On nonequilibrium quarkonium evolution in the QGP fireball

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5 Aug. 2018, Maynooth, ConfXIII
Heavy quarkonium in the QGP

The quarkonium suppression is expected to serve as a signal for the formation of a quark-gluon plasma (QGP). Matsui and Satz 1986

- Heavy quark produced at the beginning of heavy-ion collisions.
- It is sensitive to the properties and evolution of QGP.
- Final state dilepton doesn’t interact with other hadrons.

The explanation of Matsui and Satz 1986 is that

- Cornell potential $V(r) = -\frac{\alpha_s}{r} + \sigma r$ at $T = 0$
- Yukawa potential $V(r) = -\frac{\alpha_s e^{-r m_D}}{r}$ and $\sigma = 0$ for $T > T_c$

Screening is stronger at high $T$, as the Debye radius $r_D \sim 1/m_D$ decreases with increasing $T$. 

![Graph of Cornell and Yukawa potentials](image)

$V(r)$: Cornell potential

$V(r)$: Yukawa potential

$r$: Distance
Heavy quarkonium in the QGP

Static potential of heavy quarkonium in real time at finite temperature for $1/r \sim gT$ is found to have an imaginary part. Laine et al. 2007; Brambilla et al. 2008; Beraudo et al. 2008; Rothkopf et al. 2012; Burnier and Rothkopf 2012; Burnier et al. 2014.

$$V(r) = -\frac{\alpha_s e^{-r m_D}}{r} - \alpha_s m_D - i\alpha_s T \phi(r m_D)$$

with $\phi(x)$ a monotonic function with $\phi(0) = 0$ and $\phi(\infty) = 1$, which is related to the decay width of the heavy quark.
Heavy quarkonium in the QGP

Scales of Heavy quarkonium $M \gg Mv \gg Mv^2$. $v \ll 1$ makes heavy quarkonium a non-relativistic bound state.

- $M$: hard scale
- $Mv \sim p \sim 1/r$: soft scale, momentum transfer between $Q\bar{Q}$
- $Mv^2 \sim E$: ultrasoft(US) scale, binding energy

Thermodynamic scales of the QGP: $T$, $m_D \sim gT$, ⋯

QCD scale: $\Lambda_{QCD}$

Mechanisms for quarkonium evolution in QGP includes regeneration, and dissociation (color screening and scattering, and gluon absorption).

The different scales for the quarkonium evolution in a thermal medium calls for a effective field theory description of the system.
Effective field theories

The non-relativistic QCD (NRQCD) is carried out by integrating out the hard scale $M$. Caswell and Lepage 1986; Bodwin et al. 1995

- $M \gg p, E, \Lambda_{QCD}$ ($m_{c,b} = 1.27, 4.2$ GeV, $\Lambda_{QCD} = 0.2$ GeV), the matching can be done perturbatively.

- The Lagrangian is organised as an expansion of $\frac{1}{M}$.

The potential non-relativistic QCD (pNRQCD) is derived by further integrating out the soft scale $mv$ from NRQCD. Pineda and Soto 1998; Brambilla et al. 2000

- $M \gg p \gg E, \Lambda_{QCD}$, heavy quarks and anti-quarks are described with bound states of color-singlet ($S = S/\sqrt{N_c}$) and octet ($O = O^a T^a / \sqrt{T_F}$) with Coulomb potential ($h_{s,o} = \frac{p_1^2}{2M} + \frac{p_2^2}{2M} + V_{s,o}$).

- The Lagrangian is organised as an expansion of $\frac{1}{M}$ and $r$.

- To NLO in multipole expansion:

$$L_{pNRQCD} = L_{q+g} + \int d^3r \left\{ \text{Tr} \left[ S^\dagger (i\partial_0 - h_s) S + O^\dagger (iD_0 - h_s) O \right] ight.$$  

$$+ \text{Tr} \left[ O^\dagger r \cdot gE S + S^\dagger r \cdot gE O + \frac{1}{2} (O^\dagger r \cdot gE O + O^\dagger Or \cdot gE) \right] \right\}.$$
Transport equations

Quantum transport equations

- Evolution of correlators (or density matrices): Schwinger-Dyson equation, Kandoff-Baym equation Akamatsu 2015, Brambilla et al. 2017

