On the rare CP conserving $K \to \pi \ell^+ \ell^-$ Decays
$(K = K^\pm, K_S)$

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based on work in progress done in collaboration with G. D'Ambrosio and D. Greynat

OUTLINE

I. Introduction

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I. Introduction

Rare kaon decays [$K \to \pi\gamma^{(*)}$, etc.] proceed through FCNC, are suppressed in the SM $\longrightarrow$ might provide an interesting window into new physics

For a review, see V. Cirigliano et al, Rev Mod Phys 84, 399 (2012)

Particularly interesting examples

$$K^+ \to \pi^+ \nu\bar{\nu} \text{ [NA62]} \qquad\qquad K_L \to \pi^0 \nu\bar{\nu} \text{ [KOTO]}$$

- dominated by short-distances
- clean SM prediction, matrix elements from $K_{\ell 3}$

$$Br(K^+ \to \pi^+ \nu\bar{\nu})_{\mathrm{SM}} = 8.39(30) \cdot 10^{-11} \left[\frac{|V_{cb}|}{40.7 \cdot 10^{-3}}\right]^{2.8} \left[\frac{\gamma}{73.2^{\mathrm{o}}}\right]^{0.74}$$

$$Br(K_L \to \pi^0 \nu\bar{\nu})_{\mathrm{SM}} = 3.36(5) \cdot 10^{-11} \left[\frac{|V_{ub}|}{3.88 \cdot 10^{-3}}\right]^{2} \left[\frac{|V_{cb}|}{40.7 \cdot 10^{-3}}\right]^{2} \left[\frac{\sin\gamma}{\sin 73.2^{\mathrm{o}}}\right]^{2}$$

J. Brod et al., Phys Rev D 83, 034030 (2011)

A. J. Buras et al, JHEP 1511, 33 (2011)

For a review, see A. Buras et al, Rev Mod Phys 80, 965 (2008)

cf. talk by Evgueni Goudzovski

In general, long distances dominate the amplitudes

$\longrightarrow$ difficult to make predictions due to unknown hadronic matrix elements

The CP conserving decays considered here

$$K^{\pm} \to \pi^{\pm}\gamma^{*} \to \pi^{\pm}\ell^{+}\ell^{-} \qquad\qquad K_S \to \pi^{0}\gamma^{*} \to \pi^{0}\ell^{+}\ell^{-}$$

belong to this second category

- similar short-distance parts as in $K \to \pi\nu\bar{\nu}$

- analogues, in the kaon sector, of $b \to s\ell^{+}\ell^{-}$ transitions

- any LFUV effect invoked in order to explain the anomalies seen at LHCb should also manifest itself here

cf. talks by Arantza Oyanguren Campos and by Danny van Dyk

- $K_S \to \pi^{0}\ell^{+}\ell^{-}$ gives the contribution of indirect CPV to $K_L \to \pi^{0}\ell^{+}\ell^{-}$

Matrix elements $(K,\pi) = (K^{\pm},\pi^{\pm}),\ (K_S,\pi^0)$

$$\mathcal{A} = e^2 \times \bar{\mathrm{u}}(p_{\ell^-})\gamma_\sigma \mathrm{v}(p_{\ell^+}) \times \frac{(-1)}{s}\left[\eta^{\sigma\rho} - (1-\xi)\,\frac{(k-p)^\sigma (k-p)^\rho}{s}\right] \times \langle \pi(p)|j_\rho(0)|K(k)\rangle_{\mathrm{SM}}$$

$$j_\rho = \sum_{q=u,d,s} e_q \bar{q}\gamma_\rho q \qquad s \equiv (p_{\ell^+} + p_{\ell^-})^2 = (k-p)^2$$

$$\langle \pi(p)|j_\rho(x)|K(k)\rangle_{\mathrm{SM}} = \left[s(k+p)_\rho - (k-p)_\rho (M_K^2 - M_\pi^2)\right] \times \frac{W(s)}{16\pi^2 M_K^2}$$

$$\mathcal{A} = (-e^2) \times \bar{\mathrm{u}}(p_{\ell^-})\gamma_\rho \mathrm{v}(p_{\ell^+}) \times (k+p)^\rho \times \frac{W(s)}{16\pi^2 M_K^2}$$

$$W(s) = W_{\mathrm{LD}}(s;\nu) + W_{\mathrm{SD}}(s;\nu)$$

$\nu =$ short-distance renormalization scale

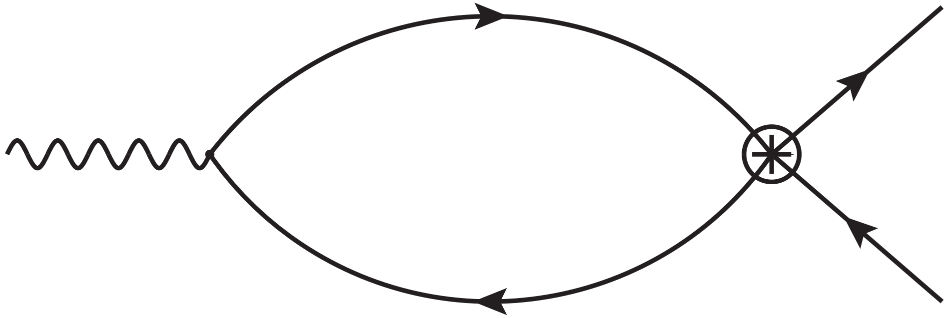
First order in G_F and in α

$$\left[s(k+p)_\rho - (M_K^2 - M_\pi^2)(k-p)_\rho\right] \times \frac{W_{\mathrm{LD}}(s;\nu)}{16\pi^2 M_K^2} = i\int d^4x\, \langle \pi(p)|T\{j_\rho(0)\mathcal{L}^{\Delta S=1}_{\mathrm{non-lept}}(x)\}|K(k)\rangle_{\mathrm{QCD}}$$

