(D_\mu[\mathscr{A}]\hat{\Theta}(x))^\dagger]. \end{aligned} \tag{4}$$

The kinetic term of the scalar field Θ is identical to the mass term of $\mathscr{W}_\mu$, according to the gauge-indep. BEH mechanism

$$\mathscr{L}_{\text{kin}} = \frac{1}{2}\text{tr}((D_\mu[\mathscr{A}]\Theta(x))^\dagger D^\mu[\mathscr{A}]\Theta(x)) = M^2 \text{tr}(\mathscr{W}_\mu(x)\mathscr{W}^\mu(x)), \ M = g\frac{v}{2}. \tag{5}$$

- The original field is decomposed into the massive vector field $\mathscr{W}_\mu$ and the residual field $\mathscr{R}_\mu$,

$$\mathscr{A}_\mu(x) = \mathscr{W}_\mu(x) + \mathscr{R}_\mu(x), \ \mathscr{R}_\mu(x) := ig^{-1}\hat{\Theta}(x)\partial_\mu\hat{\Theta}(x)^\dagger. \tag{6}$$

Then it is shown that the massive vector boson field $\mathscr{W}_\mu$ has the expression,

$$\mathscr{W}_\mu(x) = \hat{\Theta}(x)\mathscr{A}_\mu^{\hat{\Theta}^\dagger}(x)\hat{\Theta}(x)^\dagger, \tag{7}$$

where $\mathscr{A}_\mu^{\hat{\Theta}^\dagger}$ denotes the gauge transform of $\mathscr{A}_\mu$ by $\hat{\Theta} \in G$. Notice that $\mathscr{W}_\mu$ transforms according to the adjoint representation under the gauge transformation,

$$\mathscr{W}_\mu(x) \to \mathscr{W}_\mu^U(x) = U(x)\mathscr{W}_\mu(x)U(x)^\dagger, \tag{8}$$

whereas $\mathscr{A}_\mu^{\hat{\Theta}^\dagger}$ is gauge invariant,

$$\mathscr{A}_\mu^{\hat{\Theta}^\dagger}(x) \to (\mathscr{A}_\mu^{\hat{\Theta}^\dagger})^U(x) = \mathscr{A}_\mu^{\hat{\Theta}^\dagger}(x). \tag{9}$$

- The stationary reduction condition $\chi(x) = 0$ is cast into

$$\begin{aligned}\chi(x) :=& \mathscr{D}^\mu[\mathscr{A}]\mathscr{W}_\mu(x) \\ =& (\hat{\Theta}(x)\mathscr{D}^\mu[\mathscr{A}^{\hat{\Theta}^\dagger}]\hat{\Theta}(x)^\dagger)(\hat{\Theta}(x)\mathscr{A}_\mu^{\hat{\Theta}^\dagger}(x)\hat{\Theta}(x)^\dagger) \\ =& \hat{\Theta}(x)\mathscr{D}^\mu[\mathscr{A}^{\hat{\Theta}^\dagger}]\mathscr{A}_\mu^{\hat{\Theta}^\dagger}(x)\hat{\Theta}(x)^\dagger = \hat{\Theta}(x)\partial^\mu\mathscr{A}_\mu^{\hat{\Theta}^\dagger}(x)\hat{\Theta}(x)^\dagger.\end{aligned} \tag{10}$$

Therefore, imposing the reduction condition $\chi(x) := \mathscr{D}^{\mu}[\mathscr{A}]\mathscr{W}_{\mu}(x) = 0$ is equivalent to imposing the Landau gauge condition (or transverse condition) for the gauge-invariant field $\mathscr{A}_{\mu}^{\hat{\Theta}^{\dagger}}(x)$,

$$\partial^{\mu}\mathscr{A}_{\mu}^{\hat{\Theta}^{\dagger}}(x) = 0. \tag{11}$$

In the gauge-scalar model, the expectation value of the operator $O[\mathscr{A}]$ of $\mathscr{A}_{\mu}^{A}$ is given by

$$\begin{aligned}\langle O[\mathscr{A}]\rangle =& \frac{\int \mathcal{D}\mathscr{A}\mathcal{D}\Theta\mathcal{D}u e^{iS_{\mathrm{YM}}[\mathscr{A}]+iS_{\mathrm{kin}}[\mathscr{A},\Theta]+iS_{\mathrm{c}}[u,\Theta]}O[\mathscr{A}]}{\int \mathcal{D}\mathscr{A}\mathcal{D}\Theta\mathcal{D}u e^{iS_{\mathrm{YM}}[A]+iS_{\mathrm{kin}}[\mathscr{A},\Theta]+iS_{\mathrm{c}}[u,\Theta]}} \\ =& \frac{\int \mathcal{D}\mathscr{A}\mathcal{D}\Theta\delta(f(\Theta))e^{iS_{\mathrm{YM}}[\mathscr{A}]+iS_{\mathrm{kin}}[\mathscr{A},\Theta]}O[\mathscr{A}]}{\int \mathcal{D}\mathscr{A}\mathcal{D}\Theta\delta(f(\Theta))e^{iS_{\mathrm{YM}}[A]+iS_{\mathrm{kin}}[\mathscr{A},\Theta]}}.\end{aligned} \tag{12}$$

In order to introduce the stationary reduction condition (10) we insert the unity

$$1 = \int \mathcal{D}\chi^{\omega}\delta(\chi^{\omega}) = \int \mathcal{D}(\chi^{\hat{\Theta}^{\dagger}})^{\omega}\delta((\chi^{\hat{\Theta}^{\dagger}})^{\omega}) = \int \mathcal{D}\omega\delta((\partial_{\mu}\mathscr{A}_{\mu}^{h[\mathscr{A}]})^{\omega})\Delta^{\mathrm{red}}, \tag{13}$$

where $\chi^{\theta} := \chi[\mathscr{A}, \Phi^{\theta}]$ is the reduction condition written in terms of $\mathscr{A}$ and Φ^{θ} which is the local rotation of Φ by $\boldsymbol{\theta}$) and $\Delta^{\mathrm{red}} := \det\left(\frac{\delta(\partial_{\mu}\mathscr{A}_{\mu}^{h[\mathscr{A}]})^{\omega}}{\delta\boldsymbol{\omega}}\right)$ denotes the Faddeev-Popov determinant associated with the reduction condition $\chi^{\hat{\Theta}^{\dagger}} = 0$.