(Semi-)Classical transport equations

- Langevin equation, Fokker-Plank equation Blaizot et al. 2015: Momentum drag/diffusion coefficient

\[ \dot{\mathbf{r}} = \frac{\mathbf{p}}{M}, \quad M\ddot{r}_i = -F_i - \eta_{ij}p_j + \xi_i \]

\[ \langle \xi_i(t)\xi_i(t') \rangle = \delta(t - t')\lambda_{ij} \quad \text{with} \quad \lambda_{ij} = 2MT\eta_{ij} \]

\[ \left( \frac{\partial}{\partial t} + \frac{\mathbf{p} \cdot \nabla \mathbf{r}}{M} \right) P(t, \mathbf{r}, \mathbf{p}) = \nabla_{\mathbf{p}_i}[(\eta_{ij}p_j + F_i) P] + \frac{1}{2} \nabla_{\mathbf{p}_i} \nabla_{\mathbf{p}_j} [\lambda_{ij} P] \]

- Boltzmann equation Muller et al. 2017: Loss/gain term

\[ \left( \frac{d}{dt} + \frac{\mathbf{p} \cdot \nabla \mathbf{r}}{M} \right) f(t, \mathbf{r}, \mathbf{p}) = -C_- + C_+ \]

Relations from quantum to classical transport equations was discussed by e.g. Zurek 1991; Akamatsu 2015; Blaizot and Escobedo 2017; De Boni 2017

The relation between different transport equations is not fully understood. It would be important to derive those equations from QCD.
Open Quantum System

Wikipedia: In physics, an open quantum system is a quantum-mechanical system which interacts with an external quantum system, the environment.

For heavy quarkonium in high energy heavy-ion collisions: the QGP + heavy quark is considered as a large closed system.

\[ H = H_Q + H_{QGP} + H_I \]

The heavy quarkonium (HQ) evolution is described by

\[ \rho_Q(t) = \text{Tr}_{QGP}[\rho(t)] \]

- Interaction between the HQ system and the QGP is weak:

\[ \rho(t) = \rho_Q \otimes \rho_{QGP} \]

Evolution of the HQ system:

\[ i \frac{d\rho_Q(t)}{dt} = [H_Q, \rho_Q(t)] + iD\rho_Q(t) \]

- \( iD\rho_Q(t) \) describes the dissipation of the HQ system due to interaction with the QGP.
Density evolution with pNRQCD

The density matrices for color-singlet and octet are

\[ \langle \mathbf{r}_1, \mathbf{r}_2 | \rho_s(t, t) | \mathbf{r}_1', \mathbf{r}_2' \rangle \equiv \rho_s(t, \mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_1', \mathbf{r}_2') = \langle S_1^\dagger(t, \mathbf{r}_1, \mathbf{r}_2) S_2(t, \mathbf{r}_1', \mathbf{r}_2') \rangle \]

\[ \frac{\delta^{ab}}{N_c^2 - 1} \langle \mathbf{r}_1, \mathbf{r}_2 | \rho_o(t, t) | \mathbf{r}_1', \mathbf{r}_2' \rangle \equiv \frac{\delta^{ab}}{N_c^2 - 1} \rho_o(t, \mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_1', \mathbf{r}_2') = \langle O_1^{a\dagger}(t, \mathbf{r}_1, \mathbf{r}_2) O_2^b(t, \mathbf{r}_1', \mathbf{r}_2') \rangle \]
Evolution equations for the density matrix of color-singlet and octet states:\(^1\):

\[
\frac{d\rho_s(t, r; r')}{dt} = \left[ -i(h_s(r) - h_s(r')) - (\Sigma_s^\dagger(t, r') + \Sigma_s(t, r)) \right] \rho_s(t, r; r') \\
+ \Xi_{so}(t, \rho_o, r; r'),
\]

\[
\frac{d\rho_o(t, r; r')}{dt} = \left[ -i(h_o(r) - h_o(r')) - (\Sigma_o^\dagger(t, r') + \Sigma_o(t, r)) \right] \rho_o(t, r; r') \\
+ \Xi_{os}(t, \rho_s, r; r') + \Xi_{oo}(t, \rho_o, r; r').
\]

\(^1\)Higher order correction has been included on the right hand side:

\[e^{-i h_s(o)(t-t_0)} \rho_s(o)(t_0, t_0) e^{-i h_s(o)} \rightarrow \rho_s(o)(t, t),\]

which makes the evolution a Markovian process.
Self-energies in real time formalism

\[ \Sigma_s(t, \mathbf{r}_1, \mathbf{r}_2) = \frac{2g^2}{N_c} \int_0^\infty dt_1 \left\langle \text{Tr} \{I_g(t_1)T^c\} e^{-ih_0(t_1)} \text{Tr} \{T^c I_g(0)\} e^{ih_s(t_1)} \right\rangle \]