$$\mathcal{L}^{\Delta S=1}_{\mathrm{non-lept}}(x) = -\frac{G_{\mathrm{F}}}{\sqrt{2}} V_{us}^* V_{ud} \sum_{I=1}^{6} C_I(\nu) Q_I(x;\nu)$$

$\longrightarrow$ pure three-flavour (four-flavour on the lattice) QCD problem

$$\nu \frac{d}{d\nu} j_\rho = 0 \qquad \nu \frac{d}{d\nu} \mathcal{L}^{\Delta S=1}_{\rm non-lept} = 0 \qquad \text{but} \;\; \nu \frac{d}{d\nu} |T\{j_\rho(0) \mathcal{L}^{\Delta S=1}_{\rm non-lept}(x)\} \neq 0$$



Renormalization through the operator

$$\mathcal{L}^{\Delta S=1}_{\rm lept}(x) = -\frac{G_{\rm F}}{\sqrt{2}} V^*_{us} V_{ud} \left[C_{7V}(\nu) Q_{7V}(x) + C_{7A} Q_{7A}(x) \right]$$

$$Q_{7V} = (\bar{s}^i d_i)_{V-A} (\bar{\ell}\ell)_V \qquad Q_{7A} = (\bar{s}^i d_i)_{V-A} (\bar{\ell}\ell)_A$$

E. Witten, Nucl. Phys. B 122, 109 (1977)

F. J. Gilman, M. B. Wise, Phys Rev D 21, 3150 (1980)

C. Dib et al., Phys Lett B 218, 487 (1989); Phys Rev D 39, 2639 (1989)

J. Flynn, L. Randall, Nucl Phys B 326, 31 (1989) [Nucl Phys B 334, 580 (1990)]

A. J. Buras et al., Nucl Phys B 423, 349 (1994)

Several tools in order to evaluate the corresponding matrix elements:

- Chiral perturbation theory
  - one loop — G. Ecker et al., Nucl Phys B 291, 692 (1987)
  - beyond one loop — G. D'Ambrosio et al., JHEP 9808, 004 (1998)

$\longrightarrow$ limitation: unknown low-energy constants

- Chiral perturbation theory and large-N_c

S. Friot et al., Phys Lett B 595, 301 (2004)

E. Coluccio Leskov et al, Phys Rev D 93, 094031 (2016)

- Lattice QCD

G. Isidori et al., Phys Lett B 633, 75 (2006)

N. H. Christ et al, Phys Rev D 92, 094512 (2015); D 94, 114516 (2016)

cf. talk by Antonin Portelli

- $\cdots$

J. Portolés, J Phys Conf Series 800, 012030 (2017)

II. Experimental situation & Phenomenological aspects

Rather rich set of data

$K^\pm \to \pi^\pm e^+ e^-$:

- BNL AGS, ~ 500 events, K^+ only — C. Alliegro et al., Phys Rev Lett 68, 278 (1992)
- BNL AGS, 10300 events, K^+ only — R. Appel et al. [E865 Collaboration], Phys Rev Lett 83, 4482 (1999)
- CERN SPS, 7263 events, both K^+ and K^- — J. R. Batley et al. [NA48/2 Collaboration], Phys Lett B 677, 246 (2009)

$K^\pm \to \pi^\pm \mu^+ \mu^-$:

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- BNL AGS, ~ 400 events, K^+ only — H. Ma et al. [E865 Collaboration], Phys Rev Lett 84, 2580 (2000)
- FNAL, ~ 100 events, both K^+ and K^- — H. K. Park et al. [HyperCP Collaboration], Phys Rev Lett 88, 111801 (2002)
- CERN SPS, 3120 events, both K^+ and K^- — J. R. Batley et al. [NA48/2 Collaboration], Phys Lett B 697, 107 (2011)

$K_S \to \pi^0 e^+ e^-$:

- CERN SPS, 7 events — J. R. Batley et al. [NA48/1 Collaboration], Phys Lett B 576, 43 (2003)

$K_S \to \pi^0 \mu^+ \mu^-$:

- CERN SPS, 6 events — J. R. Batley et al. [NA48/1 Collaboration], Phys Lett B 599, 197 (2004)

