On nonequilibrium quarkonium evolution in the QGP fireball

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Heavy quarkonium in the QGP

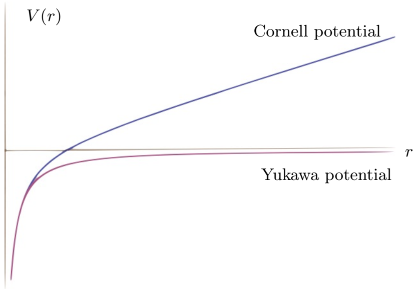
The quarkonium suppression is expected to serve as a signal for the formation of a quark-gluon plasma (QGP). Matsui and Satz 1986

- Heavy quark produced at the beginning of heavy-ion collisions.
- It is sensitive to the properties and evolution of QGP.
- Final state dilepton doesn't interact with other hadrons.

The explanation of Matsui and Satz 1986 is that

- Cornell potential $V(r) = -\frac{\alpha_s}{r} + \sigma r$ at $T = 0$
- Yukawa potential $V(r) = -\frac{\alpha_s e^{-r m_D}}{r}$ and $\sigma = 0$ for $T > T_c$

Screening is stronger at hight T, as the Debye radius $r_D \sim 1/m_D$ decreases with increasing T.

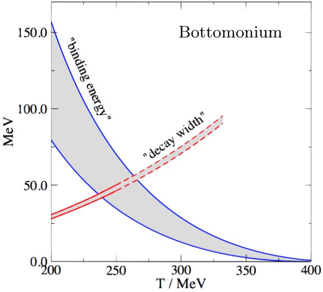


Heavy quarkonium in the QGP

Static potential of heavy quarkonium in real time at finite temperature for $1/r \sim gT$ is found to have an imaginary part. Laine et al. 2007; Brambilla et al. 2008; Beraudo et al. 2008; Rothkopf et al. 2012; Burnier and Rothkopf 2012; Burnier et al. 2014.

$$V(r) = -\frac{\alpha_s e^{-rm_D}}{r} - \alpha_s m_D - i\alpha_s T\phi(rm_D)$$

with $\phi(x)$ a monotonic function with $\phi(0) = 0$ and $\phi(\infty) = 1$, which is related to the decay width of the heavy quark.



Laine et al. 2007

Heavy quarkonium in the QGP

Scales of Heavy quarkonium $M \gg Mv \gg Mv^2$. $v \ll 1$ makes heavy quarkonium a **non-relativistic** bound state.

- M: hard scale
- $Mv \sim p \sim 1/r$: soft scale, momentum transfer between $Q\bar{Q}$
- $Mv^2 \sim E$: ultrasoft(US) scale, binding energy

Thermodynamic scales of the QGP: T, $m_{\mathrm{D}} \sim gT$, $\cdots$

QCD scale: Λ_{QCD}

Mechanisms for quarkonium evolution in QGP includes **regeneration, and dissociation** (color screening and scattering, and gluon absorption).

The different scales for the quarkonium evolution in a thermal medium calls for a effective field theory description of the system.

Effective field theories

The non-relativistic QCD (NRQCD) is carried out by integrating out the hard scale M. Caswell and Lepage 1986; Bodwin et al. 1995

- $M \gg p, E, \Lambda_{QCD}$ ($m_{c,b} = 1.27, 4.2$ GeV, $\Lambda_{QCD} = 0.2$ GeV), the matching can be done perturbatively.
- The Lagrangian is organised as an expansion of $\frac{1}{M}$.

The potential non-relativistic QCD (pNRQCD) is derived by further integrating out the soft scale mv from NRQCD. Pineda and Soto 1998; Brambilla et al. 2000

- $M \gg p \gg E, \Lambda_{QCD}$, heavy quarks and anti-quarks are described with bound states of color-singlet ($\mathrm{S} = S/\sqrt{N_c}$) and octet ($\mathrm{O} = O^a T^a/\sqrt{T_F}$) with Coulomb potential ($h_{s,o} = \frac{\mathbf{p}_1^2}{2M} + \frac{\mathbf{p}_2^2}{2M} + V_{s,o}$).
- The Lagrangian is organised as an expansion of $\frac{1}{M}$ and r.
- To NLO in multipole expansion:

$$
\begin{aligned}
L_{pNRQCD} &= L_{q+g} + \int \mathrm{d}^3 r \Big\{ \mathrm{Tr}\left[\mathrm{S}^\dagger (i\partial_0 - h_s)\mathrm{S} + \mathrm{O}^\dagger (iD_0 - h_s)\mathrm{O}\right] \\
&+ \mathrm{Tr}\left[\mathrm{O}^\dagger \mathbf{r}\cdot g\mathbf{E}\mathrm{S} + \mathrm{S}^\dagger \mathbf{r}\cdot g\mathbf{E}\mathrm{O} + \frac{1}{2}\left(\mathrm{O}^\dagger \mathbf{r}\cdot g\mathbf{E}\mathrm{O} + \mathrm{O}^\dagger \mathrm{O}\mathbf{r}\cdot g\mathbf{E}\right)\right] \Big\}.
\end{aligned}
$$

Transport equations

Quantum transport equations

- Evolution of correlators (or density matrices): Schwinger-Dyson equation, Kandoff-Baym equation Akamatsu 2015, Brambilla et at. 2017

