The static quark potential in Maximal Abelian gauge with perturbation theory

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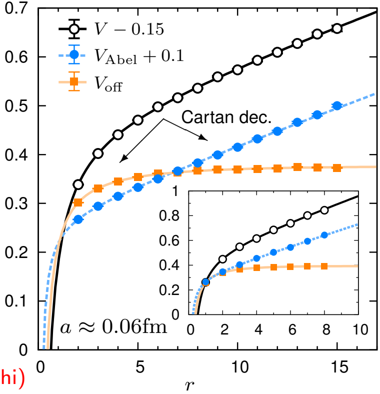
The static quark potential in the Maximal Abelian gauge (MAG)

A popular approach to explain confinement:

- chromoelectric field in vacuum behaves very similar to magnetic field in type II superconductor
- by duality argument: one may expect confinement to arise from condensation of chromomagnetic monopoles
- no monopoles as fundamental d.o.f. but they appear as singularities in a particular gauge configuration (MAG)



picture arxiv:1412.8489 (H. Suganuma, N. Sakumichi)

Key features of lattice results

- evidence for magnetic monopoles found in MAG
- Abelian dominance: linear part of potential seems to be completely contained in contribution from Abelian gluons (in particular monopole part)

Diagonal and off-diagonal generators

Example: SU(3) generators:

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad T^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$T^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad T^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Properties for general SU(N):

- $N^2 - 1$ hermitian traceless $N \times N$ matrices
- $N - 1$ diagonal generators ($\rightarrow$ Abelian subgroup)
- normalized as $\mathrm{Tr}\left[T^A, T^B\right] = \frac{1}{2}\delta^{AB}$

Color index convention

$a, b, \ldots$ for off-diagonal, $i, j, \ldots$ for diagonal, $A, B, \ldots$ for both

The Wilson loop and its Abelian projection

The Wilson loop: $W = \mathcal{P} \exp\left[-ig \int_\Box dx_\mu (A^{A\,\mu}(x) T^A)\right]$

- contour $\Box$ is a square with temporal extent T and spatial extent r
- temporal Wilson lines are static quark propagators, spatial Wilson lines ensure gauge invariance
- large time limit gives potential: $V_0(r) = \lim_{T\to 0} \frac{i}{T} \ln\langle W \rangle$ (up to ultrasoft corrections at three-loop order)
- gauge-invariant and renormalizable (mult. constant + charge renormalization)

Abelian projection: $W^{(AP)} = \mathcal{P} \exp\left[-ig \int_\Box dx_\mu (A^{i\,\mu}(x) T^i)\right]$

- only diagonal gluons contribute
- invariant under diagonal gauge transformations
- desirable to use a gauge that respects this symmetry

The Maximal Abelian gauge (MAG)

Global gauge condition:

$$\delta \int d^4x \, A^a_\mu(x) A^{a\,\mu}(x) = 0$$

Attempts to minimize the effect of off-diagonal gluons.

Infinitesimal variations $\quad \delta A^a_\mu = \frac{1}{g} D^{ab}_\mu \omega^b \quad \Rightarrow \quad$ Local gauge condition

$$\left(\partial_\mu \delta^{ab} - g f^{abi} A^i_\mu\right) A^{b\,\mu} = 0$$

Residual gauge symmetry for *diagonal* transformations U:

$$A^a_\mu T^a \to U(A^a_\mu T^a)U^\dagger \,, \qquad A^i_\mu T^i \to U(A^i_\mu T^i)U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger$$

Gauge fix also diagonal gluons:

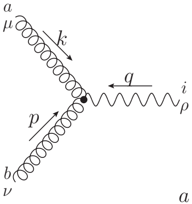
$$\partial_\mu A^{i\,\mu} = 0$$

MAG Feynman rules

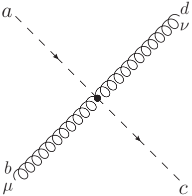
Comparison to *covariant gauges*:

- all propagators are the same in both gauges
- different vertices for diagonal or off-diagonal gluons
- additional vertices of two ghosts and two gluons
- gauge fixing parameter ξ dependent terms in three- and four-gluon vertices
- $\xi \to 0$ limit diverges for individual diagrams, but all singularities cancel

Examples:



$$= g f^{abi} \left[g_{\mu\nu}(k-p)_\rho + g_{\nu\rho}\left(\frac{1}{\xi}k + p - q\right)_\mu + g_{\rho\mu}\left(q - k - \frac{1}{\xi}p\right)_\nu \right]$$

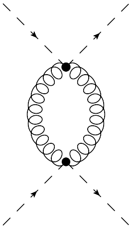


$$= -ig^2\left(f^{abi}f^{cdi} + f^{adi}f^{cbi}\right) g_{\mu\nu}$$

Gauge fixing and renormalizability

The Faddeev-Popov MA gauge fixing is not strictly renormalizable:

- four-ghost amplitude contains new divergences, e.g.
- can in principle be renormalized with four-ghost interaction
- BRST formalism allows such a term



But only Faddeev-Popov method has clear connection to MA gauge condition:

- ξ parameter from delta function: $\delta(x) = \lim_{\xi\to 0} \frac{1}{\sqrt{2\pi\xi}} e^{-x^2/2\xi}$
- ghost sector is representation of functional determinant
- only $\xi \to 0$ limit corresponds to true MAG

Conclusion

We will continue to use the Faddeev-Popov gauge fixing in order to get the correct limit for MAG.
(The extra ghost divergences do not appear at two-loop order and it is not even clear that they will contribute at higher orders.)

The two-loop calculation of the Wilson loop

Outline of the program:

- use available diagram generator
- make modifications for MAG Feynman rules:
  diagonal/off-diagonal propagators and extra vertices
- translate diagram into color factor (including subtractions for logarithm)
- evaluate color factors with Fierz identities:

$$T^i_{IJ} T^i_{KL} = \frac{1}{2}\left(\delta_{IJKL} - \frac{1}{N}\delta_{IJ}\delta_{KL}\right) \qquad T^a_{IJ} T^a_{KL} = \frac{1}{2}\left(\delta_{IL}\delta_{KJ} - \delta_{IJKL}\right)$$

- insert Feynman rules and turn diagrams into standard integrals
- solve integrals

At 2-loop order

Full Wilson loop: 2404 diagrams → 1971 log diagrams → 771 integrals
Abelian projection: 287 diagrams → 200 log diagrams → 377 integrals

The Laporta Algorithm

How to solve large number of standard integrals automatically?

$$1-\text{loop}: \qquad I_{[n_1,n_2,n_3]}(k^2) = \int \frac{d^Dp}{(2\pi)^D} \frac{1}{p_0^{n_1} p^{2n_2} (k+p)^{2n_3}} \qquad (D = 4-2\epsilon)$$

$$2-\text{loop}: \qquad I_{[n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8,n_9]}(k^2)$$

$$= \iint_{p\,q} \frac{1}{p_0^{n_1} q_0^{n_2} (p_0+q_0)^{n_3} p^{2n_4} q^{2n_5} (p+q)^{2n_6} (k+p)^{2n_7} (k+q)^{2n_8} (k+p+q)^{2n_9}}$$

Solution: integration by parts

$$0 = \int \frac{d^Dp}{(2\pi)^D} \partial_\mu p^\mu \frac{1}{p_0^{n_1} p^{2n_2} (k+p)^{2n_3}} \qquad \text{(example)}$$

$$= (D - n_1 - 2n_2 - n_3) I_{[n_1,n_2,n_3]} - n_3 I_{[n_1,n_2-1,n_3+1]} + n_3 k^2 I_{[n_1,n_2,n_3+1]}$$

- indices can be systematically lowered with this and other identities
- only a handful of "master integrals" remain (with known solutions)
  1-loop: 1 integral 2-loop: 5 integrals

Full potential

Schematic result

$$V(r) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\, \frac{4\pi C_F \alpha_s(k)}{k^2} \left[1 + \frac{\alpha_s(k)}{4\pi} a_1 + \left(\frac{\alpha_s(k)}{4\pi}\right)^2 a_2 + \mathcal{O}\left(\alpha_s^3\right)\right]$$

$$= \frac{\alpha_s(1/r)}{r} C_F \left[1 + \frac{\alpha_s(1/r)}{4\pi}\left(a_1 + 2\beta_0\gamma_E\right) + \left(\frac{\alpha_s(1/r)}{4\pi}\right)^2 \left(a_2 + 2\left(2a_1\beta_0 + \beta_1\right)\gamma_E + 4\beta_0^2\gamma_E^2 + \frac{\pi^2}{3}\beta_0^2\right) + \mathcal{O}\left(\alpha_s^3\right)\right].$$

Coefficients

$$a_1 = \frac{31}{9}N - \frac{10}{9}n_f\,, \qquad C_F = \frac{N^2-1}{2N}$$

$$a_2 = \left(\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta(3)\right)N^2 - \left(\frac{899}{81} + \frac{28}{3}\zeta(3)\right)N n_f - \left(\frac{55}{6} - 8\zeta(3)\right)\frac{N^2-1}{2N} n_f + \frac{100}{81} n_f^2$$

Abelian projected potential (in momentum space)

Schematic result

$$\widetilde{V}_{AP}(k) = \frac{4\pi C_{AP}\alpha_s(k)}{k^2}\left[1 + \frac{\alpha_s(k)}{4\pi}\left(a_1 + b_1\xi(k) + c_1\xi(k)^2\right) + \left(\frac{\alpha_s(k)}{4\pi}\right)^2\left(a_2 + b_2\xi(k) + c_2\xi(k)^2 + d_2\xi(k)^3\right) + \mathcal{O}\left(\alpha_s^3\right)\right]$$