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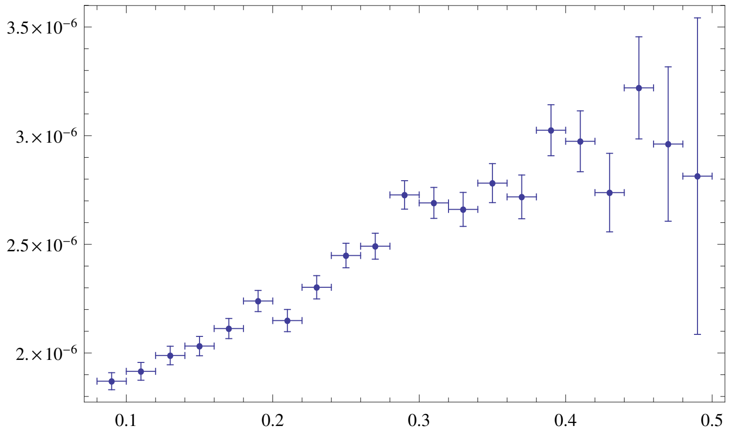
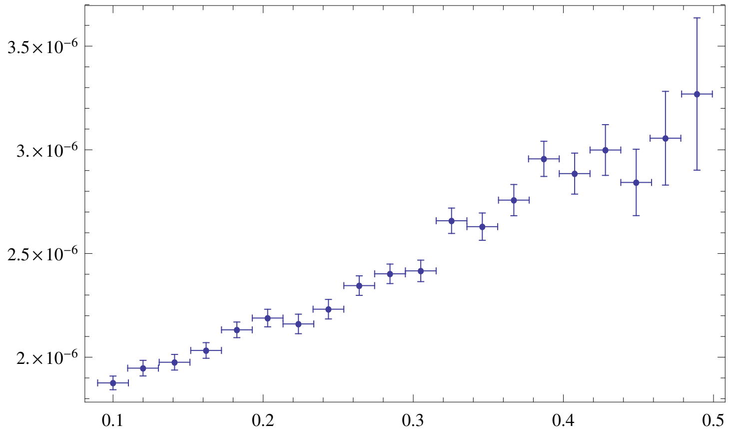
$$R_{K^\pm} \equiv \frac{\mathrm{Br}[K^\pm \to \pi^\pm \mu^+ \mu^-]}{\mathrm{Br}[K^\pm \to \pi^\pm e^+ e^-]} = \begin{cases} 0.313(71) & \text{[PDG average]} \\ 0.309(43) & \text{[NA48/2 alone]} \end{cases}$$

Differential decay rate

$$\frac{d\Gamma}{dz} = \frac{\alpha^2 M_K}{3(4\pi)^5} \lambda^{3/2}(1, z, M_\pi^2/M_K^2) \sqrt{1 - 4\frac{m_\ell^2}{z M_K^2}} \left(1 + 2\frac{m_\ell^2}{z M_K^2}\right) |W(z)|^2 \quad z \equiv s/M_K^2$$



NA48/2 (21 bins)



E865 (20 bins)

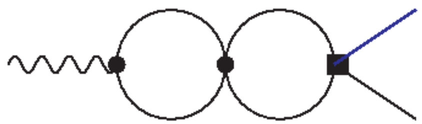
what do data tell us about $W(z)$?

Data analyzed with several parameterisations of $W(z)$

Concentrate on "beyond one loop" (BOL) model

$$W_{\text{BOL}}(z) = G_F M_K^2 (a_+ + b_+ z) + \frac{8\pi^2}{3} \frac{M_K^2}{M_\pi^2} \left[\alpha_+ + \beta_+ \frac{M_K^2}{M_\pi^2}(z - z_0)\right] \left(1 + \frac{M_K^2}{M_V^2} z\right) \left[\frac{z - 4\frac{M_\pi^2}{M_K^2}}{z} \bar{J}_{\pi\pi}(z M_K^2) + \frac{1}{24\pi^2}\right]$$

G. D'Ambrosio et al., JHEP 9808, 004 (1998)



- Similar representation for $K_S \to \pi^0 \ell^+ \ell^-$
- α_+, β_+ from slope and curvature of $K^+ \to \pi^+\pi^+\pi^-$ amplitude (more on this later)

$$\alpha_+ = -20.84(74) \cdot 10^{-8}, \quad \beta_+ = -2.88(1.08) \cdot 10^{-8}$$

J. Bijnens et al. Nucl. Phys. B 648, 317 (2003)

- Only loops with pions, loops with kaons included in the polynomial part
- Still not a complete two-loop representation
- The term proportional to $\beta_+ \times (M_K^2/M_V^2)$ is actually NNLO (not important for the fits)
- One loop corresponds to $b_+ \sim 0$, $\beta_+ = 0$, $M_V \to \infty$ (after expanding kaon loops)
- $a_+ = W_{\text{BOL}}(0)/G_F M_K^2$, $b_+ \sim W'_{\text{BOL}}(0)/G_F M_K^2$

Disclaimer:

- NA48/2 data available on hepdata, statistical errors only

$$\Delta a_+ = 0.012 \qquad \Delta b_+ = 0.053$$

- only source for the E865 data (so far) is the plot given in their paper

$$\Delta a_+ = 0.010 \qquad \Delta b_+ = 0.044$$

- uncertainties induced by $\Delta\alpha_+$ are ~ 0.001 in a_+ and ~ 0.016 in b_+

Fit to NA48/2 data (e^+e^-)

$\beta_+ \cdot 10^8$	a_+	b_+	χ^2/d.o.f
-3.96	$+0.485$	$+1.689$	29.7/19
	-0.581	-0.779	35.7/19
-2.88	$+0.491$	$+1.691$	28.3/19
	-0.585	-0.779	33.0/19
-1.80	$+0.496$	$+1.693$	27.1/19
	-0.580	-0.779	30.7/19