(Semi-)Classical transport equations

- Langevin equation, Fokker-Plank euqaion Blaizot et al. 2015 : **Momentum drag/diffusion coefficient**

$$\dot{\mathbf{r}} = \frac{\mathbf{p}}{M}\,, \qquad M\ddot{\mathbf{r}}_i = -F_i - \eta_{ij}\mathbf{p}_j + \xi_i$$

$\langle \xi_i(t)\xi_i(t')\rangle = \delta(t-t')\lambda_{ij}$ with $\lambda_{ij} = 2MT\eta_{ij}$

$$\left(\frac{\partial}{\partial t} + \frac{\mathbf{p}\cdot\nabla_{\mathbf{r}}}{M}\right)P(t,\mathbf{r},\mathbf{p}) = \nabla_{\mathbf{p}_i}[(\eta_{ij}\mathbf{p}_j + F_i)P] + \frac{1}{2}\nabla_{\mathbf{p}_i}\nabla_{\mathbf{p}_j}[\lambda_{ij}P]$$

- Boltzmann equation Muller et al. 2017: **Loss/gain term**

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathbf{p}\cdot\nabla_{\mathbf{r}}}{M}\right)f(t,\mathbf{r},\mathbf{p}) = -C_- + C_+$$

Relations from quantum to classical transport equations was discussed by e.g. Zurek 1991; Akamatsu 2015; Blaizot and Escobedo 2017; De Boni 2017

The relation between different transport equations is not fully understood. It would be important to derive those equations from QCD.

Open Quantum System

Wikipedia: In physics, an open quantum system is a quantum-mechanical system which interacts with an external quantum system, the environment.

For Heavy quarkonium in high energy heavy-ion collisions: the QGP + heavy quark is considered as a large closed system.

$$H = H_Q + H_{QGP} + H_I$$

The Heavy quarkonium (HQ) evolution is described by

$$\rho_Q(t) = \mathrm{Tr}_{QGP}[\rho(t)]$$

- Interaction between the HQ system and the QGP is weak:

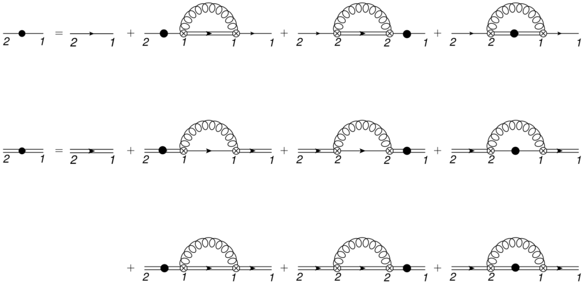
$$\rho(t) = \rho_Q \otimes \rho_{QGP}$$

Evolution of the HQ system:

$$i\frac{\mathrm{d}\rho_Q(t)}{\mathrm{d}t} = [H_Q, \rho_Q(t)] + iD\rho_Q(t)$$

- $iD\rho_Q(t)$ describes the dissipation of the HQ system due to interaction with the QGP.

Density evolution with pNRQCD



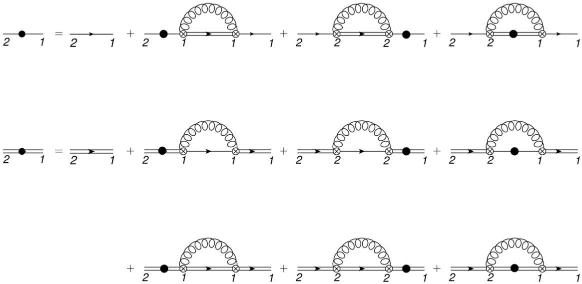
Brambilla, Escobedo et al. 2017

The density matrices for color-singlet and octet are

$$\left\langle \mathbf{r}_1, \mathbf{r}_2 | \rho_s(t,t) | \mathbf{r}_1', \mathbf{r}_2' \right\rangle \equiv \rho_s(t, \mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_1', \mathbf{r}_2') = \left\langle S_1^\dagger(t, \mathbf{r}_1, \mathbf{r}_2) S_2(t, \mathbf{r}_1', \mathbf{r}_2') \right\rangle$$

$$\frac{\delta^{ab}}{N_c^2 - 1} \left\langle \mathbf{r}_1, \mathbf{r}_2 | \rho_o(t,t) | \mathbf{r}_1', \mathbf{r}_2' \right\rangle \equiv \frac{\delta^{ab}}{N_c^2 - 1} \rho_o(t, \mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_1', \mathbf{r}_2') = \left\langle O_1^{a\dagger}(t, \mathbf{r}_1, \mathbf{r}_2) O_2^b(t, \mathbf{r}_1', \mathbf{r}_2') \right\rangle$$

