Towards the NNNLO pressure of cold and dense QCD
Matias Säppi, University of Helsinki
Collaborators: Gorda, Kurkela, Romatschke, Vuorinen
XIIIth Quark Confinement and the Hadron Spectrum
Maynooth University, Ireland. August 1 – 6 2018
Outline

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2 NNLO Logarithm
3 Double-log at NNNLO
4 Conclusions
Introduction
Goal: Describe how strongly interacting matter behaves under different conditions

- Corresponds to computing the QCD pressure

\[ p (\{\mu_i\}_i; \{m_i\}_i; T, \ldots) = T \log \int D\psi D\bar{\psi} D\phi Dc D\bar{c} \exp(-S)/V \]
Motivation

- **Goal:** Describe how strongly interacting matter behaves under different conditions
  
  - Corresponds to computing the QCD pressure
    \[ p (\{\mu_i\}_i; \{m_i\}_i; T, \ldots) = T \log \int D\psi \bar{D}\psi D\bar{A} Dc D\bar{c} \exp(-S)/V \]

- **Plenty of applications:** Early universe, heavy-ion collisions, **neutron stars**...
  
  - I’ll concentrate on the last one: Cold & Dense Matter,
    \[ p (\{\mu_i\}_i; \{m_i\}_i; T, \ldots) \rightarrow p (\mu_B) \]
Computing the pressure at large density presents challenges

- Lattice methods are non-perturbative and use first-principles theory, but suffer from the sign problem
Methodology – Problems With a Large Density

- Computing the pressure at large density presents challenges
  - Lattice methods are non-perturbative and use first-principles theory, but suffer from the sign problem
  - Traditional EFTs (eg. \(\chi\)PT) are applicable at lower densities, but suffer from inaccuracies at larger densities, and aren’t based on first principles
  - No suitable DR-framework
For the purposes of this talk, focus is on standard perturbation theory

- pQCD is based on first principles\(^1\), expansion in \(\alpha_s\) is possible at asymptotically large \(\mu_B\)

\(^1\)...We do use some HTL, but only to get rid of terms that are of higher order in the coupling itself
Methodology – Using pQCD at Large Density

- For the purposes of this talk, focus is on standard perturbation theory
  - pQCD is based on first principles\(^1\), expansion in \(\alpha_s\) is possible at asymptotically large \(\mu_B\)
  - Convergence is bad at physical densities, high-order corrections are required (hence NNNLO)
  - Suffers from IR-ambiguities, resummation needed

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In systems like the early universe & heavy-ion collisions, $\mu_B$ often fairly small

Some of the problems plaguing large $\mu_B$ disappear

- Lattice methods work if the sign problem is at most mild or can be circumvented
- pQCD admits a dimensionally reduced framework when $T/\mu_B$ is large and is understood better

Plenty of accurate results and experiments!
NNLO Logarithm
Thermal pQCD History Lesson pt. I

\[ p = c_0 + c_1 \alpha_s + c_{2,1} \alpha_s^2 \log \alpha_s + c_{2,0} \alpha_s^2 \]
\[ + c_{3,2} \alpha_s^3 \log^2 \alpha_s + c_{3,1} \alpha_s^3 \log \alpha_s + c_{3,0} \alpha_s^3 + \ldots \]

- Simplest case for dense matter: massless quarks and zero temperature
- \( c_0, c_1 \) easy to compute
- NNLO factors \( c_{2,1}, c_{2,0} \) calculated already in the 70s (!) \(^2\)

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\(^2\) Freedman & McLerran, Phys. Rev. D 16 1977
**THERMAL pQCD HISTORY LESSON pt. I**

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- \( c_0 \), \( c_1 \) easy to compute
- NNLO factors \( c_{2,1}, c_{2,0} \) calculated already in the 70s (!) \(^2\)
- Assumptions later relaxed to e.g. \( m \neq 0 \) \(^3\) and small but finite \( T \) \(^4\), but no previous NNNLO calculations

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\(^2\) Freedman & McLerran, Phys. Rev. D 16 1977

\(^3\) Kurkela et al., hep-ph/0912.1856

\(^4\) Kurkela & Vuorinen, hep-ph/1603.00750
Thermal pQCD History Lesson pt. II

\[ p = c_0 + c_1 \alpha_s + c_{2,1} \alpha_s^2 \log \alpha_s + c_{2,0} \alpha_s^2 \]
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■ Compare with ”hot” QCD (large-\(T\) limit)
■ Partial NNNLO results: \(c_{3,2} = 0\) and \(c_{3,1}\) is known \(^5\)

\(^5\)Kajantie et al. hep-ph/0211321
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- Compare with ”hot” QCD (large-\(T\) limit)
- Partial NNNLO results: \(c_{3,2} = 0\) and \(c_{3,1}\) is known \(^5\)
- Theory fundamentally different:
  - A 3d DR description is available
  - Notably, the theory has a magnetic scale, \(\alpha_s T\), which is absent at \(T = 0\)
  - As a consequence, the ”hard” contribution \(c_{3,0}\) cannot be computed in pure perturbation theory!

