Euclidean versus Minkowski short distance

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Outline of the talk

- I shall reexamine the viability of recent proposals of computing Parton Distribution Functions (PDF's) directly on the lattice ensuing from the seminal Ji (PRL 110 (2013) 262002) paper
- I'll show that unsubtracted power divergencies plague the definition of the moments associated with the Ji PDF
- I'll discuss other approaches devised to circumvent this problem
  - the Ma & Qiu (PRL 120 (2018) 022003) strategy of directly measuring matrix elements of current-current T-products
  - the reduced Ioffe-time distributions of Zhang *et al.* (PRD 96 (2017) 094503) and Orginos *et al.* (PRD 97 (2018) 074508)
  - the analysis of power subtractions employed by many groups, as presented by A.V. Radyushkin (arXiv:1807.07509v2 [hep-ph])

Disclaimer: It is impossible to give here due credit to all the papers that have appeared on this important subject. I apologize for that

Bibliography

- The talk is based on the paper
  G. C. Rossi and M. Testa,
  Phys. Rev. D **96** (2017) no.1, 014507

- See also
  G. C. Rossi and M. Testa,
  arXiv:1806.04428 [hep-lat]
  submitted to Phys. Rev. D

Minkowski metrics

The hadronic DIS cross section in the parton language reads

- $(2\pi)^4 W(q^2, q\cdot P) = \int d^4\xi e^{iq\cdot\xi}\langle P|J(0)J(\xi)|P\rangle =$

$$= \int d^4\xi e^{iq\cdot\xi}\langle P|\phi(0)\phi(\xi)|P\rangle\Delta(\xi) =$$

$$= \sum_n \int \frac{d\mathbf{k}}{2|\mathbf{k}|}|\langle n|\phi(0)|P\rangle|^2(2\pi)^4\delta^4(P+q-p_n-k)$$

- $J(\xi) = \phi(\xi)^2\,, \qquad \Delta(\xi) \equiv \int \frac{d\mathbf{k}}{2|\mathbf{k}|}e^{ik\cdot\xi} = \int d^4k\,\delta(k^2)\theta(k^0)e^{ik\xi}$

Lorentz invariance implies for the bilocal

$$\langle P|\phi(0)\phi(\xi)|P\rangle = F(P\cdot\xi, \xi^2)$$

In the canonical case $F(P\cdot\xi, \xi^2)$ is a regular function that needs to be evaluated for $\xi^2 \approx 0$

We want to compute the Fourier Transform (FT) of $F(P\cdot\xi,0)$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d(P\cdot\xi)F(P\cdot\xi,0)e^{ixP\cdot\xi}$$

$$F(P\cdot\xi,0)=\int_{-\infty}^{+\infty}dx\,f(x)e^{-ixP\cdot\xi}$$

as $f(x)$ is related to $W(q^2,q\cdot P)$ by

$$(2\pi)^4W(q^2,q\cdot P)=\int_{-\infty}^{+\infty}dx\,f(x)\int d^4\xi\,e^{-i(q+xP)\cdot\xi}\Delta(\xi)=$$

$$=(2\pi)^4\int_{-\infty}^{+\infty}dx\,f(x)\delta[(q+xP)^2]\theta[(q+xP)^0]\,,$$

finally leading in the Bjorken limit to

$$W(q^2,q\cdot P)\approx\frac{x\,f(x)}{-q^2}\,,\qquad x=\frac{-q^2}{2q\cdot P}$$

This is the standard argument relating the structure function $f(x)$ (i.e. the FT of the bilocal matrix element) to the DIS cross section, W

In the canonical case the bilocal can be Taylor expanded around $\xi=0$

- $\langle P|\phi(0)\phi(\xi)|P\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P|\phi(0) \frac{\partial}{\partial \xi^{\mu_1}} \dots \frac{\partial}{\partial \xi^{\mu_n}} \phi(\xi)\Big|_{\xi=0} |P\rangle \xi^{\mu_1} \dots \xi^{\mu_n} \equiv$