Density evolution with pNRQCD



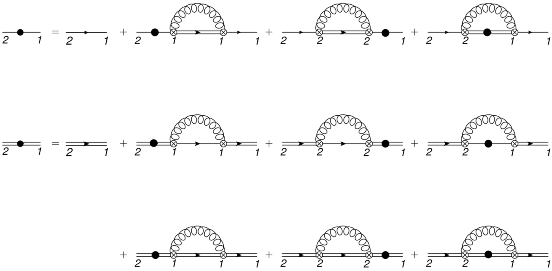
Brambilla, Escobedo et al. 2017

Evolution equations for the density matrix of color-singlet and octet states[1]:

$$
\begin{aligned}
\frac{\mathrm{d}\rho_s(t,\mathbf{r};\mathbf{r}')}{\mathrm{d}t} &= \Big[-i(h_s(\mathbf{r})-h_s(\mathbf{r}')) - (\Sigma_s^\dagger(t,\mathbf{r}')+\Sigma_s(t,\mathbf{r}))\Big]\rho_s(t,\mathbf{r};\mathbf{r}') \\
&\quad +\Xi_{so}(t,\rho_o,\mathbf{r};\mathbf{r}')\ , \\
\frac{\mathrm{d}\rho_o(t,\mathbf{r};\mathbf{r}')}{\mathrm{d}t} &= \Big[-i(h_o(\mathbf{r})-h_o(\mathbf{r}')) - (\Sigma_o^\dagger(t,\mathbf{r}')+\Sigma_o(t,\mathbf{r}))\Big]\rho_o(t,\mathbf{r};\mathbf{r}') \\
&\quad +\Xi_{os}(t,\rho_s,\mathbf{r};\mathbf{r}')+\Xi_{oo}(t,\rho_o,\mathbf{r};\mathbf{r}')\ .
\end{aligned}
$$

[1] Higher order correction has been included on the right hand side: $e^{-ih_{s(o)}(t-t0)}\rho_{s(o)}(t_0,t_0)e^{-ih_{s(o)}} \to \rho_{s(o)}(t,t)$, which makes the evolution a Markovian process.

Self-energies in real time formalism



The self-energies are expressed as[2]

$$\Sigma_s(t, \mathbf{r}_1, \mathbf{r}_2) = \frac{2g^2}{N_\mathrm{c}} \int_0^\infty \mathrm{d}t_1 \left\langle \mathrm{Tr}\left\{I_g(t_1)T^c\right\} e^{-ih_o(\mathbf{r})t_1} \mathrm{Tr}\left\{T^c I_g(0)\right\} e^{ih_s(\mathbf{r})t_1} \right\rangle$$

Gluon field in multipole expansion (with field redefinition):
$I_g = A_0^a(\mathbf{r}_1)T^a - A_0^a(\mathbf{r}_2)\tilde{T^a} \approx A_0^a(\mathbf{0})(T^a - \tilde{T^a}) - \frac{1}{2}\,\mathbf{r}\cdot\mathbf{E}(T^a + \tilde{T^a})$.

- Lindblad equation has been derived in weak and strong coupling plasma. Brambilla, Escobedo, Soto, and Vairo 2017

[2] $\int_0^{t-t_0} \mathrm{d}t_1 \to \int_0^\infty \mathrm{d}t_1$, and neglecting density matrix in 11 and 22 correlators

Quarkonium evolution in a strongly coupled plasma

Scales for strongly coupled pasma: $M \gg p \sim Mv \gg T \sim m_{\mathrm{D}} \sim gT \gg E \sim Mv^2$.

- We can expand the exponentials $e^{\pm i h_{s,o}(t-t_0)} \approx 1 \pm i h_{s,o}(t-t_0)$ with $h_{s,o} = -\frac{\nabla_{\mathbf{r}}^2}{M} + V_{s,o}$. ($h_{s,o}(t-t_0) \ll 1$ as the correlation time $t - t_0 \sim \frac{1}{T}$.)
- The LO expansion is used for deriving **Lindblad equation**.
- The LO and the NLO correspond to the **momentum diffusion** and **momentum drag** in the Fokker-Plank equation and Langevin equation in a semi-classical approximation.

$$\Sigma_s(t,\mathbf{r}) = \frac{g^2}{2N_c} \int_0^\infty \mathrm{d}t' \left\{ r_i r_j \left\langle E_i^a(t') E_j^a(0) \right\rangle + t' \left\langle r_i E_i^a(t') (i\delta V(\mathbf{r}) r_j + \dot{r}_j) E_i^a(0) \right\rangle \right\}$$

- $\delta V(\mathbf{r}) = V_s(\mathbf{r}) - V_o(\mathbf{r})$ and $\dot{\mathbf{r}} = i[\nabla_{\mathbf{r}}^2, \mathbf{r}]/M$.

Defining the real and imaginary parts of the correlator through
$\frac{g^2}{2N_c} \int_0^\infty \mathrm{d}t' \left\langle E_i^a(t) E_j^a(0) \right\rangle = \frac{1}{2}\left[\kappa_{ij}(t) + i\gamma_{ij}(t)\right]$ and
$\frac{g^2}{2N_c} \int_0^\infty \mathrm{d}t' \left\langle E_i^a(0) E_j^a(t) \right\rangle = \frac{1}{2}\left[\kappa_{ij}(t) - i\gamma_{ij}(t)\right]$, the singlet self-energy reads