The self-energies are expressed as\(^2\)

\[ \Sigma_s(t, \mathbf{r}_1, \mathbf{r}_2) = \frac{2g^2}{N_c} \int_0^\infty dt_1 \left\langle \text{Tr} \{I_g(t_1)T^c\} e^{-ih_0(t_1)} \text{Tr} \{T^c I_g(0)\} e^{ih_s(t_1)} \right\rangle \]

Gluon field in multipole expansion (with field redefinition):

\[ I_g = A^a_0(\mathbf{r}_1)T^a - A^a_0(\mathbf{r}_2)\widetilde{T}^a \approx A^a_0(0)(T^a - \widetilde{T}^a) - \frac{1}{2} \mathbf{r} \cdot \mathbf{E}(T^a + \widetilde{T}^a). \]

▶ **Lindblad equation** has been derived in weak and strong coupling plasma. 

Brambilla, Escobedo, Soto, and Vairo 2017

\[^2\int_0^{t-t_0} dt_1 \rightarrow \int_0^\infty dt_1, \text{ and neglecting density matrix in 11 and 22 correlators} \]
Quarkonium evolution in a strongly coupled plasma

Scales for strongly coupled plasma: $M \gg p \sim Mv \gg T \sim m_D \sim gT \gg E \sim Mv^2$.

- We can expand the exponentials $e^{\pm ih_{s,o}(t-t_0)} \approx 1 \pm ih_{s,o}(t-t_0)$ with $h_{s,o} = -\frac{\nabla^2_r}{M} + V_{s,o}$. ($h_{s,o}(t-t_0) \ll 1$ as the correlation time $t - t_0 \sim \frac{1}{T}$.)

- The LO expansion is used for deriving Lindblad equation.

- The LO and the NLO correspond to the momentum diffusion and momentum drag in the Fokker-Plank equation and Langevin equation in a semi-classical approximation.

\[
\Sigma_s(t, \mathbf{r}) = \frac{g^2}{2N_c} \int_0^\infty dt' \left\{ r_i r_j \langle E^a_i(t')E^a_j(0) \rangle + t' \langle r_i E^a_i(t')(i\delta V(\mathbf{r})r_j + \dot{r}_j)E^a_i(0) \rangle \right\}
\]

- $\delta V(\mathbf{r}) = V_s(\mathbf{r}) - V_o(\mathbf{r})$ and $\dot{\mathbf{r}} = i[\nabla^2_r, \mathbf{r}]/M$.

Defining the real and imaginary parts of the correlator through

\[
\frac{g^2}{2N_c} \int_0^\infty dt' \langle E^a_i(t)E^a_j(0) \rangle = \frac{1}{2} [\kappa_{ij}(t) + i\gamma_{ij}(t)] \quad \text{and} \quad \frac{g^2}{2N_c} \int_0^\infty dt' \langle E^a_i(0)E^a_j(t) \rangle = \frac{1}{2} [\kappa_{ij}(t) - i\gamma_{ij}(t)],
\]

the singlet self-energy reads

\[
\Sigma_s(t, \mathbf{r}) = \frac{1}{2} r_i r_j (\kappa_{ij}(t) + i\gamma_{ij}(t)) + \frac{1}{8T} r_i r_j \delta V(\mathbf{r}) \kappa_{ij}(t) + \frac{1}{4MT} r_i \partial_{r_j} \kappa_{ij}(t)
\]
Quantum transport equation in the Lindblad form

The Lindblad equation for the Markovian evolution of the density matrix is

$$\frac{d}{dt}\rho = -i[H, \rho] - \frac{1}{2} \sum_n [C_n^\dagger C_n \rho + \rho C_n^\dagger C_n - 2 C_n \rho C_n^\dagger]$$

$$C_i^0 = \sqrt{\frac{\kappa(t)}{N_c^2 - 1}} r_i \begin{pmatrix} 0 & 1 \\ \sqrt{N_c^2 - 1} & 0 \end{pmatrix}$$ and $$C_i^1 = \sqrt{\frac{2(N_c^2 - 4)\kappa(t)}{N_c^2 - 1}} r_i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Bottomonium Suppression with $1.8 \lesssim \frac{\kappa}{T^3} \lesssim 3.4$ and $\gamma = 0$. 