  $\equiv \sum_{n=0}^{\infty} \langle P|O_{\mu_1 \dots \mu_n}|P\rangle \xi^{\mu_1} \dots \xi^{\mu_n}$
- $\langle P|O_{\mu_1 \dots \mu_n}|P\rangle = A_n P_{\mu_1} \dots P_{\mu_n} + \textit{traces}$

where *traces* denote form factors containing some $g_{\mu_i \mu_j}$ tensor
For example, in the case of $O_{\mu_1 \mu_2}$, we have

$$\langle P|O_{\mu_1 \mu_2}|P\rangle = A_2 P_{\mu_1} P_{\mu_2} + B_2 g_{\mu_1 \mu_2}$$

- Physical PDFs are related to the A_n form factors (moments)
- B_n are spurious contributions that need to be subtracted out
- In Minkowski region this is automatically achieved by taking $\xi^2 = 0$
- In the Euclidean case the situation is more complicated

Euclidean metrics

- Eliminating trace terms problematic if only Euclidean data are available
- To make contact with Minkowski physics we may want to consider

  the equal time correlator $\langle P|\phi(0)\phi(\xi)|P\rangle\Big|_{\xi=(0,0,0,z)} = F(P_z z, -z^2)$

- $F(P_z z, -z^2)$ depends on two independent variables $\alpha \equiv P_z z$ & $\beta \equiv -z^2$
- The desired structure function is recovered from the (obvious) formula

$$f(x) = \lim_{\beta\to 0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\alpha, \beta) e^{ix\alpha} d\alpha$$

- Thus to remove *trace* terms in Euclidean region we must know $\langle P|\phi(0)\phi(z)|P\rangle$ for $P_z \to \infty$ as $z \to 0$, while keeping $\alpha = P_z z$ fixed
- In lattice simulations this requirement poses problems
  - momenta are bounded from above by a^{-1} (inverse lattice spacing)
  - this in turn limits the minimal value that z can take to be $O(\alpha\, a)$

Renormalization - I

What about renormalization issues?

- DIS scaling in QCD is controlled by computable logarithmic corrections
- $O_{\mu_1 \ldots \mu_n}$ require not just a simple multiplicative renormalization
- as $\langle P|O_{\mu_1 \ldots \mu_n}|P\rangle$ matrix elements are power divergent
- We need to resolve the mixing with lower dimensional (trace) operators to make finite A_n and B_n form factors
- In particular, to be able to take the limit $P_z \to \infty$ (necessary to eliminate the contamination from higher twists) one needs to make the B_n's finite
- The only renormalization considered in the original Ji paper was the multiplicative "matching condition", according to which
- one starts by considering the regularized quantity

$$\tilde{F}(x, P_z; \Lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d(zP_z) e^{ix(zP_z)} \langle P_z|\phi(0)\phi(z)|P_z\rangle\Big|_{\Lambda}$$

Renormalization - II

- Renormalization is carried out by introducing $F(x, P_z; \mu)$ through

$$\tilde{F}(x, P_z; \Lambda) = \int_x^{+\infty} \frac{dx'}{x'} Z(\frac{x}{x'}; \Lambda, \mu) F(x', P_z; \mu)$$

  where $Z(\frac{x}{x'}; \Lambda, \mu)$ is a logarithmically divergent renormalization function (computed in PT) such that $F(x, P_z; \mu)$ is UV finite

- The convolution property of the Mellin transform implies

$$\int_{-\infty}^{+\infty} dx\, x^n \tilde{F}(x, P_z; \Lambda) = \int_{-\infty}^{+\infty} dx' x'^{\,n} Z(x'; \Lambda, \mu) \int_{-\infty}^{+\infty} dx\, x^n F(x, P_z; \mu) \equiv$$

$$\equiv Z_n\left(\frac{\Lambda}{\mu}\right) \int_{-\infty}^{+\infty} dx\, x^n F(x, P_z; \mu)$$