• We proceed to eliminate the scalar field Φ or Θ by solving the reduction condition to obtain the pure gauge theory from the complementary gauge-scalar model.

The general form of the transverse and gauge-invariant gauge field $\mathscr{A}_\mu^{h[\mathscr{A}]}$ satisfying (11) can be obtained explicitly by order by order expansion in powers of the gauge field $\mathscr{A}$ up to the Gribov copies. Indeed, $\mathscr{A}_\mu^h$ satisfying the transverse condition, $\partial_\mu \mathscr{A}_\mu^{h[\mathscr{A}]} = 0$, is obtained as a power series in $\mathscr{A}$,

$$\begin{aligned}\mathscr{A}_\mu^{h[\mathscr{A}]} =& \mathscr{A}_\mu^T - ig\frac{\partial_\mu}{\partial^2}\left[\mathscr{A}_\nu, \partial_\nu \frac{\partial\cdot\mathscr{A}}{\partial^2}\right] - \frac{i}{2}g\frac{\partial_\mu}{\partial^2}\left[\partial\cdot\mathscr{A}, \frac{1}{\partial^2}\partial\cdot\mathscr{A}\right] \\ &+ ig\left[\mathscr{A}_\mu, \frac{1}{\partial^2}\partial\cdot\mathscr{A}\right] + \frac{i}{2}g\left[\frac{1}{\partial^2}\partial\cdot\mathscr{A}, \frac{\partial_\mu}{\partial^2}\partial\cdot\mathscr{A}\right] + \mathcal{O}(\mathscr{A}^3),\end{aligned} \tag{14}$$

where we have defined the transverse field $\mathscr{A}_\mu^T$ in the lowest order term linear in $\mathscr{A}$ as $\mathscr{A}_\mu^T := \mathscr{A}_\mu - \partial_\mu \frac{\partial\cdot\mathscr{A}}{\partial^2}$. Then we find that the transverse field $\mathscr{A}_\mu^{h[\mathscr{A}]}$ is rewritten into

$$\begin{aligned}\mathscr{A}_\mu^{h[\mathscr{A}]} &= \left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)\Psi_\nu, \\ \Psi_\nu =& \mathscr{A}_\nu - ig\left[\frac{1}{\partial^2}\partial\cdot\mathscr{A}, \mathscr{A}_\nu\right] + \frac{i}{2}g\left[\frac{1}{\partial^2}\partial\cdot\mathscr{A}, \partial_\nu\frac{1}{\partial^2}\partial\cdot\mathscr{A}\right] + \mathcal{O}(\mathscr{A}^3).\end{aligned} \tag{15}$$

Under an infinitesimal gauge transformation δ_λ defined by $\delta_\lambda \mathscr{A}_\mu = \mathscr{D}_\mu[\mathscr{A}]\lambda := \partial_\mu\lambda - ig[\mathscr{A}_\mu, \lambda]$, Ψ_ν transforms as

$$\delta_\lambda \Psi_\nu = \partial_\nu\left(\lambda - \frac{i}{2}g\left[\frac{\partial\cdot\mathscr{A}}{\partial^2}, \lambda\right]\right) + \mathcal{O}(g^2). \tag{16}$$

Therefore, $\mathscr{A}_\mu^h$ given by (15) is left invariant by infinitesimal gauge transformations order by order of the expansion,

$$\delta_\lambda \mathscr{A}_\mu^{h[\mathscr{A}]}(x) = 0. \tag{17}$$

We can give a recursive construction of the transverse field $\mathscr{A}_\mu^{h[\mathscr{A}]}$ and the proof of gauge invariance of the resulting $\mathscr{A}_\mu^{h[\mathscr{A}]}$.
Therefore, the "mass term" of gauge-invariant field $\mathscr{A}_\mu^h$ is used to writen the kinetic term of the scalar field:

$$
\begin{aligned}
S^*_{\rm kin}[\mathscr{A}] &= \int d^D x \; M^2 {\rm tr}(\mathscr{A}_\mu^{h[\mathscr{A}]} \mathscr{A}^{\mu h[\mathscr{A}]}). \\
&= \int d^D x \; M^2 {\rm tr}\left\{ \mathscr{A}_\mu^T \mathscr{A}^{\mu T} - ig \mathscr{A}_\mu^T \left[\frac{\partial \cdot \mathscr{A}}{\partial^2}, \partial_\mu \frac{\partial \cdot \mathscr{A}}{\partial^2} \right] \right\} + \mathcal{O}(\mathscr{A}^4). \qquad (18)
\end{aligned}
$$

In this way, we have eliminated the scalar field by solving the reduction condition.

Only when we adopt the covariant Landau gauge $\partial \cdot \mathscr{A} = 0$ as the gauge-fixing condition, the infinite number of nonlocal terms disappear so that $S^*_{\rm kin}$ reduces to the naive mass term of $\mathscr{A}$,

$$
S_m[\mathscr{A}] = \int d^D x \; M^2 {\rm tr}(\mathscr{A}_\mu(x) \mathscr{A}_\mu(x)). \qquad (19)
$$

In the Landau gauge, thus, the complementary gauge-scalar model with the reduction condition reduces to the massive Yang-Mills theory with the naive mass term.