Brambilla et al. 2017
Quantum transport equation in the Lindblad form

The Lindblad equation for the Makovian evolution of the density matrix is

\[
\frac{d}{dt} \rho = -i [H, \rho] - \frac{1}{2} \sum_n \left[ C_n^\dagger C_n \rho + \rho C_n^\dagger C_n - 2C_n \rho C_n^\dagger \right]
\]

\[
C_i^0 = \sqrt{\frac{\kappa(t)}{N_c^2 - 1}} r^i \begin{pmatrix} 0 & 1 \\ \sqrt{N_c^2 - 1} & 0 \end{pmatrix} \quad \text{and} \quad C_i^1 = \sqrt{\frac{2(N_c^2 - 4) \kappa(t)}{N_c^2 - 1}} r^i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

Bottonium Suppression with \(1.8 \lesssim \frac{\kappa}{T^3} \lesssim 3.4\) and \(\gamma = 0\).
Semi-classical approximation in a strongly coupled plasma

Semi-classical limit
- In the static \((M \to \infty)\) limit \(r = r'\).
- The semi-classical limit is derived through Taylor expansion of small \(r^- = r - r'\).

With \(r^- = r - r'\) and \(r^+ = \frac{r^+ + r'}{2}\), the semi-classical approximation of the singlet self-energy reads

\[
\Sigma_{ss}^{\dagger}(t, r^+, r^-) \equiv \Sigma_s(t, r^+, r^-) + \Sigma_s(t, r^+, r^-) \approx (r^+_i r^+_j + \frac{1}{4} r^-_i r^-_j) \kappa_{ij}(t) + i r^+_i r^-_j \gamma_{ij}(t)
\]

\[
+ \left[ \frac{1}{8T} (2 r^+_i r^-_j + r^+_i r^+_j r^- k \partial_{r^+_k}) \delta V(r^+) + \frac{1}{2MT} (r^+_i \partial_{r^+_i} + r^-_i \partial_{r^-_i}) \right] \kappa_{ij}(t)
\]
Fokker-Planker equation

The Fokker-Planker equation can be worked out by Wigner transformation of the semi-classical results.

\[ \rho(t, r, r') = \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot r'} \rho(t, r^+, p) \]
Fokker-Planker equation

In the basis of \( \rho_0 = \rho_s + \rho_o \) and \( \rho_8 = \rho_s - \rho_o/(N_c^2 - 1) \),

- Static limit (Similar to Blaizot and Escobedo 2017 with Non-relativistic QCD):
  \[
  \frac{d\rho_0(t, r^+, p)}{dt} = 0 \quad \leftarrow \text{color equilibrium states } \rho_s = \frac{\rho_o}{N_c^2 - 1}
  \]
  \[
  \frac{d\rho_8(t, r^+, p)}{dt} = - \frac{N_c^2}{N_c^2 - 1} \kappa(t)(r^+)^2 \rho_8(t, r^+, p) \quad \leftarrow \text{decreases in time}
  \]

- Diagonalising the semi-classical equations of \( \rho_0 \) and \( \rho_8 \), the maximum entropy state evolves w.r.t.
  \[
  \frac{d}{dt} + \frac{2p \cdot \nabla r^+}{M} = L' \rho_0'(t, r^+, p) = L' \rho_0'(t, r^+, p)
  \]

\[
L' = \frac{N_c^2}{2N_c^2 - 1} [((N_c^2 - 1)\partial_i V_o'(r^+) + \partial_i V_s'(r^+)]
\]

\[
+ \nabla_{p_i} \nabla_{p_j} \left( \frac{1}{4} \kappa_{ij}(t) + \frac{(N_c^2 - 1)^2 \partial_i \delta V'(r^+)}{r^+_i r^+_j \kappa_{i'j'}(t) N_c^6} \right) + \frac{1}{2MT} \nabla_{p_i} p_{j'} \kappa_{ij}(t)
\]

\[
+ \nabla_{p_i} \nabla_{p_j} \left\{ \frac{i(N_c^2 - 1)\partial_i \delta V'(r^+)}{4TN_c^4 r^+_i r^+_j \kappa_{i'j'}(t)} \right. \right.
\]

\[
+ \frac{r^+_n \kappa_{in}(t) \delta V(r^+)(r^+_k \kappa_{jk}(t) \delta V(r^+) + r^+_k r^+_l \kappa_{kl}(t) \partial_j \delta V(r^+))}{16r^+_i r^+_j \kappa_{i'j'}(t) N_c^2 T^2}
\]

with \( \partial_i V'_s(r^+) = \partial_i V_s(r^+) + r^+_i \gamma(t) \), \( \partial_i V'_o(r^+) = \partial_i V_o(r^+) + r^+_i \frac{N_c^2 - 2}{2(N_c^2 - 1)} \gamma(t) \),
and \( \partial_i \delta V' = \partial_i V'_s - \partial_i V'_o \) and \( \partial_i \delta V = \partial_i V_s - \partial_i V_o \).
Langevin equation