Two close-by minima, $a_+ > 0$ and $b_+ > 0$ slightly favoured (but $b_+ > 3a_+$ difficult to understand)

Experimental analysis explored only the minimum around the negative solution (theoretical prejudice)

Fit to E865 data (e^+e^-)

$\beta_+ \cdot 10^8$	a_+	b_+	χ^2/d.o.f
-3.96	$+0.479$	$+1.602$	55.6/18
	-0.599	-0.632	13.2/18
-2.88	$+0.485$	$+1.600$	50.5/18
	-0.592	-0.635	12.2/18
-1.80	$+0.492$	$+1.597$	45.9/18
	-0.586	-0.638	11.7/18

The case $a_+ < 0$, $b_+ < 0$, $a_+ \sim b_+$ quite clearly favoured

Combined fit (e^+e^-)

$\beta_+ \cdot 10^8$	a_+	b_+	$\chi^2/\mathrm{d.o.f}$
-3.96	$+0.484$	$+1.625$	101.1/39
	-0.598	-0.682	65.2/39
-2.88	$+0.490$	$+1.624$	94.8/39
	-0.591	-0.683	61.7/39
-1.80	$+0.496$	$+1.623$	89.1/39
	-0.585	-0.685	58.8/39

Negative solution clearly favoured, but χ^2 not very good
(improves as β_+ moves towards higher values $\longrightarrow$ see later)

Combined fit (e^+e^-)

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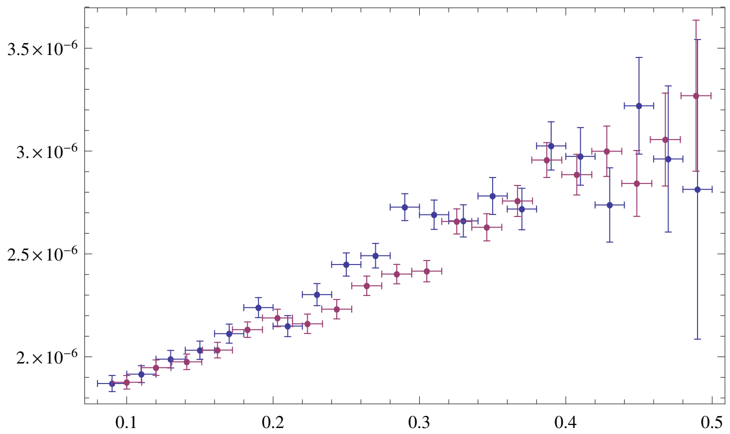
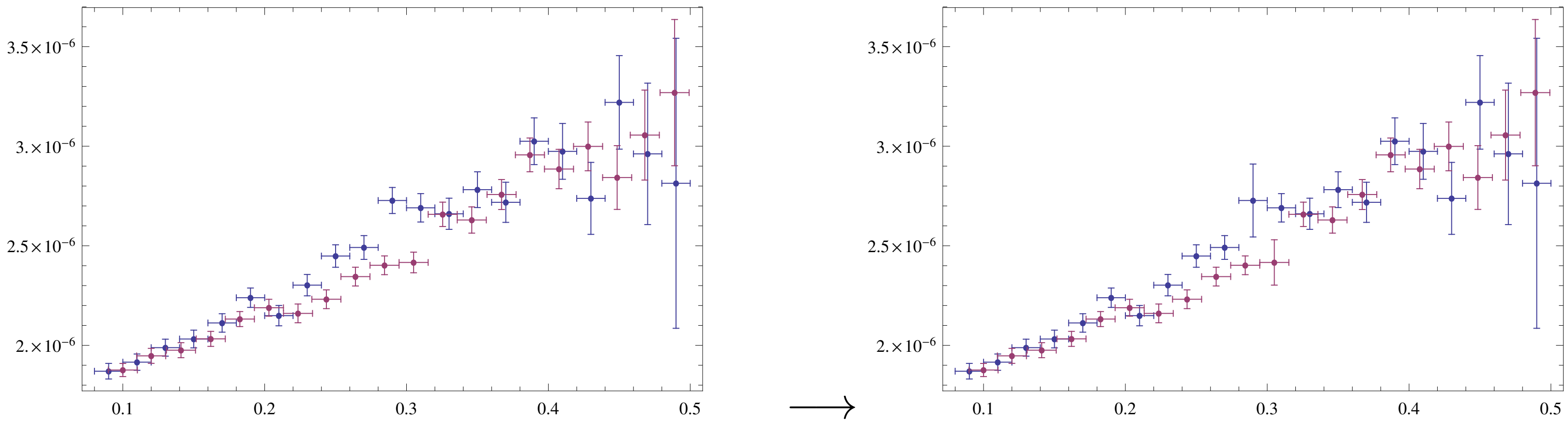
-1.80	$+0.496$	$+1.623$	89.1/39
	-0.585	-0.685	58.8/39



NA48/2 + E865

Combined fit (e^+e^-)