The expression of the massive vector field $\mathscr{W}_\mu$ is given

$$
\begin{aligned}
\mathscr{W}_\mu =& \mathscr{A}_\mu^T + \frac{1}{2} ig \frac{1}{\partial^2} \partial_\mu \left[\frac{\partial \cdot \mathscr{A}}{\partial^2}, \partial \cdot \mathscr{A} \right] - ig \frac{1}{\partial^2} \partial_\mu \left[\mathscr{A}_\lambda, \partial_\lambda \frac{\partial \cdot \mathscr{A}}{\partial^2} \right] \\
& - i \frac{1}{2} g \left[\frac{\partial \cdot \mathscr{A}}{\partial^2}, \partial_\mu \frac{\partial \cdot \mathscr{A}}{\partial^2} \right] + \mathcal{O}(\mathscr{A}^3). \qquad (20)
\end{aligned}
$$

§ Positivity violation in the massive Yang-Mills theory

We consider the **Schwinger function** defined by

$$\Delta(t) := \Delta(t, \boldsymbol{p})|_{\boldsymbol{p}=0} := \int d^3x e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \mathscr{D}(t, \boldsymbol{x})|_{\boldsymbol{p}=0} = \int_{-\infty}^{+\infty} \frac{dp_4}{2\pi} e^{ip_4 t} \tilde{\mathscr{D}}(\boldsymbol{p} = 0, p_4). \tag{1}$$

Suppose the Euclidean propagator $\tilde{\mathscr{D}}(p)$ in momentum space has a **spectral representation**, or the **Källen-Lehmann representation** with the nonnegative **spectral function** ρ, i.e., $\rho(\sigma^2) \geq 0$ for all σ^2:

$$\tilde{\mathscr{D}}(p) = \int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 + p^2} \Longrightarrow \Delta(t) = \int_0^\infty d\sigma \rho(\sigma^2) e^{-\sigma t}. \tag{2}$$

This means that the nonnegative $\rho(\sigma^2)$ leads to the positive $\Delta(t)$ for all $t \geq 0$:

$$\rho(\sigma^2) \geq 0 \text{ for all } \sigma^2 \geq 0 \Longrightarrow \Delta(t) > 0 \text{ for all } t \geq 0. \tag{3}$$

Therefore, non-positivity of $\Delta(t)$ at some value of t leads to the positivity violation,

$$\Delta(t) \leq 0 \text{ for some } t \geq 0 \Longrightarrow \rho(\sigma^2) < 0 \text{ for some } \sigma^2 \geq 0. \tag{4}$$

The corresponding states cannot appear in the physical particle spectrum. This is consistent with gluon confinement.

(i) For the free massive propagator, $\Delta(t)$ is positive for any t:

$$\tilde{\mathscr{D}}(p) = \frac{1}{p^2 + m^2} \Longrightarrow \Delta(t) = \int_{-\infty}^{+\infty} \frac{dp_4}{2\pi} e^{ip_4 t} \frac{1}{p_4^2 + m^2} = \frac{1}{2m} e^{-m|t|} > 0. \tag{5}$$

Therefore, there is no reflection-positivity violation for the free massive propagator, as expected. This case corresponds to the spectral function of $\rho(\sigma^2) = \delta(\sigma^2 - m^2) = \frac{1}{2m}\delta(\sigma - m) > 0$.

$\odot$ convariant Landau gauge:

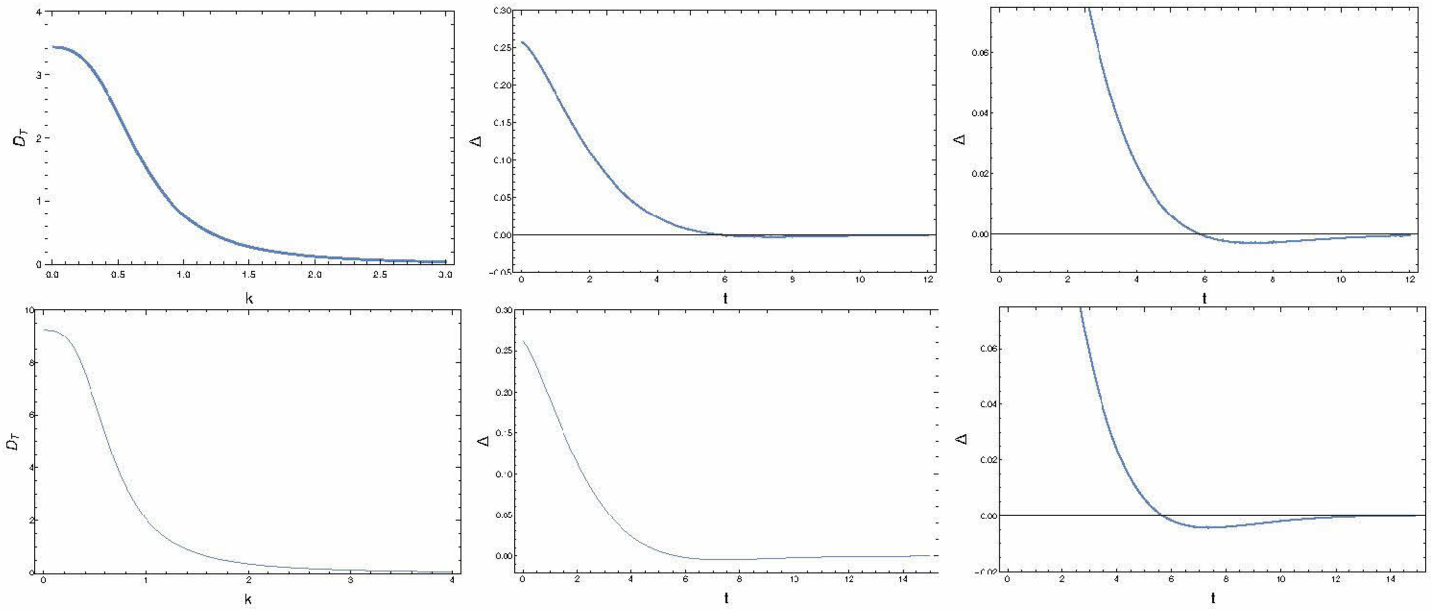
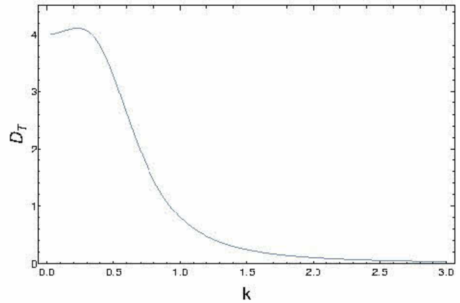
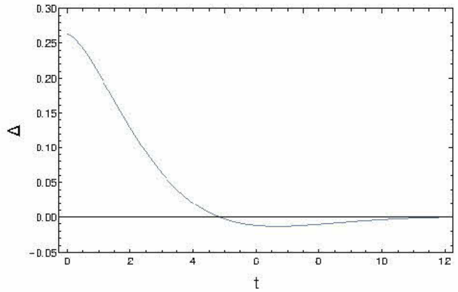