- Moments of $\tilde{F}$ renormalize multiplicatively and independently one from the others

Renormalization - III

- If one could take the limit $P_z \to \infty$ the relation

$$\int_{-\infty}^{+\infty} dx \tilde{F}(x, P_z; \Lambda) x^n = Z_n\left(\frac{\Lambda}{\mu}\right) \int_{-\infty}^{+\infty} dx\, x^n F(x, P_z; \mu)$$

would allow computing the moments of the physical PDFs

But performing this limit turns out to be “problematic”, as we now argue

- Taking the n-th derivative with respect to z at $z = 0$ of (see slide 8)

$$\langle P|\phi(0)\phi(z)|P\rangle\Big|_{\Lambda} = \int_{-\infty}^{+\infty} dx\, e^{-ixzP_z} \tilde{F}(x, P_z; \Lambda)$$

one gets

$$(-i)^n \int_{-\infty}^{+\infty} dx\ x^n \tilde{F}(x, P_z; \Lambda) = \frac{1}{(P_z)^n} \langle P|\phi(0)\frac{\partial^n \phi}{\partial z^n}(0)|P\rangle\Big|_{\Lambda}$$

hence

$$\int_{-\infty}^{+\infty} dx\, x^n F(x, P_z; \mu) = \frac{(-i)^n}{Z_n(\Lambda/\mu)} \int_{-\infty}^{+\infty} dx\ x^n \tilde{F}(x, P_z; \Lambda) =$$

$$= \frac{1}{(P_z)^n} \langle P| \frac{1}{Z_n(\Lambda/\mu)} \phi(0) \frac{\partial^n \phi}{\partial z^n}(0)|P\rangle\Big|_{\Lambda}$$

Renormalization - IV

- The "matching" procedure has led to the relation

$$\int_{-\infty}^{+\infty} dx\, x^n F(x, P_z; \mu) = \frac{1}{(P_z)^n} \langle P| \frac{1}{Z_n(\Lambda/\mu)} \phi(0) \frac{\partial^n \phi}{\partial z^n}(0)|P\rangle\Big|_\Lambda$$

As $P_z \to \infty$ it should yield the "measurable, UV finite" PDF moments

- This is not so however
  - In the r.h.s. we have power divergent equal-point operators
    - that feature $(P_z a)^{-2k}$ divergent terms
  - Since P_z can never exceed a^{-1}
    - (one would never take momenta larger than the UV cutoff)
    - such power divergent terms need to be subtracted out
    - thus multiplicative renormalization is not enough

PDF from the current-current T-product?

To overcome these difficulties Ma & Qiu propose to compute directly on the lattice

$$\sigma(\omega,\xi^2) = \langle P|T(J(0)J(\xi))|P\rangle\,, \qquad \omega = P\cdot\xi$$

They use OPE, valid for small ξ^2, to rewrite σ in the form

$$\sigma(\omega,\xi^2) = \sum_n W_n(\xi^2;\mu)\,\xi^{\mu_1}\xi^{\mu_2}\ldots\xi^{\mu_n}\langle P|O_{\mu_1\mu_2\ldots\mu_n}(\mu)|P\rangle$$

After introducing the definition

$$\langle P|O_{\mu_1\mu_2\ldots\mu_n}(\mu)|P\rangle = A_n(\mu)(P_{\mu_1}P_{\mu_2}\ldots P_{\mu_n} - traces)$$

with

$$A_n(\mu) = \int \frac{dx}{x}\, x^n f(x;\mu)$$

$\sigma(\omega,\xi^2)$ can be cast in the form

$$\sigma(\omega,\xi^2) = \int \frac{dx}{x} f(x;\mu)K(x\omega,\xi^2,x^2;\mu) + \mathrm{O}(\xi^2\Lambda^2_{QCD})$$

$$K(x\omega,\xi^2,x^2;\mu) = \sum_n x^n W_n(\xi^2;\mu)\,\xi^{\mu_1}\xi^{\mu_2}\ldots\xi^{\mu_n}(P_{\mu_1}P_{\mu_2}\ldots P_{\mu_n} - traces)$$