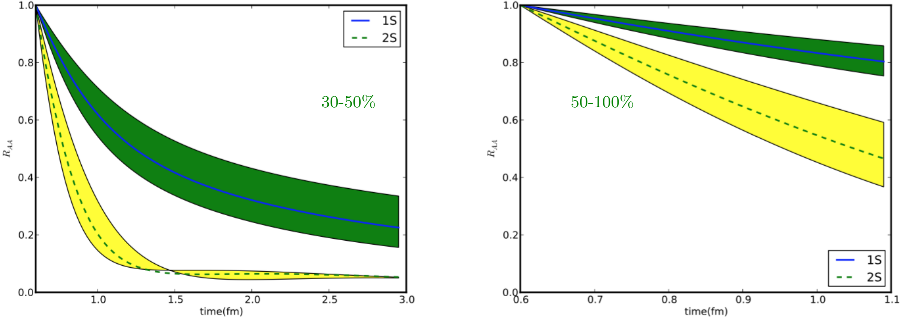
$$\Sigma_s(t,\mathbf{r}) = \frac{1}{2} r_i r_j (\kappa_{ij}(t) + i\gamma_{ij}(t)) + \frac{1}{8T} r_i r_j \delta V(\mathbf{r}) \kappa_{ij}(t) + \frac{1}{4MT} r_i \partial_{r_j} \kappa_{ij}(t)$$

Quantum transport equation in the Lindblad form

The Lindblad equation for the Makovian evolution of the density matrix is

$$\frac{d}{dt}\rho = -i[H,\rho] - \frac{1}{2}\sum_n [C_n^\dagger C_n \rho + \rho C_n^\dagger C_n - 2C_n \rho C_n^\dagger]$$

$$C_i^0 = \sqrt{\frac{\kappa(t)}{N_c^2-1}} r^i \begin{pmatrix} 0 & 1 \\ \sqrt{N_c^2-1} & 0 \end{pmatrix} \text{ and } C_i^1 = \sqrt{\frac{2(N_c^2-4)\kappa(t)}{N_c^2-1}} r^i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$



Brambilla et al. 2017

Bottonium Suppression with $1.8 \lesssim \frac{\kappa}{T^3} \lesssim 3.4$ and $\gamma = 0$.

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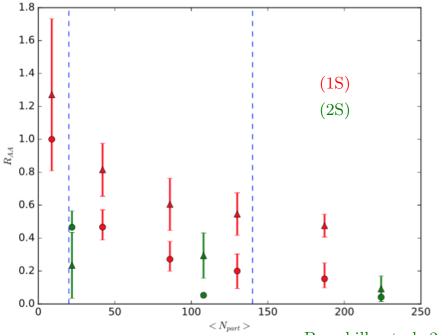

Brambilla et al. 2017

Bottonium Suppression with $1.8 \lesssim \frac{\kappa}{T^3} \lesssim 3.4$ and $\gamma = 0$.

Semi-classical approximation in a strongly coupled plasma

Semi-classical limit

- In the static ($M \to \infty$) limit $\mathbf{r} = \mathbf{r}'$.
- The semi-classical limit is derived through Taylor expansion of small $\mathbf{r}^- = \mathbf{r} - \mathbf{r}'$.

With $\mathbf{r}^- = \mathbf{r} - \mathbf{r}'$ and $\mathbf{r}^+ = \frac{\mathbf{r}+\mathbf{r}'}{2}$, the semi-classical approximation of the singlet self-energy reads

$$\Sigma_{ss^\dagger}(t, \mathbf{r}^+, \mathbf{r}^-) \equiv \Sigma_s(t, \mathbf{r}^+, \mathbf{r}^-) + \Sigma_s^\dagger(t, \mathbf{r}^+, \mathbf{r}^-) \approx (r_i^+ r_j^+ + \frac{1}{4} r_i^- r_j^-)\kappa_{ij}(t) + i r_i^+ r_j^- \gamma_{ij}(t)$$
$$+ \left[\frac{1}{8T} (2 r_i^+ r_j^- + r_i^+ r_j^+ r_k^- \partial_{r_k^+}) \delta V(\mathbf{r}^+) + \frac{1}{2MT} (r_i^+ \partial_{r_i^+} + r_i^- \partial_{r_i^-}) \right] \kappa_{ij}(t)$$

Fokker-Planker equation

The Fokker-Planker equation can be worked out by Wigner transformation of the semi-classical results.

$$\rho(t, \mathbf{r}, \mathbf{r}^{'}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, e^{i\mathbf{p}\cdot\mathbf{r}^-} \rho(t, \mathbf{r}^+, \mathbf{p})$$

Fokker-Planker equation

In the basis of $\rho_0 = \rho_s + \rho_o$ and $\rho_8 = \rho_s - \rho_o/(N_c^2 - 1)$,

- Static limit (Similar to Blaizot and Escobedo 2017 with Non-relativistic QCD):

$$\frac{\mathrm{d}\rho_0(t, \mathbf{r}^+, \mathbf{p})}{\mathrm{d}t} = 0 \quad \leftarrow \text{color equilibrium states } \rho_s = \frac{\rho_o}{N_c^2 - 1}$$

$$\frac{\mathrm{d}\rho_8(t, \mathbf{r}^+, \mathbf{p})}{\mathrm{d}t} = -\frac{N_c^2}{N_c^2 - 1}\kappa(t)(r^+)^2\rho_8(t, \mathbf{r}^+, \mathbf{p}) \quad \leftarrow \text{decreases in time}$$