Figure 4: The gluon propagator and the resulting Schwinger function in the convariant Landau gauge for the SU(3) Yang-Mills theory under the renormalization condition
upper: [TW] with $g = 4.9$ and $M = 0.54$ GeV.
lower: [OS] with $g = 2.4$ and $M = 0.33$ GeV.

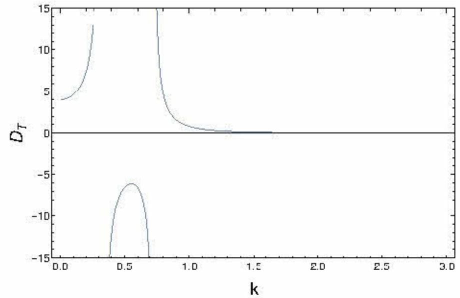
§ Positivity violation/restoration transition of the complementary gauge-scalar model

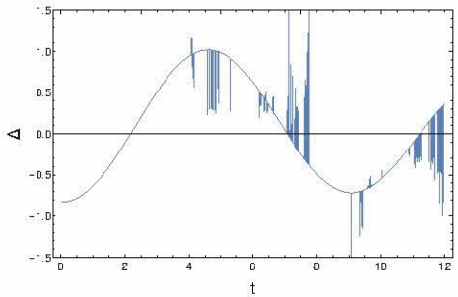
For larger coupling constant g, $g = 6$ and $M = 0.5$ GeV, $\rightarrow$ positivity violation $-$ (blue)



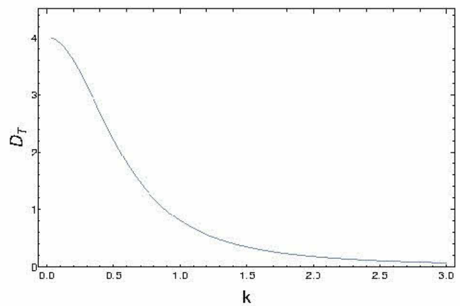


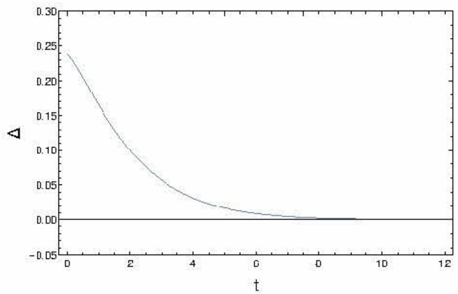
For quite large coupling constant g, $g = 11$ and $M = 0.5$ GeV, $\rightarrow$ artifact $\times$ (orange)





For smaller coupling constant g, $g = 3$ and $M = 0.5$ GeV, $\rightarrow$ positivity restoration $+$ (red)





We examine the positivity violation/restoration in the two-dimensional phase diagram of the complementary $SU(2)$ gauge-scalar model.