Since $K(x\omega, \xi^2, x^2; \mu)$ can be computed in PT, it is claimed that the full $f(x; \mu)$ can be obtained as the one-dimensional FT

$$\frac{1}{4\pi} \int \frac{d\omega}{\omega} \, e^{-ix\omega} \sigma(\omega, \xi^2) = f(x; \mu)$$

if lattice data are inserted for $\sigma(\omega, \xi^2)$

The difficulties with this approach are similar to the one we have identified before

- The equation above should be more correctly written

$$\frac{1}{4\pi} \int \frac{d\omega}{\omega} \, e^{-ix\omega} \sigma(\omega, \xi^2) = f(x; \mu) + O(\xi^2 \Lambda^2_{QCD})$$

- To give higher twists a vanishing weight one should take, besides $\xi^0 = 0$, also $\xi^3 = z \to 0$ in order to maintain the Euclidean constraint $\xi^2 \to 0$
- If one does so, however, to keep the integration variable $\omega = P_z \, z$ fixed, one needs to send $P_z \to \infty$ as $z \to 0$
- Again, it looks problematic to send $P_z \to \infty$ because the accessible values of P_z are limited by the lattice UV cutoff.

Moment resummation from current-current T-product

- Euclidean lattice data can instead in principle give access to PDF moments (similarly to what it was proposed to do in configuration-space renormalization, see C. Dawson *et al.* Nucl. Phys. B **514** (1998) 313)
- We can assume to be able to disentangle numerically all the PDF moments starting from $\langle P|J(0)J(\xi)|P\rangle|^{lattice}$ by fitting its (singular) ξ^2 dependence (J. Karpie, K. Orginos, S. Zafeiropoulos 1807.10933 [het-lat])
- Can one NP-ly resum the moment series and reconstruct the full PDF?
- The Mellin theory tells us that this step actually requires the knowledge of moments for complex values of n – something we do not have
- Lacking this information, one can show that the only possible alternative for a formal moment resummation is provided by the Ji expression

- To see this let's introduce the one-dimensional FT

$$\tilde{f}(P\cdot\xi;\mu) = \int \frac{dx}{2\pi} e^{ixP\cdot\xi} f(x;\mu)$$

- The derivatives of $\tilde{f}(P\cdot\xi;\mu)$ at $\xi = 0$ are related to the moments of $f(x;\mu)$

$$\frac{1}{i^n} \frac{\partial^n \tilde{f}(P\cdot\xi;\mu)}{\partial\xi^{\mu_1}\partial\xi^{\mu_2}\ldots\partial\xi^{\mu_n}}\Big|_{\xi=0} = P_{\mu_1}P_{\mu_2}\ldots P_{\mu_n}\frac{1}{2\pi}\int dx x^n f(x;\mu) =$$
$$= P_{\mu_1}P_{\mu_2}\ldots P_{\mu_n}A_n(\mu)$$

- Recalling

$$\langle P|O_{\mu_1\mu_2\ldots\mu_n}(0)|P\rangle = A_n P_{\mu_1}P_{\mu_2}\ldots P_{\mu_n} + traces\,,$$
$$O_{\mu_1\mu_2\ldots\mu_n} = \phi(0)\frac{\partial^n\phi(\xi)}{\partial\xi^{\mu_1}\partial\xi^{\mu_2}\ldots\partial\xi^{\mu_n}}\Big|_{\xi=0}$$

  and ignoring for a moment renormalization issues, we immediately get

$$\tilde{f}(P\cdot\xi;\mu) = \langle P|\phi(0)\phi(\xi)|P\rangle$$

  which is precisely the Ji formula
- *Multiplicative* renormalization of moments can be dealt with by means of the "matching condition" as discussed in slide 9
- The conclusion is that the knowledge of moments is not enough to reconstruct the full PDF: one ends up with the Ji formula