- Diagonalising the semi-classical equations of ρ_0 and ρ_8, the maximum entropy state evolves w.r.t. $(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{2\mathbf{p}\cdot\nabla_{\mathbf{r}^+}}{M})\rho_0'(t, \mathbf{r}^+, \mathbf{p}) = L'\rho_0'(t, \mathbf{r}^+, \mathbf{p})$

$$\begin{aligned}
L' &= \frac{\nabla_{\mathbf{p}_i}[(N_c^2 - 1)\partial_i V_o'(\mathbf{r}^+) + \partial_i V_s'(\mathbf{r}^+)]}{N_c^2} \\
&+ \nabla_{\mathbf{p}_i}\nabla_{\mathbf{p}_j}\left(\frac{1}{4}\kappa_{ij}(t) + \frac{(N_c^2 - 1)^2\partial_i\delta V'(\mathbf{r}^+)\partial_j\delta V'(\mathbf{r}^+)}{r_{i'}^+ r_{j'}^+ \kappa_{i'j'}(t) N_c^6}\right) + \frac{1}{2MT}\nabla_{\mathbf{p}_i}\mathbf{p}_j\kappa_{ij}(t) \\
&+ \nabla_{\mathbf{p}_i}\nabla_{\mathbf{p}_j}\left\{\frac{i(N_c^2 - 1)\partial_i\delta V'(\mathbf{r}^+)(2r_k^+\kappa_{jk}(t)\delta V(\mathbf{r}^+) + r_k^+ r_l^+\kappa_{kl}(t)\partial_j\delta V(\mathbf{r}^+))}{4TN_c^4 r_{i'}^+ r_{j'}^+\kappa_{i'j'}(t)}\right. \\
&+ \left.\frac{r_n^+\kappa_{in}(t)\delta V(\mathbf{r}^+)(r_k^+\kappa_{jk}(t)\delta V(\mathbf{r}^+) + r_k^+ r_l^+\kappa_{kl}(t)\partial_j\delta V(\mathbf{r}^+))}{16 r_{i'}^+ r_{j'}^+\kappa_{i'j'}(t) N_c^2 T^2}\right\}
\end{aligned}$$

with $\partial_i V_s'(\mathbf{r}^+) = \partial_i V_s(\mathbf{r}^+) + r_i^+\gamma(t)$, $\partial_i V_o'(\mathbf{r}^+) = \partial_i V_o(\mathbf{r}^+) + r_i^+ \frac{N_c^2 - 2}{2(N_c^2 - 1)}\gamma(t)$, and$\partial_i\delta V' = \partial_i V_s' - \partial_i V_o'$ and $\partial_i\delta V = \partial_i V_s - \partial_i V_o$.

Langevin equation

The Corresponding Langevin equation (strongly coupled pNRQCD with contribution from ultra-soft gluon) is

$$\dot{\mathbf{r}} = \frac{2\mathbf{p}}{M}\,, \qquad \frac{M\ddot{\mathbf{r}}_i}{2} = -F_i - \eta_{ij}\mathbf{p}_j + \xi_i + \Theta_i \tag{1}$$

$\eta_{ij} = \frac{1}{2MT}\,\kappa_{ij}(t)$, and $\langle \xi_i(t)\xi_j(t')\rangle = \delta(t-t')\lambda_{ij}$ with $\lambda_{ij} = MT\eta_{ij}$.

The random force $\langle \Theta_i(t)\Theta_j(t')\rangle = \delta(t-t')\frac{2(N_c^2-1)^2\partial_i\delta V'(\mathbf{r}^+)\partial_j\delta V'(\mathbf{r}^+)}{r^+_{i'}r^+_{j'}\kappa_{i'j'}(t)N_c^6}$ + **R.F from potential change of bound states**.

The **external force** $\mathbf{F} = \frac{(N_c^2-1)\nabla V_o'(\mathbf{r}^+)+\nabla V_s'(\mathbf{r}^+)}{N_c^2}$ with $V_s = -\frac{C_F}{r}$, $V_o = \frac{1}{2N_c r}$.

Langevin equation

The Langevin equation from non-relativistic QCD with only Coulomb gluon field(Blaizot and Escobedo 2017) is

$$\mathbf{F} = 0$$

$\eta_{ij} = \frac{C_F}{2MT}\,\partial_i\partial_j W(0)$, and $\langle \xi_i(t)\xi_i(t')\rangle = \delta(t-t')\lambda_{ij}$ with $\lambda_{ij} = MT\eta_{ij}$

$\langle \Theta_i(t)\Theta_j(t')\rangle = \delta(t-t')\,\frac{C_F}{N_c^2}\,\frac{\partial_i V(\mathbf{r}^+)\partial_j V(\mathbf{r}^+)}{W(\mathbf{r}^+)-W(0)}$, with
$-W(\mathbf{r}) = -W(-\mathbf{r}) = \int_0^\infty \mathrm{d}t\,\langle A_0(t,\mathbf{r})A_0(0,0)\rangle + \langle A_0(0,\mathbf{r})A_0(t,0)\rangle$ and
$V(r) - V(0) = \frac{\alpha_s e^{-rm_D}}{r} + \alpha_s m_D$, $W(r) - W(0) = \alpha_s T\phi(rm_D)$ (for $1/r \sim m_D$).