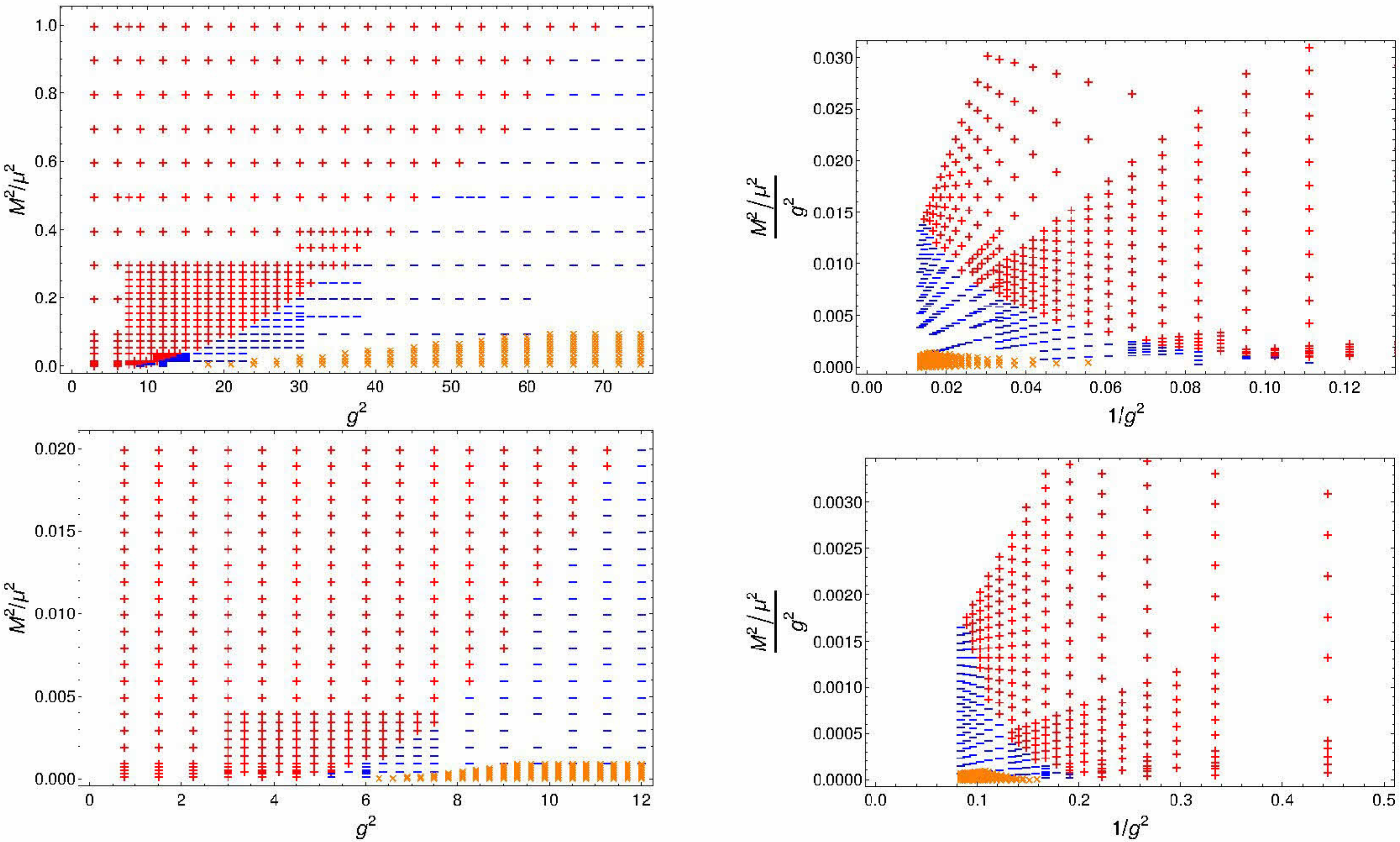


Figure 5: Distribution of positivity violation/restoration in the covariant Landau gauge $\alpha = 0$ for the SU(2) Yang-Mills theory (Left) in the plane $(g^2, \frac{M^2}{\mu^2})$, (Right) in the plane $(1/g^2, \frac{M^2/\mu^2}{g^2})$.
upper renormalization condition [TW]
lower renormalization condition [OS]

$\odot$ Comparison with the Gribov-Zwanziger theory

The $SU(2)$ gauge-scalar model in the covariant Landau gauge was analyzed according to the Gribov-Zwanziger framework [Capri et al. (2012), 1212.1003[hep-th]].

In the case of the fundamental scalar, according to the value of a defined for $M = gv/2$ by

$$a := \frac{g^2 v^2}{4\mu^2} \exp\left[-1 + \frac{32\pi^2}{3g^2}\right] = \frac{M^2}{\mu^2} \exp\left[-1 + \frac{32\pi^2}{3g^2}\right], \tag{1}$$

phases are classified as

$$\begin{cases} \text{(i) } a < 1/e \ \ \rightarrow \text{Gribov type} \Rightarrow \text{Confinement phase} \\ \text{(ii) } 1/e < a < 1/2 \ \ \rightarrow \text{Yukawa type} \\ \text{(iii) } a > 1/2 \ \ \Rightarrow \text{Higgs phase} \end{cases} . \tag{2}$$

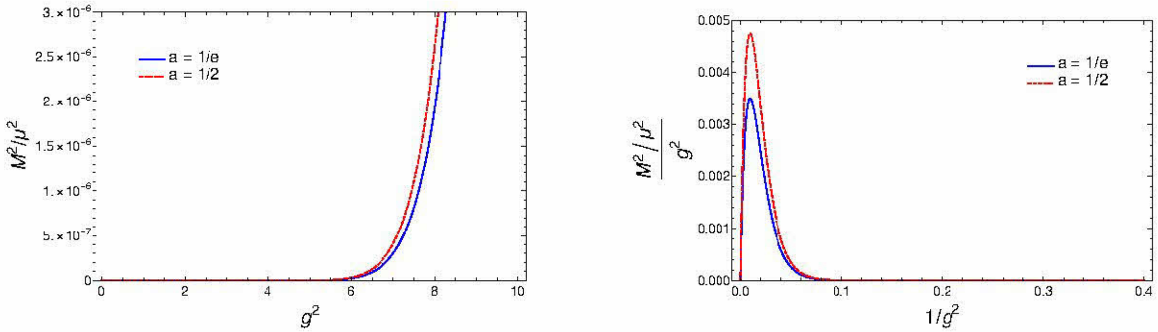


Figure 6: The phase structure of the gauge-fundamental scalar model distinguished by the parameter $a = 1/e$ and $1/2$ Confinement phase is below the blue line while Higgs phase is above the red line.

⊙ Analysis of the phase boundary

By taking the logarithm of (1), we find the linear relation between $\ln \frac{M^2}{\mu^2}$ and $1/g^2$:

$\ln \frac{M^2}{\mu^2} = 1 + \ln a - \frac{32\pi^2}{3g^2}$.