$\beta_+ \cdot 10^8$	a_+	b_+	χ^2/d.o.f
-3.96	$+0.483$	$+1.632$	86.7/39
	-0.598	-0.678	48.8/39
-2.88	$+0.489$	$+1.630$	60.4/39
	-0.592	-0.680	45.4/39
-1.80	$+0.495$	$+1.629$	74.8/39
	-0.585	-0.682	42.8/39



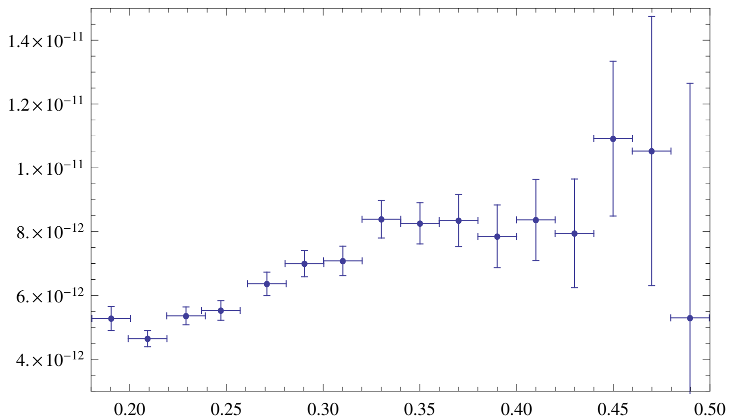
NA48/2 + E865

Rescale error bar of one data point in each set (notice that $z_{\rm thr} \equiv 4M_\pi^2/M_K^2 \sim 0.3$)

Quite reasonable χ^2, negative solution clearly favoured

Fit to NA48/2 data ($\mu^+\mu^-$)

$\beta_+ \cdot 10^8$	a_+	b_+	χ^2/d.o.f
-3.96	$+0.372$	$+2.102$	11.9/15
	-0.611	-0.746	15.9/15
-2.88	$+0.384$	$+2.081$	12.1/15
	-0.598	-0.768	15.2/15
-1.80	$+0.397$	$+2.060$	12.4/15
	-0.585	-0.790	14.5/15



NA48/2

As in the NA48/2 data for the e^+e^- channel, the data show a slight preference for the positive solution

Fits were performed for fixed values of α_+ and β_+

- α_+ is the slope of the $K^+ \to \pi^+\pi^+\pi^-$ amplitude
- and β_+ corresponds to one of its curvatures

$$A^{K^+\to\pi^+\pi^+\pi^-}(s_1, s_2, s_3) = (2\alpha_1 - \alpha_3) + (\beta_1 - \frac{\beta_3}{2} + \sqrt{3}\,\gamma_3)Y + 2(\zeta_1 + \zeta_3)\left(Y^2 + \frac{X^2}{3}\right) - (\xi_1 + \xi_3 - \xi_3')\left(Y^2 - \frac{X^2}{3}\right)$$

$$Y \equiv \frac{s_3 - s_0}{M_\pi^2} \quad X \equiv \frac{s_2 - s_1}{M_\pi^2} \quad s_0 \equiv M_K^2 + 3M_\pi^2 \quad s_1 + s_2 + s_3 = 3s_0$$

- Values

$$\alpha_+ = -20.84(74) \cdot 10^{-8}, \quad \beta_+ = -2.88(1.08) \cdot 10^{-8}$$

result from a global fit to all $K \to \pi\pi\pi$ decay distributions

J. Bijnens et al. Nucl. Phys. B 648, 317 (2003)

- α_+ determined rather tightly, fit less sensitive to curvatures

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What do the $K^+ \to \pi^+\ell^+\ell^-$ data tell us about β_+?

$\longrightarrow$ fit values of $a_+,\ b_+,\ \beta_+$

Separate fits

$$a_+ = -0.582(37),\quad b_+ = -0.639(95),\quad -4.9 \le \beta_+ \cdot 10^8 \le +3.0,\quad \chi^2/\mathrm{d.o.f} = 11.6/17\ \ [e^+e^-,\ \mathrm{E865}]$$

$$a_+ = -0.535(45),\quad b_+ = -0.771(130),\quad -0.4 \le \beta_+ \cdot 10^8 \le +13.4,\quad \chi^2/\mathrm{d.o.f} = 22.4/18\ \ [e^+e^-,\ \mathrm{NA48/2}]$$

$$a_+ = -0.40(17),\quad b_+ = -1.10(36),\quad +0.7 \le \beta_+ \cdot 10^8 \le +27.8,\quad \chi^2/\mathrm{d.o.f} = 9.8/14\ \ [\mu^+\mu^-,\ \mathrm{NA48/2}]$$

Combined NA48/2+E865 e^+e^- fit

rescaled

$$a_+ = -0.563(29),\quad b_+ = -0.691(77),\quad -1.4 \le \beta_+ \cdot 10^8 \le +5.9,\quad \chi^2/\mathrm{d.o.f} = 34.8/38\ \ [\mathrm{NA48/2} + \mathrm{E865}]$$

not rescaled

$$a_+ = -0.561(33),\quad b_+ = -0.694(89),\quad -1.8 \le \beta_+ \cdot 10^8 \le +6.8,\quad \chi^2/\mathrm{d.o.f} = 49.2/38\ \ [\mathrm{NA48/2} + \mathrm{E865}]$$

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data favour higher values of β_+, $\sim 3\sigma$ away from $K \to \pi\pi\pi$ determination