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- Ultra-soft gluons in QGP introduce extra force in the Langevin equation of ρ_0'.
- Potential change of the bound state introduces a random force.
- When the force is large, it can't be considered as a small random force, thus the Langevin equation is not valid.

Time Evolution of Correlators

The density is related to the correlator through

$$G_s(t, \mathbf{r}, \mathbf{r}^{'}) = \langle \mathbf{r}|\rho_s(t,t)|\mathbf{r}'\rangle = \left\langle S_1^\dagger(t,\mathbf{r}) S_2(t,\mathbf{r}')\right\rangle,$$

$$\delta^{ab} G_o(t, \mathbf{r}_o, \mathbf{r}_o') = \frac{\delta^{ab}}{N_c^2 - 1} \langle \mathbf{r}_o|\rho_o(t,t)|\mathbf{r}_o'\rangle = \left\langle O_1^{a\dagger}(t,\mathbf{r}_1,\mathbf{r}_2) O_2^b(t,\mathbf{r}_1',\mathbf{r}_2')\right\rangle.$$

The evolution of correlators are

$$\begin{aligned}
\frac{\mathrm{d}G_s(t,\mathbf{r};\mathbf{r}')}{\mathrm{d}t} &= \int \mathrm{d}^3 r" \left[-i(h_s(\mathbf{r}) - h_s(\mathbf{r}')) - (\Sigma_s^\dagger(t,\mathbf{r}'',\mathbf{r}') + \Sigma_s(t,\mathbf{r},\mathbf{r}''))\right] \\
&\quad \times G_s(t,\mathbf{r}'';\mathbf{r}') + \Xi_{so}(t, G_o, \mathbf{r}; \mathbf{r}') \\
\frac{\mathrm{d}G_o(t,\mathbf{r};\mathbf{r}')}{\mathrm{d}t} &= \int \mathrm{d}^3 r" \left[-i(h_o(\mathbf{r}) - h_o(\mathbf{r}')) - (\Sigma_o^\dagger(t,\mathbf{r}'',\mathbf{r}') + \Sigma_o(t,\mathbf{r},\mathbf{r}''))\right] \\
&\quad \times G_o(t,\mathbf{r}'';\mathbf{r}') + \Xi_{os}(t, G_s, \mathbf{r}; \mathbf{r}') + \Xi_{oo}(t, G_o, \mathbf{r}; \mathbf{r}')
\end{aligned}$$

$h_s(o) = \mathbf{p}^2/M + V_s(o)(\mathbf{r})$ with $V_s(\mathbf{r}) = -C_\mathrm{F}\frac{\alpha_\mathrm{s}}{r}$ and $V_o(\mathbf{r}) = \frac{\alpha_\mathrm{s}}{2N_\mathrm{c} r}$.

Assumptions in deriving Boltzmann equation

$$G_s(t,\mathbf{r},\mathbf{r}^{'}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, e^{i\mathbf{p}\cdot\mathbf{r}^-}\, G_s(t,\mathbf{r}^+,\mathbf{p})$$

$$\int \mathrm{d}^3r^{"}\Sigma_s(t,\mathbf{r},\mathbf{r}^{"})G_s(t,\mathbf{r}^{"},\mathbf{r}^{'}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, e^{i\mathbf{p}\cdot\mathbf{r}^-}\, \Sigma_s(t, \frac{\mathbf{r}+\mathbf{r}^{"}}{2}, \mathbf{p})G_s(t, \frac{\mathbf{r}^{"}+\mathbf{r}^{'}}{2}, \mathbf{p})$$

$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, e^{i\mathbf{p}\cdot\mathbf{r}^-}\, e^{\frac{i}{2}\left(\partial_p^{\Sigma_s}\cdot\partial_r^{G_s} - \partial_r^{\Sigma_s}\cdot\partial_p^{G_s}\right)}\Sigma_s(t,\mathbf{r},\mathbf{p})G_s(t,\mathbf{r},\mathbf{p})$$

The system is a weak nonequilibrium system with weak and instantaneous local interactions with the medium (Kandoff-Baym ansatz).

- (a) The dependence of G_s and Σ_s is slow enough that one can use $\frac{\mathbf{r}+\mathbf{r}^{"}}{2} \approx \frac{\mathbf{r}^{"}+\mathbf{r}^{'}}{2} \approx \frac{\mathbf{r}+\mathbf{r}^{'}}{2} \approx \mathbf{r}$.
- (b) The evolution of potential is slow enough to be neglected ($h_{s/o} \approx \mathbf{p}^2/M$).
- (c) The spectral function of the system is a product of a δ-function and the distribution function ($G_{12} = 2\pi\delta(E - M/2)f$).

