Studying running coupling effects on the numerical solutions of the JIMWLK equation

Piotr Korcyl

in collaboration with Krzysztof Cichy, Piotr Kotko, Krzysztof Kutak

XIIIth Quark Confinement and the Hadron Spectrum, August 5, 2018
Hadrons’ internal structure

Standard Model of elementary particles: electrons, muons, quarks, gluons, photons, $W^\pm$, $Z$, Higgs, ...

Credit: Brookhaven National Lab website

**Experiment:** HERA, LHC, BNL, JLab, SLAC, Fermilab, Electron-Ion Collider study hadron structure functions and try to discover the origin of mass,

**Theory:** Quantum Chromodynamics (QCD) is the theory describing the
Proton structure function

Large $x$

Lattice QCD is able to provide $x$-dependent structure functions, such as the PDFs. The method seems to be applicable for large $x > 0.1$. Reaching smaller values of $x$ would require lattices large lattices.

Small $x$

At small $x$ where gluons dominate one introduces CGC - effective theory which can be used among other applications to describe the structure of hadrons. The basic equations are B-JIMWLK and BK. In the dilute region - i.e. $x$ is moderate and $k_t$ is large enough one can formally neglect nonlinearities and use BFKL. The initial conditions can be obtained for instance from McLerran-Venugopalan model.

Check

As for finding the signatures of nonlinearities of CGC type one first needs to fit total cross section like $F_2$ to extract the parameters of the initial condition. Before this can be achieved, several systematic effects need to be taken into account.
Proton structure function

Fit to the HERA F2 data: Mäntysaari, Schenke, arxiv:1806.06783
Solving numerically the JIMWLK equation

Parallel code

Following the works by Rummukainen, Weigert, Lappi, Mäntysaari, Marquet, Petreska, Roiesnel we wrote a highly parallel implementation of the stochastic approach to the JIMWLK equation:

- Fourier acceleration: as many parts as possible in momentum space
- parallel FFTW library
- MPI parallelization in one dimension
- openmp parallelization of the volume loops
McLerran-Venugopalan model

We generate the initial condition following Rummukainen and Weigert:

- **dipole correlation function**
  \[ C(x - y) = \langle \text{Tr} U_x^\dagger U_y \rangle \]

- **light-cone Y-M equations**
  \[ U_x = \exp \left( -i \frac{g \rho(x)}{\nabla^2} \right) \]

- **random gaussian sources**
  \[ \langle \rho^a(x) \rho^b(y) \rangle = \delta^{a,b} \delta(x - y) g^2 \mu^2 \]

- **lattice discretization**
  \[ \langle \rho^a_k(x) \rho^b_l(y) \rangle = \delta^{a,b} \delta^{k,l} \delta(x - y) \frac{g^2 \mu^2}{N_y} \]
Initial condition

Example

\[ k^2 C(k) \]

\[ k_T \]

\[ L = 128 \]
\[ L = 256 \]
\[ L = 512 \]

Piotr Korcyl
JIMWLK equation as a stochastic process

**Langevin equation**

We implement the stochastic process following Lappi, Mäntysaari (Eur. Phys. J. C 73 (2013) 2307), Marquet, Petreska and Roiesnel (JHEP 1610 (2016) 065)

\[
U_x(s + \delta s) = \exp \left[ -\sqrt{\delta s} \sum_y U_y(s) \left( \vec{K}(x - y) \vec{\xi}(y) \right) U_y^\dagger(s) \right] \times \\
\times U_x(s) \times \\
\times \exp \left[ \sqrt{\delta s} \sum_y \vec{K}(x - y) \vec{\xi}(y) \right]
\]

where

\[
s = \frac{\alpha_s}{\pi^2} y, \quad y = \ln \frac{x_0}{x_2}
\]

The noise vectors \( \vec{\xi} \) are uncorrelated, gaussian random variables with \( \sigma = 1 \) variance.
Including the running coupling

The running of the coupling can be included in the "square root" prescription

\[
U_x(s + \delta s) = \exp \left[ - \sqrt{\delta y} \sum_y U_y(s) \left( \alpha_s(|x - y|) \vec{K}(x - y) \vec{\xi}(y) \right) U_y^+(s) \right] \times \\
\times U_x(s) \times \\
\times \exp \left[ \sqrt{\delta y} \sum_y \alpha_s(|x - y|) \vec{K}(x - y) \vec{\xi}(y) \right]
\]
Including the running coupling