III. Theoretical aspects

Robustness of determinations of a_+ and b_+

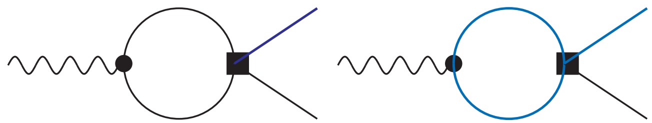
Impact of remaining two-loop contributions, not contained in $W_{\text{BOL}}(z)$

Predictions for a_+ and b_+?

$$W_{\rm 1loop}(z) \;=\; G_F M_K^2 a_+^{\rm 1loop} + \frac{8\pi^2}{3}\frac{M_K^2}{M_\pi^2}\left\{\alpha_+^{\rm 1loop}\left[\frac{z-4\frac{M_\pi^2}{M_K^2}}{z}\,\bar J_{\pi\pi}(zM_K^2)+\frac{1}{24\pi^2}\right]\right.$$
$$\left. + \tilde\alpha_+^{\rm 1loop}\left[\frac{z-4}{z}\,\bar J_{KK}(zM_K^2)+\frac{1}{24\pi^2}\right]\right\}$$

G. Ecker, A. Pich, E. de Rafael, Nucl Phys B 291, 692 (1987)

B. Ananthanarayan, I. S. Imsong, J. Phys. G 39, 095002 (2012)



At order one loop,

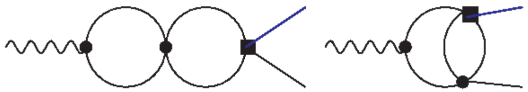
$$\alpha_+^{\rm 1loop} = \tilde\alpha_+^{\rm 1loop} = \left(-\frac{G_F}{\sqrt{2}}V_{us}^* V_{ud}\right) M_\pi^2\left(g_8 - \frac{13}{3}g_{27}\right) = -0.36 M_\pi^2 G_F = -8.16\cdot 10^{-8}$$

$$a_+^{\rm 1loop} = W_{\rm 1loop}(0) = \left(-\frac{1}{\sqrt{2}}V_{us}^* V_{ud}\right)\left(g_8 w_+^{(8)} - \frac{13}{3}g_{27}w_+^{(27)}\right) - \underbrace{\frac{\alpha_+^{\rm 1loop}+\tilde\alpha_+^{\rm 1loop}}{6M_\pi^2 G_F}}_{-0.12}$$

Expanding the kaon loop to the term linear in z gives

$$b_+^{\rm 1loop} = \frac{\tilde\alpha_+^{\rm 1loop}}{60 M_\pi^2 G_F}\frac{M_K^2}{M_\pi^2} \sim -0.075$$

Complete two-loop representation of $W(z)$ (pion loops)



Rewrite $W_{1\text{loop}}(z)$ as the dispersive representation

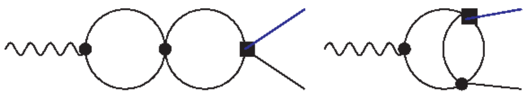
$$W_{1\text{loop}}(z) = G_F M_K^2 a_+^{1\text{loop}} + \frac{zM_K^2}{\pi}\int_0^\infty \frac{dx}{x}\frac{\text{Im}\,W_{1\text{loop}}(x/M_K^2)|_{\pi\pi}}{x - zM_K^2 - i0}$$

$$\frac{\text{Im}\,W_{1\text{loop}}(s/M_K^2)|_{\pi\pi}}{16\pi^2 M_K^2} = \theta(s-4M_\pi^2)\times\frac{s-4M_\pi^2}{s}\lambda_{K\pi}^{-1/2}(s)\times F_V^\pi(s)|_{\mathcal{O}(E^2)}\times f_1^{\pi^+\pi^-\to K^+\pi^-}(s)|_{\mathcal{O}(E^2)}$$

where

$$F_V^\pi(s)|_{\mathcal{O}(E^2)} = 1 \qquad f_1^{\pi^+\pi^-\to K^+\pi^-}(s)|_{\mathcal{O}(E^2)} = \frac{\alpha_+^{1\text{loop}}}{96\pi M_\pi^2}\times\lambda_{K\pi}^{1/2}(s)\sqrt{1-\frac{4M_\pi^2}{s}}$$

Complete two-loop representation of $W(z)$ (pion loops)



At two loops this becomes

$$W_{2\text{loop}}(z) = G_F M_K^2 (a_+^{2\text{loop}} + b_+^{2\text{loop}} z) + \frac{z^2 M_K^4}{\pi} \int_0^\infty \frac{dx}{x^2} \frac{\text{Abs}\, W_{2\text{loop}}(x/M_K^2)|_{\pi\pi}}{x - z M_K^2 - i0}$$

with

$$\frac{\text{Abs}\, W_{2\text{loop}}(s/M_K^2)|_{\pi\pi}}{16\pi^2 M_K^2} = \theta(s - 4M_\pi^2) \times \frac{s - 4M_\pi^2}{s} \lambda_{K\pi}^{-1/2}(s) \Big[\text{Re}\, F_V^\pi(s)|_{\mathcal{O}(E^4)} \times f_1^{\pi^+\pi^- \to K^+\pi^-}(s)|_{\mathcal{O}(E^2)} + F_V^\pi(s)|_{\mathcal{O}(E^2)} \times [\text{Disp}\, f_1^{\pi^+\pi^- \to K^+\pi^-}(s)|_{\mathcal{O}(E^4)} - f_1^{\pi^+\pi^- \to K^+\pi^-}(s)|_{\mathcal{O}(E^2)}] \Big]$$