The Corresponding Langevin equation (strongly coupled pNRQCD with contribution from ultra-soft gluon) is

\[ \dot{r} = \frac{2p}{M}, \quad \frac{M\ddot{r}_i}{2} = -F_i - \eta_{ij}p_j + \xi_i + \Theta_i \]  \tag{1} 

\[ \eta_{ij} = \frac{1}{2MT}\kappa_{ij}(t), \text{ and } \langle \xi_i(t)\xi_j(t') \rangle = \delta(t-t')\lambda_{ij} \text{ with } \lambda_{ij} = MT\eta_{ij}. \]

The random force \( \langle \Theta_i(t)\Theta_j(t') \rangle = \delta(t-t')\frac{2(N_c^2-1)^2\partial_i\delta V'(r^+)\partial_j\delta V'(r^+)}{r_i^+r_j^+\kappa_{i'j'}(t)N_c^6} + \text{R.F from potential change of bound states.} \)

The external force \( F = \frac{(N_c^2-1)\nabla V_o'(r^+)+\nabla V_s'(r^+)}{N_c^2} \) with \( V_s = -\frac{C_F}{r}, \quad V_o = \frac{1}{2N_c r}. \)
The Langevin equation from non-relativistic QCD with only Coulomb gluon field (Blaizot and Escobedo 2017) is

$$ F = 0 $$

$$ \eta_{ij} = \frac{C_F}{2MT} \partial_i \partial_j W(0), \text{ and } \langle \xi_i(t)\xi_i(t') \rangle = \delta(t - t')\lambda_{ij} \text{ with } \lambda_{ij} = MT\eta_{ij} $$

$$ \langle \Theta_i(t)\Theta_j(t') \rangle = \delta(t - t')\frac{C_F}{N_c^2} \frac{\partial_i V(r^+)\partial_j V(r^+)}{W(r^+)-W(0)}, \text{ with} $$

$$ -W(r) = -W(-r) = \int_0^\infty dt \langle A_0(t,r)A_0(0,0) \rangle + \langle A_0(0,r)A_0(t,0) \rangle \text{ and} $$

$$ V(r) - V(0) = \frac{\alpha_s e^{-r m_D}}{r} + \alpha_s m_D, \text{ and } W(r) - W(0) = \alpha_s T \phi(r m_D) \text{ (for } 1/r \sim m_D). $$
Langevin equation

The Corresponding Langevin equation (strongly coupled pNRQCD with contribution from ultra-soft gluon) is

\[
\dot{r} = \frac{2p}{M}, \quad \frac{M\ddot{r}_i}{2} = -F_i - \eta_{ij} p_j + \xi_i + \Theta_i
\]  

(1)

\[\eta_{ij} = \frac{1}{2MT} \kappa_{ij}(t), \text{ and } \langle \xi_i(t)\xi_j(t') \rangle = \delta(t-t')\lambda_{ij} \text{ with } \lambda_{ij} = MT\eta_{ij}.\]

The random force \(\langle \Theta_i(t)\Theta_j(t') \rangle = \delta(t-t')\frac{2(N_c^2-1)^2\partial_i\delta V'(r^+)\partial_j\delta V'(r^+)}{r_i^+r_j^+\kappa_{i'j'}(t)N_c^6} + \text{R.F from potential change of bound states.}\)

The external force \(F = \frac{(N_c^2-1)\nabla V_o'(r^+)+\nabla V_s'(r^+)}{N_c^2} \text{ with } V_s = -\frac{C_F}{r}, V_o = \frac{1}{2N_cr}.\)

- Ultra-soft gluons in QGP introduce extra force in the Langevin equation of \(\rho_0'.\)
- Potential change of the bound state introduces a random force.
- When the force is large, it can’t be considered as a small random force, thus the Langevin equation is not valid.
Time Evolution of Correlators

The density is related to the correlator through

\[ G_s(t, \mathbf{r}, \mathbf{r}') = \langle \mathbf{r} | \rho_s(t, t) | \mathbf{r}' \rangle = \langle S_1^\dagger(t, \mathbf{r}) S_2(t, \mathbf{r}') \rangle, \]

\[ \delta^{ab} G_o(t, \mathbf{r}_o, \mathbf{r}'_o) = \frac{\delta^{ab}}{N_c^2 - 1} \langle \mathbf{r}_o | \rho_o(t, t) | \mathbf{r}'_o \rangle = \langle O_1^{a\dagger}(t, \mathbf{r}_1, \mathbf{r}_2) O_2^b(t, \mathbf{r}'_1, \mathbf{r}'_2) \rangle. \]