Time evolution of Correlators

Implementating (a) and (b):

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{2\mathbf{p}\cdot\nabla_{\mathbf{r}}}{M}\right) G_s(t,\mathbf{r},\mathbf{p}) = -2\,\mathrm{Re}\{\Sigma_s(t,\mathbf{r},\mathbf{p})\} G_s(t,\mathbf{r},\mathbf{p}) + \Xi_{so}(t,\mathbf{r},\mathbf{p})$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{2\mathbf{p}\cdot\nabla_{\mathbf{r}}}{M}\right) G_o(t,\mathbf{r},\mathbf{p}) = -2\,\mathrm{Re}\{\Sigma_o(t,\mathbf{r},\mathbf{p})\} G_o(t,\mathbf{r},\mathbf{p}) + \Xi_{os}(t,\mathbf{r},\mathbf{p}) + \Xi_{oo}(t,\mathbf{r},\mathbf{p})$$

For a strongly coupled system with $p \gg T \sim gT \gg E$

$$\begin{aligned}
\Sigma_s(t,\mathbf{r},\mathbf{p}) &= \frac{g^2}{2N_c}\int_0^\infty ds \left\langle \mathbf{r},\mathbf{p}|r^i e^{-ih_o s} r^j e^{ih_s s}|\mathbf{r},\mathbf{p}\right\rangle \left\langle E^{a,i}(s,\mathbf{0})E^{a,j}(0,\mathbf{0})\right\rangle \\
&= ig^2 C_F \frac{\int \mathrm{d}^3 r_o \int \mathrm{d}^3 p_o}{(2\pi)^3} \frac{\int \mathrm{d}^4 k}{(2\pi)^4} \frac{\langle \mathbf{r},\mathbf{p}|r^i|\mathbf{r}_o,\mathbf{p}_o\rangle \langle \mathbf{r}_o,\mathbf{p}_o|r^j|\mathbf{r},\mathbf{p}\rangle}{E_o - E_s + k_0} k_0^2 D^{>ij}(k) \\
&= ig^2 C_F \int \frac{\mathrm{d}^3 p_o}{(2\pi)^3} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\langle \mathbf{r},\mathbf{p}|r^i|\mathbf{r}_o,\mathbf{p}_o\rangle \langle \mathbf{r}_o,\mathbf{p}_o|r^j|\mathbf{r},\mathbf{p}\rangle}{E_o - E_s + k_0} \\
&\quad \times k_0^2 D^{>ij}(k)(2\pi)^3 \delta^3(\mathbf{p}-\mathbf{p}_o-\mathbf{k})
\end{aligned} \tag{2}$$

where we have used

$$\left\langle E^{a,i}(s,\mathbf{0})E^{a,j}(0,\mathbf{0})\right\rangle = (N_c^2-1)\int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-ik_0 s} k_0^2 D^{>ij}(k)$$

in dimensional regularization.

Time evolution of Correlators

Implementating (a) and (b):

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{2\mathbf{p}\cdot\nabla_{\mathbf{r}}}{M}\right) G_s(t,\mathbf{r},\mathbf{p}) = -2\,\mathrm{Re}\{\Sigma_s(t,\mathbf{r},\mathbf{p})\} G_s(t,\mathbf{r},\mathbf{p}) + \Xi_{so}(t,\mathbf{r},\mathbf{p})$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{2\mathbf{p}\cdot\nabla_{\mathbf{r}}}{M}\right) G_o(t,\mathbf{r},\mathbf{p}) = -2\,\mathrm{Re}\{\Sigma_o(t,\mathbf{r},\mathbf{p})\} G_o(t,\mathbf{r},\mathbf{p}) + \Xi_{os}(t,\mathbf{r},\mathbf{p}) + \Xi_{oo}(t,\mathbf{r},\mathbf{p})$$

With $D^{>}_{ij}(k) = (\delta_{ij} - k_i k_j/k^2) D^{>}(k)$, with $D^{>}(k) = n_{\mathrm{B}}(k_0) + 1$ we get

$$\Sigma_s(t,\mathbf{r},\mathbf{p}) = i g^2 C_{\mathrm{F}} \int \frac{\mathrm{d}^3 p_o}{(2\pi)^3} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{|\langle \mathbf{r},\mathbf{p}|\epsilon^*_\lambda(\mathbf{k})\cdot\mathbf{r}|\mathbf{r}_o,\mathbf{p}_o\rangle|^2}{E_o - E_s + k_0} \times k_0^2 D^{>}(k)(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}_o - \mathbf{k})$$

$\epsilon^*_\lambda(\mathbf{k})\epsilon_\lambda(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$.

The factor $\frac{i}{E_o - E_s + k_0} = G_o^{11}$ is the free octet propagator, which in a thermal medium will be $\frac{i}{E_o - E_s + k_0} + (f_o(E_o,\mathbf{r}_o,\mathbf{p}_o) + \frac{1}{2})2\pi\delta(E_o - E_s + k_0)$ (**Kandoff-Baym** ansatz (c)).