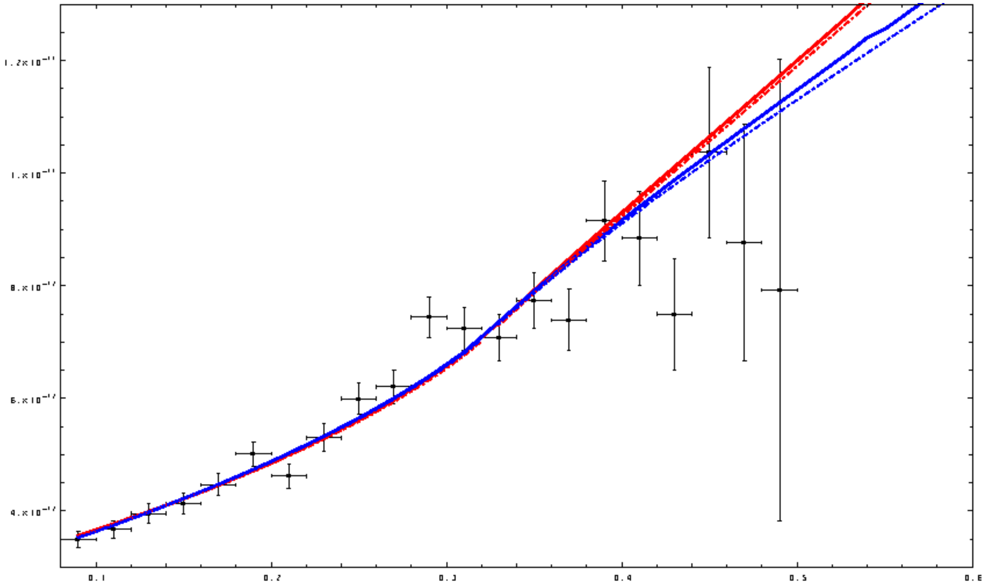
Requires to compute also $\text{Disp}\, f_1^{\pi^+\pi^- \to K^+\pi^-}(s)|_{\mathcal{O}(E^4)} \ldots$

J. Stern et al, Phys Rev D 47, 3814 (1993)
M. K. et al, Nucl Phys B 457, 513 (1995)
K. Kampf et al., Phys Rev D 84, 114015 (2011)
S. Descotes-Genon, M. K., Eur Phys J C 72, 1962 (2012)
V. Bernard et al., Eur Phys J C 73, 2478 (2013)

. . . somewhat tedious, but straightforward

note: $\text{Disp} \neq \text{Re}, \text{Abs} \neq \text{Im}$

Comparing $W_{2\text{loop}}(z)$ and $W_{\text{BOL}}(z)$



solid lines: $|W_{2\text{loop}}(z)|^2$ full two loops

dash-dotted lines $|W_{\text{BOL}}(z)|^2$

red curves: $a_+ = -0.585$, $b_+ = -0.779$, $\beta_+ = -2.88 \cdot 10^{-8}$

blue curves: $a_+ = -0575$, $b_+ = -0.779$, $\beta_+ = -0.99 \cdot 10^{-8}$

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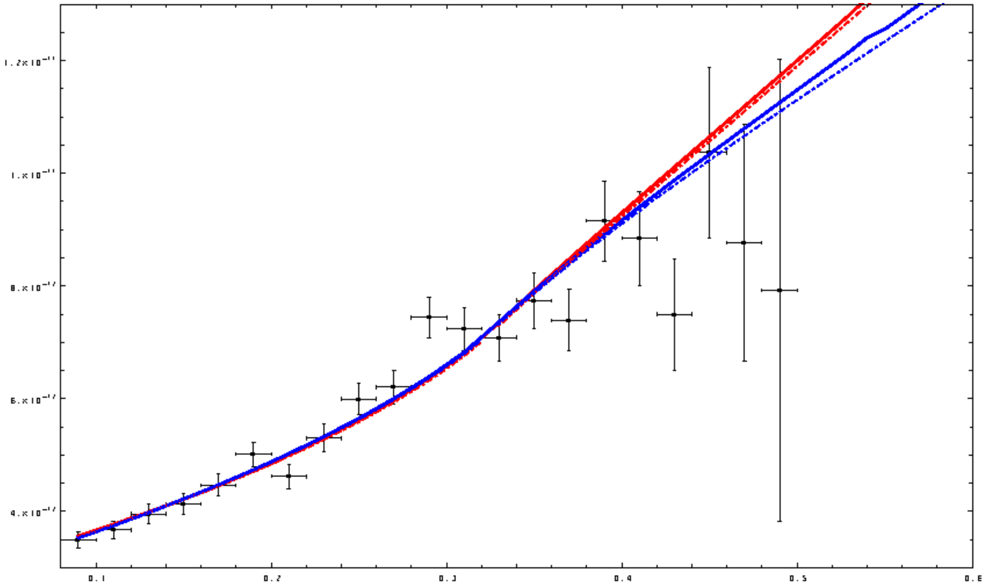