Coefficients

$$a_1 = \frac{205}{36}N - \frac{10}{3}n_f\,, \qquad b_1 = \frac{3}{2}N\,, \qquad c_1 = \frac{1}{4}N\,, \qquad C_{AP} = \frac{N-1}{2N}$$

$$a_2 = \left(\frac{90391}{1296} - \frac{57}{8}\zeta(3)\right)N^2 + \left(\frac{347}{24} - \frac{115}{4}\zeta(3)\right)N - \left(\frac{1736}{81} + 4\zeta(3)\right)Nn_f - \left(\frac{55}{6} - 8\zeta(3)\right)\frac{N^2-1}{2N}n_f + \frac{100}{81}n_f^2\,,$$

$$b_2 = \left(\frac{211}{16} + \frac{5}{4}\zeta(3)\right)N^2 + \left(\frac{367}{24} - \frac{7}{2}\zeta(3)\right)N - \frac{5}{3}Nn_f\,,$$

$$c_2 = \left(\frac{191}{48} - \frac{1}{8}\zeta(3)\right)N^2 + \left(\frac{145}{24} + \frac{1}{4}\zeta(3)\right)N\,, \qquad d_2 = \frac{9}{16}N^2 + \frac{7}{8}N$$

Performing the Fourier transform

- Fourier transform introduces extra terms due to the scale dependence of the coupling constant (with $A = \alpha(1/r)/4\pi$ and $L = \ln(k^2r^2)$)

$$\alpha_s(k) = \alpha_s(1/r)\Big[1 - A\beta_0 L - A^2\left(\beta_1 L - \beta_0^2 L^2\right) + \mathcal{O}\left(\alpha_s^3\right)\Big]$$

with $\beta_0 = \frac{11}{3}N - \frac{2}{3}n_f$ and $\beta_1 = \frac{34}{3}N^2 - \frac{10}{3}Nn_f - 2\frac{N^2-1}{2N}n_f$

- at 2-loop order charge renormalization does not remove all divergences
- remaining divergences can be absorbed by renormalization of ξ-parameter
- also ξ becomes scale dependent:

$$\xi(k) = \xi(1/r)\left[1 - A\left(\frac{\zeta_0^{(-1)}}{\xi(1/r)} + \zeta_0^{(0)} + \zeta_0^{(1)}\xi(1/r)\right)L + \mathcal{O}\left(\alpha_s^2\right)\right]$$

with $\zeta_0^{(-1)} = 3$, $\zeta_0^{(0)} = -\frac{13}{6}N + 3 + \frac{2}{3}n_f$, and $\zeta_0^{(1)} = \frac{1}{2}N + 1$

- this scale dependence affects Fourier transform of Abelian projected potential

Abelian projected potential (in position space)

Final expression:

$$
\begin{aligned}
V_{AP}(r) = \frac{\alpha_s(1/r)}{r} C_{AP} \Big[& 1 + A\left(a_1 + 2\beta_0\gamma_E + b_1\xi(1/r) + c_1\xi(1/r)^2\right) \\
& + A^2 \Bigg(a_2 + \left(4a_1\beta_0 + 2b_1\zeta_0^{(-1)} + 2\beta_1\right)\gamma_E + 4\beta_0^2\gamma_E^2 + \frac{\pi^2}{3}\beta_0^2 \\
& + \left(b_2 + \left(4b_1\beta_0 - 2b_1\zeta_0^{(0)} + 4c_1\zeta_0^{(-1)}\right)\gamma_E\right)\xi(1/r) \\
& + \left(c_2 + \left(4c_1\beta_0 + 4c_1\zeta_0^{(0)} + 2b_1\zeta_0^{(1)}\right)\gamma_E\right)\xi(1/r)^2 \\
& + \left(d_2 + 4c_1\zeta_0^{(1)}\gamma_E\right)\xi(1/r)^3 \Bigg) + \mathcal{O}\left(\alpha_s^3\right) \Big]
\end{aligned}
$$

Discussion

Some observations

- full potential is gauge invariant and agrees with known result
- Abelian projected potential similar in structure (no sign of linear part)
- renormalization and scale dependence of ξ-parameter prohibits $\xi \to 0$ limit
- taking bare $\xi_0 \to 0$ would re-introduce divergences
- extra ghost interaction in BRST gauge fixing leads to (finite) difference in two-loop $\mathcal{O}(\xi)$ terms

Conclusions

- perturbative MA gauge may not coincide with lattice gauge
- difficulties in taking the "true" MA gauge limit may hint at missing monopole contributions