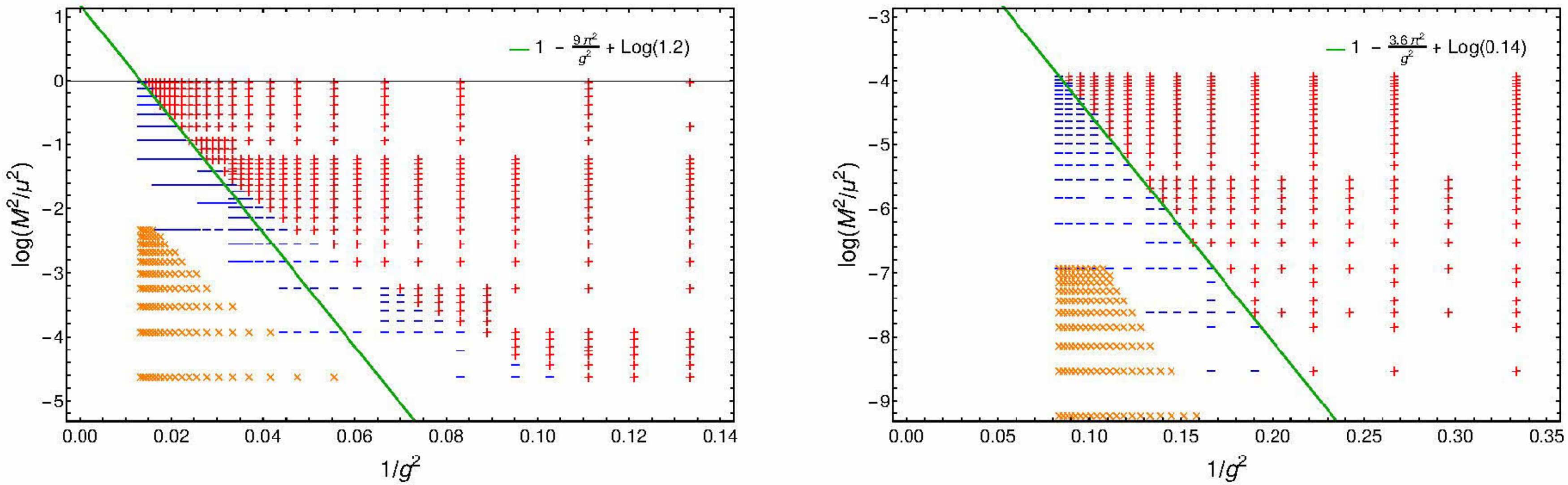


Figure 7: Positivity violation/restoration in the plane $(1/g^2, \ln \frac{M^2}{\mu^2})$ in the covariant Landau gauge $\alpha = 0$ for the renormalization (Left) [TW] (Right) [OS]

$$[TW] \quad \ln \frac{M^2}{\mu^2} = 1 + \ln 1.2 - \frac{9\pi^2}{g^2}, \tag{3}$$

$$[OS] \quad \ln \frac{M^2}{\mu^2} = 1 + \ln 0.14 - \frac{3.6\pi^2}{g^2}. \tag{4}$$

For (TW), it is a good approximation only in the small coupling region. For (OS), it is rather good approximation in all the available region. Therefore, we can find a similar phase structure. to that obtained in the Gribov-Zwanziger approach.

However, it should be remarked that the Gribov-Zwanziger approach reproduces the **scaling solution**, while our approach intends to reproduce the **decoupling solution**.

We have examined the positivity violation/restoration in the phase diagram (g^2, v^2) of the $SU(2)$ gauge-scalar model.

• If $g^2 \to 0$, the theory has no interaction and the propagator is nothing but the free massive propagator $D(k) = \frac{1}{k^2+M^2}$. Therefore, the Schwinger function must be positive in the small g^2 region for any value of M. In fact, our results are consistent with this observation.

• For large g^2 and small M^2/μ^2, namely, for small $1/g^2$ and small $v^2 \simeq (M^2/\mu^2)/g^2$, the Schwinger function exhibits positivity violation. This region corresponds to the confining phase in the gauge-scalar model.

• For small g^2 and large M^2/μ^2, namely, for large $1/g^2$ and large $v^2 \simeq (M^2/\mu^2)/g^2$, the Schwinger function is positive. This region corresponds to the Higgs phase in the gauge-scalar model.

(ii) We consider the propagator of the **Gribov-Stingl type** in Euclidean space,

$$\tilde{\mathscr{D}}(p) = \frac{d_0 + d_1 p^2}{c_0 + c_1 p^2 + c_2 p^4}, \quad p^2 \geq 0, \quad c_0, c_1, c_2, d_0, d_1 \in \mathbb{R}. \tag{5}$$

which has the IR limit: $\tilde{\mathscr{D}}(0) = \frac{d_0}{c_0}$. In the case of $c_2 \neq 0$ [1] ($c_0 \neq 0$ is assumed to obtain a finite value $\tilde{\mathscr{D}}(0) = \frac{d_0}{c_0} < \infty$), the gluon propagator $\tilde{\mathscr{D}}(p)$ has a pair of complex conjugate poles at

$$p^2 = z, z^*, \quad z := x + iy, \quad x := -\frac{1}{2}\frac{c_1}{c_2}, \; y := \sqrt{\frac{c_0}{c_2} - \frac{1}{4}\left(\frac{c_1}{c_2}\right)^2}, \tag{7}$$

as long as irrespective of the value of d_1.

$$c_1^2 < 4c_0c_2 \quad \text{or} \quad \frac{c_0}{c_2} > \frac{1}{4}\left(\frac{c_1}{c_2}\right)^2, \tag{8}$$

In particular, $c_1 = 0$ corresponds to the **Gribov type**, with a pair of pure imaginary poles:

$$p^2 = z, z^* = \pm iy, \; z := iy, \quad y = \sqrt{\frac{c_0}{c_2}}. \tag{9}$$

---

[1] In the case of $c_2 = 0$,

$$\tilde{\mathscr{D}}(p) = \frac{d_0 + d_1 p^2}{c_0 + c_1 p^2} = \frac{d_1}{c_1} + \frac{d_0 - \frac{c_0 d_1}{c_1}}{c_0 + c_1 p^2}, \tag{6}$$

there is no reflection-positivity violation, as far as c_0 and c_1 have the same sign, $c_0/c_1 > 0$. For $c_0/c_1 \leq 0$, the propagator has a real pole, which is excluded.