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Fitting the data with $W_{2\mathrm{loop}}(z)$ instead of $W_{\text{BOL}}(z)$ will not modify the values of a_+ and b_+

Predicting $a_+,\ b_+$? Going beyond the low-energy expansion

requires an unsubtracted dispersion relation

$$W(z)|_{\pi\pi} = \frac{1}{\pi}\int_0^\infty dx \frac{\mathrm{Abs}\, W(x/M_K^2)|_{\pi\pi}}{x - zM_K^2 - i0}$$

with

$$\frac{\mathrm{Abs}\, W(s/M_K^2)|_{\pi\pi}}{16\pi^2 M_K^2} = \theta(s-4M_\pi^2) \times \frac{s-4M_\pi^2}{s}\lambda_{K\pi}^{-1/2}(s) \times F_V^{\pi *}(s) \times f_1^{\pi^+\pi^-\to K^+\pi^-}(s)$$

Then a_+ and b_+ are given by spectral sum rules

$$G_F M_K^2 a_+|_{\pi\pi} = W(0)|_{\pi\pi} = \frac{1}{\pi}\int_0^\infty \frac{dx}{x}\,\mathrm{Abs}\, W(x/M_K^2)|_{\pi\pi}$$

and

$$\begin{aligned} G_F M_K^2 b_+|_{\pi\pi} &= W'(0)|_{\pi\pi} - \frac{1}{60}\left(\frac{M_K^2}{M_\pi^2}\right)^2\left(\alpha_+ - \beta_+\frac{s_0}{M_\pi^2}\right) \\ &= \frac{M_K^2}{\pi}\int_0^\infty \frac{dx}{x^2}\,\mathrm{Abs}\, W(x/M_K^2)|_{\pi\pi} - \frac{1}{60}\left(\frac{M_K^2}{M_\pi^2}\right)^2\left(\alpha_+ - \beta_+\frac{s_0}{M_\pi^2}\right) \end{aligned}$$

requires $F_V^{\pi *}(s)$ and $f_1^{\pi^+\pi^-\to K^+\pi^-}(s)$ beyond low-energy expansion

Predicting $a_+,\ b_+$? Going beyond the low-energy expansion

Simple approach: unitarize both using the inverse amplitude method

T. N. Truong, Phys Rev Lett 61, 2526 (1988)

A. Dobado et al, Phys Lett B 235, 134 (1990)

T. Hannah, Phys Rev D 55, 5613 (1997)

A. Dobado, J. R. Pelaez, Phys Rev D 56, 3057 (1997)

J. Nieves et al., Phys Rev D 65, 036002 (2002)

$$a_+|_{\pi\pi} = -(1.574^{+0.003}_{-0.020}) \qquad b_+|_{\pi\pi} = -(0.622^{+0.012}_{-0.017}) \qquad \text{for } \beta_+ = -0.85 \cdot 10^{-8}$$

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note: position of the ρ resonance in $f_1^{\pi^+\pi^-\to K^+\pi^-}(s)$ much too low for $\beta_+ = -2.88(1.08) \cdot 10^{-8}$... (phase goes through $\pi/2$ at $s \sim M_\rho^2/2$!)

IV. Summary - Conclusions

- $K^{\pm} \to \pi^{\pm}\ell^{+}\ell^{-}$ and $K_S \to \pi^{0}\ell^{+}\ell^{-}$ offer a window to BSM physics (as, in general, other rare kaon decays)
- Unfortunately, they are long-distance dominated
- Rather precise data on decay distribution quite helpful
- Indications that $\beta_{+} \sim -1 \cdot 10^{-8}$ rather than $-2.88(1.08) \cdot 10^{-8}$

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- Anatomy of $W(z)$:

$$W(z) = W(z)|_{\pi\pi} + W(z;\nu)|_{\rm res} + W(z;\nu)|_{\rm SD}$$

- $W(z)|_{\pi\pi}$:

- contribution from P-wave $\pi\pi$ intermediate state

- requires $F_V^\pi(s)$

$\longrightarrow$ IAM unitarized form factor decreases too fast for high s

$\longrightarrow$ representations of $F_V^\pi(s)$ with correct QCD behaviour available

- requires also $f_1^{K\pi\to\pi\pi}(s)$

$\longrightarrow$ can in principle also be improved used KT-type iteration

Interesting observation: situation quite similar to the form factor $f_+^{\eta\pi}(s)$ for the second-class decay $\tau \to \eta\pi^+\nu_\tau$ S. Descotes-Genon, B. Moussallam, Eur Phys J C 74, 2946 (2014)

... only short-distance behaviour different

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- describe higher-threshold intermediate states ($K\pi$, $K\bar{K}$,...) through exchanges of narrow-width resonances

- required to match the short-distance behaviour $f_+(s) \times \ln(-s/\nu^2)$

- infinite set of resonances

$$\mathrm{Im}\, W(z;\nu)|_{\rm res} = f_+(s) \times \sum_n A(n)\delta(s - nM^2)$$

- possible to find $A(n)$ to match the correct SD behaviour!

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Should be possible to obtain estimates of a_+ and b_+

Thank you for your attention