Other approaches and variations thereof

1. An interesting alternative is offered by the use of the reduced Ioffe-time distributions advocated by Zhang *et al.* (PRD 96 (2017) 094503) and Orginos *et al.* (PRD 97 (2018) 074508)

$$\mathfrak{M}(zP_z, z^2) = \frac{F(zP_z, z^2)}{F(0, z^2)}$$

$$F(P\,z, z^2) = \langle P|\phi(0)\phi(\xi)|P\rangle\Big|_{\xi=(0,0,0,z)}$$

coupled to some perturbative subtraction

2. I'm now going to discuss the pro's and con's of this approach with the help of the formulation provided by A.V. Radyushkin 1807.07509v2 [hep-ph]

Ioffe-time distributions and power divergent mixings

- One can prove the formula (in the notations of A.V. Radyushkin 1807.07509v2 [hep-ph])

  - $$Q(y,P) = f(y,\mu^2) - \frac{\alpha_s}{2\pi} C_F \int_0^1 \frac{du}{u} f(\frac{y}{u},\mu^2)\Big[B(u)\ln(\frac{\mu^2}{P^2}) + C(u)\Big] +$$
    $$+\frac{\alpha_s}{2\pi} C_F \int_{-1}^1 dx\, f(x,\mu^2) L(y,x) + O(P^{-2}) + O(\alpha_s^2)$$
  - where $L(y,x) = \frac{P}{2\pi}\int_0^1 du\, B(u) \int_{-\infty}^{+\infty} dz\, e^{-i(y-ux)zP} \ln(z^2 P^2)$

  - The last term produces (unwanted) contributions in the $|y| > 1$ region (responsible for power divergent moments)
  - One can thus think of subtracting out by hand these terms writing
    - $$f(y,\mu^2) = \Big[Q(y,P) - \frac{\alpha_s}{2\pi} C_F \int_{-1}^1 dx\, f(x,\mu^2) L(y,x)\Big] +$$
      $$+\frac{\alpha_s}{2\pi} C_F \int_0^1 \frac{du}{u} f(\frac{y}{u},\mu^2)\Big[B(u)\ln(\frac{\mu^2}{P^2}) + C(u)\Big] + O(P^{-2}) + O(\alpha_s^2)$$

$$f(y,\mu^2) = \Big[Q(y,P) - \frac{\alpha_s}{2\pi} C_F \int_{-1}^{1} dx\, f(x,\mu^2) L(y,x)\Big] +$$

$$+\frac{\alpha_s}{2\pi} C_F \int_0^1 \frac{du}{u} f(\frac{y}{u},\mu^2)\Big[B(u)\ln(\frac{\mu^2}{P^2}) + C(u)\Big] + \mathrm{O}(P^{-2}) + \mathrm{O}(\alpha_s^2)$$

The difficulties posed by this procedure, which is widely used in actual simulations, are as follows

- Subtraction needs to be carried out before removing the cutoff
- The term in square parenthesis has a smooth $P \to \infty$ limit
  - but $\mathrm{O}(\alpha_s^2)$ corrections don't: at small lattice spacings they matter
- In the r.h.s. the PDF, $f(y,\mu^2)$, one is looking for appears
  1. in practice it gets replaced by lattice $Q(y,P)$ to leading order in α_s
     - $Q(y,P)$ does not have the proper support properties
     - one thus needs to enforce them by hand (non-localities introduced)
  2. then the question arises
     - are the moments of the PDF built in this way the matrix elements of the renormalized local DIS operators one finds in the Bjorken limit?