We find [Kondo, Phys.Rev. D84 (2011) 061702, e-Print: arXiv:1103.3829 [hep-th]]

$$\Delta(t) = \frac{1}{2c_2|z|^{3/2}\sin(2\varphi)} e^{-t|z|^{1/2}\sin\varphi}[d_0\cos(t|z|^{1/2}\cos\varphi - \varphi) + d_1|z|\cos(t|z|^{1/2}\cos\varphi + \varphi)], \tag{10}$$

where

$$\begin{aligned} z =& |z|e^{2i\varphi}, \ |z| = (c_0/c_2)^{1/2}, \\ \cos(2\varphi) =& 2\cos^2\varphi - 1 = -\sqrt{c_1^2/(4c_0c_2)}, \\ \sin(2\varphi) =& 2\sin\varphi\cos\varphi = \sqrt{1 - c_1^2/(4c_0c_2)}. \end{aligned} \tag{11}$$

Here c_0 and c_2 must have the same sign.

The Schwinger function $\Delta(t)$ is oscillatory in t and is negative over finite intervals in the Euclidean time t. Therefore, the reflection positivity is violated for the gluon propagator of the Gribov-Stingl form (5) irrespective of d_1, as long as $0 < c_1^2 < 4c_0c_2$,

$$0 < \frac{c_1^2}{4c_0c_2} < 1. \tag{12}$$

⊙ Fitting to the Gribov-Stingl form

• For the renormalization condition [TW] (TW) at $\mu = 1$ GeV, the lattice results with the parameters $g = 4.9, M = 0.54$ GeV are reproduced by the Gribov-Stingl form with the parameters:

$$c_0 = 0.2896 \pm 0.0003,\ c_1 = 0.417 \pm 0.004,$$
$$c_2 = 1.05 \pm 0.02,\ d_1 = 0.35 \pm 0.01,$$
$$\chi^2_{\text{reduced}} = 1.34189, \Longrightarrow \frac{c_1^2}{4c_0c_2} = 0.143 \pm 0.005. \tag{13}$$

• For the renormalization condition [OS] (OS) at $\mu = 4$ GeV, the lattice results with the parameters $g = 2.47, M = 0.329$ GeV are reproduced by the Gribov-Stingl form with the parameters:

$$c_0 = 0.1074 \pm 0.0002,\ c_1 = 0.144 \pm 0.002,$$
$$c_2 = 0.55 \pm 0.01,\ d_1 = 0.62 \pm 0.03,$$
$$\chi^2_{\text{reduced}} = 1.31751, \Longrightarrow \frac{c_1^2}{4c_0c_2} = 0.088 \pm 0.005. \tag{14}$$

• All results exhibit positivity violation, $0 < \frac{c_1^2}{4c_0c_2} < 1$. This is consistent with the result obtained by directly analyzing the Schwinger function.

§ Conclusion and discussion

• The mass-deformed Yang-Mills theory in the covariant Landau gauge well reproduces the decoupling solution of the pure Yang-Mills theory in the Landau gauge for gluon and ghost propagators in the low-momentum region by choosing a suitable value of parameters g and M.

• The mass-deformed Yang-Mills theory in the covariant Landau gauge has the gauge-invariant extension, namely, the complementary gauge-scalar model with a radially fixed fundamental scalar field subject to an appropriate reduction condition.
In other words, the gauge-scalar model with a radially fixed fundamental scalar field subject to the reduction condition can be gauge-fixed to obtain the mass-deformed Yang-Mills theory in the covariant Landau gauge.

[a non-gauge theory = a gauge-fixed version of the gauge-invariant theory]

• The gauge-invariant extension of a non-gauge theory is performed through the gauge-independent description of the BEH mechanism.

• The mass-deformed Yang-Mills theory with parameters g and M can be analytically continued to the larger region in the phase diagram of the complementary gauge-scalar model which has positivity violation/restoration corresponding to gluon confinement/deconfinement phase as suggested from the Fradkin-Shenker continuity.

In the complementary gauge-scalar model, the scalar field Φ and the gauge field $\mathscr{A}$ are not independent field variables, because we intend to obtain the massive pure Yang-Mills theory which does not contain the scalar field Φ. Therefore, the scalar field Φ is to be eliminated in favor of the gauge field $\mathscr{A}$.

This is in principle achieved by solving the reduction condition as an off-shell equation, which is different from solving the equation of motion for the scalar field Φ in the Stueckelberg formalism [Stueckelberg(1938), Kunimasa-Goto (1967), Cornwall (1974,1982), Delbourgo and Thompson (1986)]. Consequently, the resulting massive Yang-Mills theory in the covariant gauge-fixing term and the associated Faddeev-Popov ghost term becomes power-counting renormalizable in the perturbative framework, as shown in the next section.

Moreover, the entire theory is invariant under the usual Becchi-Rouet-Stora-Tyutin (BRST) transformation δ_{BRST}. The nilpotency $\delta_{BRST}\delta_{BRST} = 0$ of the usual BRST transformations ensures the unitarity of the theory in the physical subspace of the total state vector space determined by zero BRST charge according to Kugo and Ojima [Kugo-Ojima (1979)] .

This situation should be compared with the Curci-Ferrari model [Curci-Ferrari (1976)] which is not invariant under the ordinary BRST transformation, but instead can be made invariant under the modified BRST transformation δ'_{BRST}. However, this fact does not guarantee the unitarity due to the lack of usual nilpotency of the modified BRST transformation satisfying $\delta'_{BRST}\delta'_{BRST}\delta'_{BRST} = 0$.