Alternatively the running of the coupling can be included as a modification of the $\xi$ vectors (Lappi, Mäntysaari, Eur. Phys. J. C (2013) 73)

$$U_x(s + \delta s) = \exp \left[ - \sqrt{\delta s} \sum_y U_y(s) \left( \bar{K}(x - y) \bar{\eta}(y) \right) U_y^\dagger(s) \right] \times$$

$$\times U_x(s) \times$$

$$\times \exp \left[ \sqrt{\delta s} \sum_y \bar{K}(x - y) \bar{\eta}(y) \right]$$

where now

$$\langle \eta_x^{a,i} \eta_y^{b,j} \rangle = \delta^{a,b} \delta^{i,j} \int \frac{d^2 k}{(2\pi)^2} e^{i k (x - y)} \alpha_s(k)$$
Adding the running coupling to the JIMWLK equation

**Including the running coupling**

**Two possibilities:**

- momentum space

\[ \langle \eta_p^{a,i} \eta_q^{b,j} \rangle = \delta^{a,b} \delta^{i,j} \delta(q - p) \alpha_s(p) \]

\[ \Rightarrow \text{diagonal in momentum space: for each } p \text{ generate uncorrelated gaussian variable with variance } \sigma = \alpha_s(p) \]

- position space

\[ \langle \eta_x^{a,i} \eta_y^{b,j} \rangle = \delta^{a,b} \delta^{i,j} \alpha_s(x - y) \]

\[ \Rightarrow \text{correlated random variables for each } x \text{ and } y: \text{generate the correlation matrix } \Sigma, \text{use Cholesky decomposition to get } A \text{ such that } AA^T = \Sigma, \text{transform uncorrelated gaussian variables: } \eta = A \xi \]
Adding the running coupling to the JIMWLK equation

\( \alpha_s(k) \) and \( \alpha_s(r) \)

\[
\alpha_s(k) = \frac{4\pi}{\beta \ln \left\{ \left[ \left( \frac{\mu_0^2}{\Lambda_{QCD}^2} \right)^{\frac{1}{c}} + \left( \frac{k^2}{\Lambda_{QCD}^2} \right)^{\frac{1}{c}} \right]^{c} \right\}}
\]

\[
\alpha_s(r) = \frac{4\pi}{\beta \ln \left\{ \left[ \left( \frac{\mu_0^2}{\Lambda_{QCD}^2} \right)^{\frac{1}{c}} + \left( \frac{4e^{-2\gamma E}}{r^2 \Lambda_{QCD}^2} \right)^{\frac{1}{c}} \right]^{c} \right\}}
\]


- \( \beta = 11 - 2N_f/3 \), \( N_f = 3 \),
- \( c = 0.2 \), \( \mu_0L = 15 \), \( \Lambda_{QCD}L = 6 \)
- the running coupling freezes at the value \( \alpha_0 = 0.76 \)
- we regularize \( r = 0 \) case by setting \( \alpha_s(0) = 0.0001 \)

They have shown that both definitions give compatible results for the BK equation using the "square root" prescription.
Adding the running coupling to the JIMWLK equation

Evolution with constant coupling constant

Graph showing $k^2 C(k)$ vs. $k_T$ with initial condition and evolution, constant $\alpha_s$.
Adding the running coupling to the JIMWLK equation

Constant coupling vs. running coupling in momentum space

- Initial condition
- Constant $\alpha_s$
- Running $\alpha_s$, sqrt prescription

Piotr Korcyl
Adding the running coupling to the JIMWLK equation

"Square root" vs. Lappi’s prescription

![Graph showing k^2 C(k) vs. k_T with different prescriptions for running alpha_s.]

- Initial condition
- Running alpha_s, momentum space
- Running alpha_s, sqrt prescription
Adding the running coupling to the JIMWLK equation

Running coupling in momentum vs. position space

Connection with observations at NLO BK (Ducloue et al., arXiv:1807.04971)?
Conclusions

Summary

• we have implemented the numerical framework for solving the JIMWLK equation expressed as a Langevin equation
• we implemented both the "square root" and Lappi’s prescription to include the effects of the running coupling
• both prescriptions can be implemented either in position or momentum space
• we find that for the same evolution parameters, each prescription gives different evolution speeds

Outlook

• study other remaining systematics
• reconstruct cross-section from the correlation function
• implement and perform the fit to experimental data
Fourier transforms

\[ \langle \eta(x)\eta(y) \rangle = \delta(x - y) \]

\[ \eta(p) = \eta(p)\eta(p) = \sum_{x,y} e^{ip(x+y)} \eta(x)\eta(y) \]

\[ C(z) = \sum_{p} e^{-ipz} C(p) = \]

\[ = \sum_{p} \sum_{x,y} e^{ip(z-x-y)} \eta(x)\eta(y) = \]

\[ = \sum_{x,y} \delta(z - x - y) \eta(x)\eta(y) = \]

\[ = \sum_{x} \eta(x)\eta(z - x) \]

\[ \langle C(z) \rangle = \sum_{x} \langle \eta(x)\eta(z - x) \rangle = \sum_{x} \delta(2x - z) \]