The results for different Feynman diagrams

The results for singlets are

$$
\begin{aligned}
2\Re\{\Sigma_s(t,\mathbf{r},\mathbf{p})\} &= 2g^2C_{\mathrm{F}}\int\frac{\mathrm{d}^3p_o}{(2\pi)^3}\int\frac{\mathrm{d}^3k}{2k(2\pi)^3}\,|\,\langle\mathbf{r},\mathbf{p}|\epsilon^*_\lambda(\mathbf{k})\cdot\mathbf{r}|\mathbf{r}_o,\mathbf{p}_o\rangle\,|^2 \\
&\quad\times k^2\left[(f_o(E_o,\mathbf{r}_o,\mathbf{p}_o)+\frac{1}{2})(n_{\mathrm{B}}(k)+1)\right] \\
&\quad(2\pi)^4\delta(E_s-E_o-k)\delta^3(\mathbf{p}-\mathbf{p}_o-\mathbf{k}) \\
\Xi_{so}(t,\mathbf{r},\mathbf{p}) &= \frac{g^2(N_{\mathrm{c}}^2-1)}{N_{\mathrm{c}}}\int\frac{\mathrm{d}^3p_o}{(2\pi)^3}\int\frac{\mathrm{d}^3k}{2k(2\pi)^3}\,|\,\langle\mathbf{r},\mathbf{p}|\epsilon^*_\lambda(\mathbf{k})\cdot\mathbf{r}|\mathbf{r}_o,\mathbf{p}_o\rangle\,|^2 \\
&\quad\times k^2\left[(f_s(E_s,\mathbf{r},\mathbf{p})+\frac{1}{2})n_{\mathrm{B}}(k)\right] \\
&\quad(2\pi)^4\delta(E_s-E_o+k)\delta^3(\mathbf{p}-\mathbf{p}_o+\mathbf{k})G_o(t,\mathbf{r}_o,\mathbf{p}_o)
\end{aligned}
$$

This two terms can be combined by changing $(k_0,\mathbf{k})$ to $-(k_0,\mathbf{k})$ and using $n_{\mathrm{B}}(-k) = -(n_{\mathrm{B}}(k)+1)$.

Similar calculation can be done for octet self-energies.

The Boltzmann equations

$$G_s(t,\mathbf{r},\mathbf{p}) = f_s(t,\mathbf{r},\mathbf{p}) 2\pi\delta(E_s - \tfrac{M}{2}) \text{ and } G_o(t,\mathbf{r},\mathbf{p}) = f_o(t,\mathbf{r}_o,\mathbf{p}_p) 2\pi\delta(E_o - \tfrac{M}{2})$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{2\mathbf{p}\cdot\nabla_{\mathbf{r}}}{M}\right) f_s(t,\mathbf{r},\mathbf{p}) = -C_-^s + C_+^s$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{2\mathbf{p_o}\cdot\nabla_{\mathbf{r}_o}}{M}\right) f_o(t,\mathbf{r}_o,\mathbf{p}_o) = (-C_-^s(s\leftrightarrow o) + C_+^s(s\leftrightarrow o) + C_i)/(N_c^2-1)$$

$$C_-^s = g^2 C_F \int \frac{\mathrm{d}^3 p_o}{(2\pi)^3} \int \frac{\mathrm{d}^3 k}{2k(2\pi)^3} \left| \langle \mathbf{r},\mathbf{p}|\epsilon_\lambda^*(\mathbf{k})\cdot\mathbf{r}|\mathbf{r}_o,\mathbf{p}_o\rangle \right|^2 \Leftarrow \text{Dissociation of Singlets}$$
$$\times k^2 f_s(E_s,\mathbf{r},\mathbf{p})(n_B(k)+1)(2\pi)^4\delta(E_s - E_o - k)\delta^3(\mathbf{p}-\mathbf{p}_o-\mathbf{k})$$

$$C_+^s = g^2 C_F \int \frac{\mathrm{d}^3 p_o}{(2\pi)^3} \int \frac{\mathrm{d}^3 k}{2k(2\pi)^3} \left| \langle \mathbf{r},\mathbf{p}|\epsilon_\lambda^*(\mathbf{k})\cdot\mathbf{r}|\mathbf{r}_o,\mathbf{p}_o\rangle \right|^2 \Leftarrow \text{Regeneration of Singlets}$$
$$\times k^2 f_o(t,\mathbf{r}_o,\mathbf{p}_o)(n_B(k)+1)(2\pi)^4\delta(E_s - E_o - k)\delta^3(\mathbf{p}-\mathbf{p}_o-\mathbf{k})$$

$$C_i = \frac{g^2(N_c^2-4)C_F}{2} \int \frac{\mathrm{d}^3 p_o'}{(2\pi)^3} \int \frac{\mathrm{d}^3 k}{2k(2\pi)^3} \left| \langle \mathbf{r}_o,\mathbf{p}_o|\epsilon_\lambda^*(\mathbf{k})\cdot\mathbf{r}|\mathbf{r}_o',\mathbf{p}_o'\rangle \right|^2 \Leftarrow \text{Octets transition}$$
$$\times k^2 (f_o'(E_o',\mathbf{r}_o',\mathbf{p}_o') - f_o(E_o,\mathbf{r}_o,\mathbf{p}_o))(n_B(k)+1)(2\pi)^4\delta(E_o - E_o' - k)\delta^3(\mathbf{p}_o-\mathbf{p}_o'-\mathbf{k})$$

Summary

(Semi-)classical description of quarkonia evolution in a strongly coupled plasma has been derived from pNRQCD for a strongly coupled system.

- Langevin equation and Fokker-Plank equation are derived in a semi-classical approximation.
- Boltzmann equation is worked out for a weakly nonequilibrium system with weak interaction with the QGP.

In addition,

- it would be interesting to numerically solve those equations.
- Similar calculation can be done for a weakly coupled system.
- Comparing to experimental data (with proper initial condition and medium description) will reveal the properties of QGP and formation of heavy quarkonia.

Thank you!