We start from the spectral representation of the propagator,

$$\tilde{\mathscr{D}}(k^2) = \int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 + k^2}. \tag{1}$$

This means that, if the positivity holds $\rho(\sigma^2) \geq 0$, the propagator $\tilde{\mathscr{D}}(k^2)$ is a positive function of k^2,

$$\rho(\sigma^2) \geq 0 \text{ for all } \sigma^2 \geq 0 \Longrightarrow \tilde{\mathscr{D}}(k^2) > 0 \text{ for all } k^2 \geq 0. \tag{2}$$

Therefore, non-positivity of the propagator $\tilde{\mathscr{D}}(k^2)$ at some value of k^2 leads to the positivity violation,

$$\tilde{\mathscr{D}}(k^2) \leq 0 \text{ for some } k^2 \geq 0 \Longrightarrow \rho(\sigma^2) < 0 \text{ for some } \sigma^2 \geq 0. \tag{3}$$

For the first derivative, we find

$$\tilde{\mathscr{D}}^{(1)}(k^2) := \frac{d}{dk^2}\tilde{\mathscr{D}}(k^2) = -\int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{(\sigma^2 + k^2)^2}. \tag{4}$$

This means that, if the positivity holds $\rho(\sigma^2) \geq 0$, the propagator $\tilde{\mathscr{D}}(k^2)$ is monotonically decreasing function of k^2,

$$\rho(\sigma^2) \geq 0 \text{ for all } \sigma^2 \geq 0 \Longrightarrow \tilde{\mathscr{D}}^{(1)}(k^2) < 0 \text{ for all } k^2 \geq 0. \tag{5}$$

Therefore, non-monotonicity of the propagator $\tilde{\mathscr{D}}(k^2)$ at some value of k^2 leads to the positivity violation,

$$\tilde{\mathscr{D}}^{(1)}(k^2) \geq 0 \text{ for some } k^2 \geq 0 \Longrightarrow \rho(\sigma^2) < 0 \text{ for some } \sigma^2 \geq 0. \tag{6}$$

For the one-loop result, it is easy to check that the first derivative of the vacuum polarization function takes the infinitely large negative value,

$$\frac{d}{ds}\hat{\Pi}(s)|_{s=0} = -\infty. \tag{7}$$

This comes from the $s \ln s$ term appearing in the vacuum polarization function $\hat{\Pi}$ which has the power-series expansion in s around $s = 0$ apart from $s^n \ln s$ terms,

$$\hat{\Pi}(s) = \frac{g^2 C_2(G)}{48\pi^2}\frac{1}{4}\left[C + \sum_{n=1,3} c_n s^n \ln s + O(s)\right], \quad s := \frac{k^2}{M^2}, \tag{8}$$

with non-vanishing coefficients, $c_1 = 1$, $c_3 = -1/2$, and a constant C depending on the renormalization condition adopted.

Therefore, even if $\mathscr{D}_{\mathrm{T}}(k^2) < \infty$ including $\mathscr{D}_{\mathrm{T}}(0) < \infty$ at $k^2 = 0$, the first derivative diverges positively at the origin,

$$\mathscr{D}_{\mathrm{T}}^{(1)}(0) = -\, M^4 \mathscr{D}_{\mathrm{T}}(0)^2 \left[1 + \frac{d}{ds}\hat{\Pi}(s)|_{s=0}\right] = +\infty. \tag{9}$$

Thus, the positivity is violated for any choice of the two parameters g and M, as far as the one-loop expression is accepted at face value for representing the propagator in the deep momentum limit $k^2 = 0$.

We require analyticity or finiteness of the derivatives of the propagator at $k^2 = 0$,

$$\tilde{\mathscr{D}}^{(n)}(k^2 = 0) < \infty. \tag{10}$$

This requirement removes the $s^n \ln s$ terms.

We have shown that the gluon propagator is well fitted by the Gribov-Stingl form, once the $s^n \ln s$ term is removed. For the propagator of the Gribov-Stingl form,

$$\tilde{\mathscr{D}}(k^2) = \frac{d_0 + d_1 k^2}{c_0 + c_1 k^2 + c_2 k^4}, \ k^2 \geq 0, \ c_0, c_1, c_2, d_0, d_1 \in \mathbb{R}, \tag{11}$$

the first derivative is given by

$$\tilde{\mathscr{D}}^{(1)}(k^2) = \frac{(c_0 d_1 - c_1 d_0) - 2c_2 d_0 k^2 - c_2 d_1 k^4}{(c_0 + c_1 k^2 + c_2 k^4)^2}. \tag{12}$$

According to the fitted data, the coefficients satisfy the relations,

$$c_0 d_1 - c_1 d_0 < 0, \ c_2 d_0 > 0, \ c_2 d_1 > 0, \tag{13}$$

which implies that the propagator is monotonically decreasing for all $k^2 \geq 0$,

$$\tilde{\mathscr{D}}^{(1)}(k^2) < 0 \text{ for all } k^2 \geq 0. \tag{14}$$

On the other hand, it is shown that the Schwinger function obtained from the gluon propagator of the Gribov-Stingle type exhibits positivity violation and hence $\rho(\sigma^2) \leq 0$ for some σ^2. Nevertheless, this does not mean non-monotonicity of the propagator, since (14) holds. Therefore, the converse of (5): monotonicity means positivity, is not true, namely, the positivity violation does not necessarily mean the non-monotonicity, although non-monotonicity of the propagator $\tilde{\mathscr{D}}(k^2)$ at some value of k^2 leads to the positivity violation as implied from (5). The monotonicity and positivity violation can be compatible.