The evolution of correlators are

\[
\frac{dG_s(t, \mathbf{r}; \mathbf{r}')}{dt} = \int d^3r'' \left[ -i(h_s(\mathbf{r}) - h_s(\mathbf{r}')) - (\Sigma_s^\dagger(t, \mathbf{r}'', \mathbf{r}') + \Sigma_s(t, \mathbf{r}, \mathbf{r}'')) \right] \\
\times G_s(t, \mathbf{r}''; \mathbf{r}') + \Xi_{so}(t, G_o, \mathbf{r}; \mathbf{r}')
\]

\[
\frac{dG_o(t, \mathbf{r}; \mathbf{r}')}{dt} = \int d^3r'' \left[ -i(h_o(\mathbf{r}) - h_o(\mathbf{r}')) - (\Sigma_o^\dagger(t, \mathbf{r}'', \mathbf{r}') + \Sigma_o(t, \mathbf{r}, \mathbf{r}'')) \right] \\
\times G_o(t, \mathbf{r}''; \mathbf{r}') + \Xi_{os}(t, G_s, \mathbf{r}; \mathbf{r}') + \Xi_{oo}(t, G_o, \mathbf{r}; \mathbf{r}')
\]

\[ h_s(o) = \mathbf{p}^2/M + V_s(o)(\mathbf{r}) \text{ with } V_s(\mathbf{r}) = -C_F \frac{\alpha_s}{r} \text{ and } V_o(\mathbf{r}) = \frac{\alpha_s}{2N_c r}. \]
Assumptions in deriving Boltzmann equation

\[ G_s(t, \mathbf{r}, \mathbf{r}') = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p} \cdot \mathbf{r}^\prime} G_s(t, \mathbf{r}^+, \mathbf{p}) \]

\[ \int d^3r'' \Sigma_s(t, \mathbf{r}, \mathbf{r}'') G_s(t, \mathbf{r}'', \mathbf{r}') = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p} \cdot \mathbf{r}^-''} \Sigma_s(t, \frac{\mathbf{r} + \mathbf{r}''}{2}, \mathbf{p})G_s(t, \frac{\mathbf{r}'' + \mathbf{r}'}{2}, \mathbf{p}) \]

\[ = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p} \cdot \mathbf{r}^-''} e^{\frac{i}{2} (\partial_{\mathbf{p}} \Sigma_s \cdot \partial_{\mathbf{r}} G_s - \partial_{\mathbf{r}} \Sigma_s \cdot \partial_{\mathbf{p}} G_s)} \Sigma_s(t, \mathbf{r}, \mathbf{p})G_s(t, \mathbf{r}, \mathbf{p}) \]

The system is a weak nonequilibrium system with weak and instantaneous local interactions with the medium (Kandoff-Baym ansatz).

- (a) The dependence of \( G_s \) and \( \Sigma_s \) is slow enough that one can use
  \[ \frac{\mathbf{r} + \mathbf{r}''}{2} \approx \frac{\mathbf{r}'' + \mathbf{r}'}{2} \approx \frac{\mathbf{r} + \mathbf{r}'}{2} \approx \mathbf{r}. \]
- (b) The evolution of potential is slow enough to be neglected (\( h_{s/o} \approx \mathbf{p}^2/M \)).
- (c) The spectral function of the system is a product of a \( \delta \)-function and the distribution function (\( G_{12} = 2\pi\delta(E - M/2)f \)).
Time evolution of Correlators

Implementing (a) and (b):

\[
\left( \frac{d}{dt} + \frac{2\mathbf{p} \cdot \nabla_{\mathbf{r}}}{M} \right) G_s(t, \mathbf{r}, \mathbf{p}) = -2 \text{Re}\{\Sigma_s(t, \mathbf{r}, \mathbf{p})\} G_s(t, \mathbf{r}, \mathbf{p}) + \Xi_{so}(t, \mathbf{r}, \mathbf{p})
\]

\[
\left( \frac{d}{dt} + \frac{2\mathbf{p} \cdot \nabla_{\mathbf{r}}}{M} \right) G_o(t, \mathbf{r}, \mathbf{p}) = -2 \text{Re}\{\Sigma_o(t, \mathbf{r}, \mathbf{p})\} G_o(t, \mathbf{r}, \mathbf{p}) + \Xi_{os}(t, \mathbf{r}, \mathbf{p}) + \Xi_{oo}(t, \mathbf{r}, \mathbf{p})
\]