Conclusions

- In this talk I have rediscussed the viability of the proposal of directly extracting PDF's from lattice simulations
- Apparently there are still missing ingredients in such a program related to the problem of subtracting power divergent trace terms
- In summary at this moment
  - the original Ji formalism of using the "matched" bilocal operator does not allow accessing the full PDF from lattice simulations
  - direct simulations of the current-current T-product surely allow extracting PDF moments (see e.g. Ma & Qiu and J. Karpie, K. Orginos, S. Zafeiropoulos)
  - perhaps more promising is the idea of subtracting by hand in PT from lattice data, terms that ruin the compactness of the PDF support and are responsible for power divergent moments

THANKS FOR YOUR ATTENTION

Back-up slides

NP subtraction: a mathematical toy-model

- A way to provide an intuition of the impact of power divergent mixings is to consider the toy-model representation

$$\langle P|\phi(0)\phi(z)|P\rangle\Big|_{\Lambda} = \int dk\, e^{-\frac{k^2}{\Lambda^2}} e^{ikz} g(P_z z, k)$$

- The factor e^{ikz} describes trace operator divergent terms $\to$ expanding in z leads to extra powers of $k \to (\Lambda z)^n$ divergencies
- Crossing out e^{ikz} is an effective way of subtracting them
- If we do so we get for the PDF

$$f(\omega;\Lambda) = P_z \int_{-\infty}^{\infty} dz\, e^{i\omega P_z z} \int dk\, e^{-\frac{k^2}{\Lambda^2}} g(P_z z, k) = \int dk\, e^{-\frac{k^2}{\Lambda^2}}\, \tilde{g}(\omega, k)$$

where $\tilde{g}(\omega, k) = \int dy\, e^{i\omega y} g(y, k)$

- If we don't, we get

$$\widehat{f}(\omega;\Lambda) = P_z \int_{-\infty}^{\infty} dz\, e^{i\omega P_z z} \int dk\, e^{-\frac{k^2}{\Lambda^2}} e^{ikz} g(P_z z, k) =$$

$$= \int dk\, e^{-\frac{k^2}{\Lambda^2}} \tilde{g}(\omega + \frac{k}{P_z}, k) \stackrel{\Lambda\to\infty}{\to} \widehat{f}(\omega) = \int dk\, \tilde{g}(\omega + \frac{k}{P_z}, k)$$

rather than

$$f(\omega;\Lambda) = \int dk\, e^{-\frac{k^2}{\Lambda^2}}\, \tilde{g}(\omega, k) \stackrel{\Lambda\to\infty}{\to} f(\omega) = \int dk\, \tilde{g}(\omega, k)$$

- In the last formulae we have taken the limit $\Lambda \to \infty$
- Mixings with trace operators do not show up as (power) divergencies in $\widehat{f}(\omega)$
- Rather at finite P_z they deform the expression of the latter
- The limit $P_z \to \infty$ cannot be taken inside the integral
- unless the integrand is "well behaved", i.e. made finite!

Support truncation "by hand" destroys locality

- Recall the definition

$$F(\omega, P_z) = \int_{-\infty}^{+\infty} dz'\, e^{iz' P_z \omega} \langle P|\phi(0)\phi(z')|P\rangle$$

- If we truncate the ω-support of $F(\omega, P_z)$ by hand, we get

$$\int_{-1}^{+1} d\omega\, e^{-izP_z\omega} F(\omega, P_z) = \frac{P_z}{2\pi} \int_{-1}^{+1} d\omega \int_{-\infty}^{+\infty} dz'\, e^{-i(z-z')P_z\omega} \langle P|\phi(0)\phi(z')|P\rangle =$$

$$= \frac{1}{\pi} \int_{-\infty}^{+\infty} dz' \, \frac{\sin P_z(z-z')}{z-z'} \langle P|\phi(0)\phi(z')|P\rangle \stackrel{?}{\to} \langle P|\phi(0)\phi(z)|P\rangle$$

- This quantity, the FT of which gives rise to a support-truncated $F(\omega, P_z)$, is not the matrix element of a bilocal operator
- $\lim_{P_z\to\infty} \frac{1}{\pi} \frac{\sin P_z(z-z')}{z-z'} = \delta(z-z')$ cannot be taken
  - because of the P_z-dependence of the quantity under the integral
  - unless the latter is the FT of a function with $[-1, +1]$ support!