For a strongly coupled system with \( p \gg T \sim gT \gg E \)

\[
\Sigma_s(t, \mathbf{r}, \mathbf{p}) = \frac{g^2}{2N_c} \int_{\infty}^0 ds \left\langle \mathbf{r}, \mathbf{p} | r^i \, e^{-i\omega_0 s} \, r^j \, e^{i\delta s} | \mathbf{r}, \mathbf{p} \right\rangle \left\langle E^{a,i}(s, 0) E^{a,j}(0, 0) \right\rangle
\]

\[
= ig^2 C_F \int \frac{d^3 r_o}{(2\pi)^3} \int \frac{d^3 p_o}{(2\pi)^3} \int d^4 k \frac{\left\langle \mathbf{r}, \mathbf{p} | r^i | \mathbf{r}_o, \mathbf{p}_o \right\rangle \left\langle \mathbf{r}_o, \mathbf{p}_o | r^j | \mathbf{r}, \mathbf{p} \right\rangle}{E_o - E_s + k_0} k_0^2 D^{>ij}(k)
\]

\[
= ig^2 C_F \int \frac{d^3 p_o}{(2\pi)^3} \int \frac{d^4 k}{(2\pi)^4} \left\langle \mathbf{r}, \mathbf{p} | r^i | \mathbf{r}_o, \mathbf{p}_o \right\rangle \left\langle \mathbf{r}_o, \mathbf{p}_o | r^j | \mathbf{r}, \mathbf{p} \right\rangle \frac{k_0^2 D^{>ij}(k)(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}_o - \mathbf{k})}{E_o - E_s + k_0}
\]

where we have used

\[
\left\langle E^{a,i}(s, 0) E^{a,j}(0, 0) \right\rangle = (N_c^2 - 1) \int \frac{d^4 k}{(2\pi)^4} e^{-ik_0 s} k_0^2 D^{>ij}(k)
\]

in dimensional regularization.
Time evolution of Correlators

Implementating (a) and (b):

\[
\left( \frac{d}{dt} + \frac{2p \cdot \nabla r}{M} \right) G_s(t, r, p) = -2 \text{Re}\{\Sigma_s(t, r, p)\} G_s(t, r, p) + \Xi_{so}(t, r, p)
\]

\[
\left( \frac{d}{dt} + \frac{2p \cdot \nabla r}{M} \right) G_o(t, r, p) = -2 \text{Re}\{\Sigma_o(t, r, p)\} G_o(t, r, p) + \Xi_{os}(t, r, p)
\]

\[+ \Xi_{oo}(t, r, p)\]

With \( D_{ij}(k) = (\delta_{ij} - k_i k_j / k^2) D^>(k), \) with \( D^>(k) = n_B(k_0) + 1 \) we get

\[
\Sigma_s(t, r, p) = i g^2 C_F \int \frac{d^3 p_o}{(2\pi)^3} \int \frac{d^4 k}{(2\pi)^4} \left| \langle r, p | \epsilon^*_\lambda(k) \cdot r | r_o, p_o \rangle \right|^2 \]

\[
\times k_0^2 D^>(k)(2\pi)^3 \delta^3(p - p_o - k)\]

\[\epsilon^*_\lambda(k)\epsilon_\lambda(k) = \delta_{ij} - k_i k_j / k^2.\]

The factor \( \frac{i}{E_o - E_s + k_0} = G_1^{11} \) is the free octet propagator, which in a thermal medium will be \( \frac{i}{E_o - E_s + k_0} + (f_o(E_o, r_o, p_o) + \frac{1}{2})2\pi \delta(E_o - E_s + k_0) \) (Kandoff-Baym ansatz (c)).
The results for different Feynman diagrams

The results for singlets are

$$2\Re\{\Sigma_s(t, r, p)\} = 2g^2 C_F \int \frac{d^3 p_o}{(2\pi)^3} \int \frac{d^3 k}{2k(2\pi)^3} |\langle r, p|\epsilon_\lambda^*(k) \cdot r|r_o, p_o\rangle|^2$$

$$\times k^2 \left[ (f_o(E_o, r_o, p_o) + \frac{1}{2})(n_B(k) + 1) \right]$$

$$\times (2\pi)^4 \delta(E_s - E_o - k)\delta^3(p - p_o - k)$$

$$\Xi_{so}(t, r, p) = \frac{g^2(N_c^2 - 1)}{N_c} \int \frac{d^3 p_o}{(2\pi)^3} \int \frac{d^3 k}{2k(2\pi)^3} |\langle r, p|\epsilon_\lambda^*(k) \cdot r|r_o, p_o\rangle|^2$$

$$\times k^2 \left[ (f_s(E_s, r, p) + \frac{1}{2})n_B(k) \right]$$

$$\times (2\pi)^4 \delta(E_s - E_o + k)\delta^3(p - p_o + k)G_o(t, r_o, p_o)$$

This two terms can be combined by changing $(k_0, k)$ to $-(k_0, k)$ and using $n_B(-k) = -(n_B(k) + 1)$.

Similar calculation can be done for octet self-energies.
The Boltzmann equations

\[ G_s(t, \mathbf{r}, \mathbf{p}) = f_s(t, \mathbf{r}, \mathbf{p})2\pi \delta(E_s - \frac{M}{2}) \] and \[ G_o(t, \mathbf{r}, \mathbf{p}) = f_o(t, \mathbf{r}_o, \mathbf{p}_o)2\pi \delta(E_o - \frac{M}{2}) \]

\[
\left( \frac{d}{dt} + \frac{2\mathbf{p} \cdot \nabla \mathbf{r}}{M} \right) f_s(t, \mathbf{r}, \mathbf{p}) = -C_-^s + C_+^s
\]

\[
\left( \frac{d}{dt} + \frac{2\mathbf{p}_o \cdot \nabla \mathbf{r}_o}{M} \right) f_o(t, \mathbf{r}_o, \mathbf{p}_o) = (-C_-^s(s \leftrightarrow o) + C_+^s(s \leftrightarrow o) + C_i)/(N_c^2 - 1)
\]

\[ C_-^s = g^2 C_F \int \frac{d^3 p_o}{(2\pi)^3} \int \frac{d^3 k}{2k(2\pi)^3} |\langle \mathbf{r}, \mathbf{p}|\epsilon^*_\lambda(\mathbf{k}) \cdot \mathbf{r}|\mathbf{r}_o, \mathbf{p}_o \rangle|^2 \ \Leftrightarrow \ \text{Dissociation of Singlets}
\]
\[ \times k^2 f_s(E_s, \mathbf{r}, \mathbf{p})(n_B(k) + 1)(2\pi)^4 \delta(E_s - E_o - k)\delta^3(\mathbf{p} - \mathbf{p}_o - \mathbf{k})
\]

\[ C_+^s = g^2 C_F \int \frac{d^3 p_o}{(2\pi)^3} \int \frac{d^3 k}{2k(2\pi)^3} |\langle \mathbf{r}, \mathbf{p}|\epsilon^*_\lambda(\mathbf{k}) \cdot \mathbf{r}|\mathbf{r}_o, \mathbf{p}_o \rangle|^2 \ \Leftrightarrow \ \text{Regeneration of Singlets}
\]
\[ \times k^2 f_o(t, \mathbf{r}_o, \mathbf{p}_o)(n_B(k) + 1)(2\pi)^4 \delta(E_s - E_o - k)\delta^3(\mathbf{p} - \mathbf{p}_o - \mathbf{k})
\]

\[ C_i = \frac{g^2(N_c^2 - 4)C_F}{2} \int \frac{d^3 p'_o}{(2\pi)^3} \int \frac{d^3 k}{2k(2\pi)^3} |\langle \mathbf{r}_o, \mathbf{p}_o|\epsilon^*_\lambda(\mathbf{k}) \cdot \mathbf{r}|\mathbf{r}_o', \mathbf{p}_o' \rangle|^2 \ \Leftrightarrow \ \text{Octets transition}
\]
\[ \times k^2 (f'_o(E'_o, \mathbf{r}_o', \mathbf{p}_o') - f_o(E_o, \mathbf{r}_o, \mathbf{p}_o))(n_B(k) + 1)(2\pi)^4 \delta(E_o - E'_o - k)\delta^3(\mathbf{p}_o - \mathbf{p}_o' - \mathbf{k})
\]
Summary

(Semi-)classical description of quarkonia evolution in a strongly coupled plasma has been derived from pNRQCD for a strongly coupled system.

▶ Langevin equation and Fokker-Plank equation are derived in a semi-classical approximation.

▶ Boltzmann equation is worked out for a weakly nonequilibrium system with weak interaction with the QGP.

In addition,

▶ it would be interesting to numerically solve those equations.

▶ Similar calculation can be done for a weakly coupled system.

▶ Comparing to experimental data (with proper initial condition and medium description) will reveal the properties of QGP and formation of heavy quarkonia.

Thank you!