\(^5\)Kajantie et al. hep-ph/0211321
Scale Hierarchy at Finite $\mu_B$ & $T = 0$

- Naïve diagrammatic expansion valid close to vacuum, where the fields are not screened

- Hard Scale $P \sim \mu_B$:
  - Energetic excitations $\rightarrow$ medium effects suppressed
  - Standard Feynman graph expansion OK
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  - Low-energy excitations → medium effects significant
  - Resum IR-sensitive objects (gluons) to all orders
Scale Hierarchy at Finite $\mu_B$ & $T = 0$

- Naïve diagrammatic expansion valid close to vacuum, where the fields are not screened
- In a dense medium, medium effects break the formalism
  → Resummation of diagram classes required
- In the intermediate region, when ratios of scales are involved, nonanalytic terms $\mathcal{O}(\alpha_s^m \log^n \alpha_s)$ are generated from resummed diagrams

- Hard Scale $P \sim \mu_B$:
  - Energetic excitations
    → medium effects suppressed
  - Standard Feynman graph expansion OK

- Semisoft Scale $\alpha_s^{1/2} \mu_B \ll P \ll \mu_B$:
  - ”Somewhat” energetic excitations
    → medium effects present
  - Corrections to naïve diagrams not suppressed enough, but approximations possible

- Soft Scale $P \sim \alpha_s^{1/2} \mu_B$:
  - Low-energy excitations
    → medium effects significant
  - Resum IR-sensitive objects (gluons) to all orders
First logs at $\mathcal{O}(\alpha_s^2)$: The gluonic ring sum
1-loop Ring Sum

- First logs at $\Theta(\alpha_s^2)$: The gluonic ring sum

- Well-known: Contributes a non-analytic term at NNLO: $c_{2,1} \neq 0$
Even the 1-loop self-energy is complicated.

IR DoFs admit an effective description, and only they require resummation!
HTL Approximation

- Even the 1-loop self-energy is complicated
- IR DoFs admit an effective description, and only they require resummation!
- Hard Thermal Loops\(^6\) suitable since hard modes dominate the self-energies

Physical motivation: Accounts for the medium

\(^6\)Hard Dense Loops, really, but abbreviated HTL here
Approximating the Ring Sum

Applying the HTL approximation leads to the following:

\[ \begin{align*}
& \Pi + \Pi + \Pi + \Pi + \ldots \\
= & \left( \begin{array}{c}
\Pi \\
_{\text{HTL,}} \\
_{P < \Lambda}
\end{array} \right) + \left( \begin{array}{c}
\Pi \\
_{P > \Lambda}
\end{array} \right) + \mathcal{O}(\alpha_s^3)
\end{align*} \]

- The logarithms, that is, the semisoft region, can be taken to come from the UV-limit of the first term or the IR-limit of the second one, we choose the first.
- Using HTL gives the correct \(c_{2,1}\) : There is a Logarithmic enhancement at NNLO from the semisoft modes.
HTL Propagators & $m_E$

- After scalarisation, transversal and longitudinal gluons:

$$G_T(P) = \frac{-1}{P^2 + \frac{m_E^2}{d-1} - \frac{p^2}{p^2} \Pi_{HTL}(P)}, \quad G_L(P) = \frac{1}{p^2 + \Pi_{HTL}(P)}$$

- HTL structure is given by the function

$$\Pi_{HTL}(P) = m_E^2 \left( 1 - \int_{S^{d-1}} \Omega_v \frac{iP_0}{iP_0 - p \cdot v} \right)$$

- Details unimportant, point is that the propagators are nontrivial
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Without quark masses, the only scale is the effective mass

\[ m_E^2 \equiv \text{Tr} \Pi \overset{T \to 0}{\sim} \alpha_s \mu^2 \]
Double-log at NNNLO
The next order is $\mathcal{O}(\alpha_s^3)$, NNNLO. How do we get there?

Calculating everything up to $c_{3,0}$ is daunting, but...

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Gorda, Kurkela, Romatschke, MS, Vuorinen, hep-ph/1807.04120
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...At $\mathcal{O}(\alpha_s^2)$ the ring sum contribution was separate from everything else

$\implies$ Try to compute the non-analytic terms $c_{3,1}, c_{3,2}$ first

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$\Rightarrow$ Try to compute the non-analytic terms $c_{3,1}, c_{3,2}$ first

- Again, non-hard modes require establishing a suitable resummation scheme
- Analogous with hot QCD development

Computation of $c_{3,2}$ is complete $^7$

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$^7$Gorda, Kurkela, Romatschke, MS, Vuorinen, hep-ph/1807.04120
Finding the Logarithms

- There are now generally double logarithms $c_{3,2} \neq 0$, and single logarithms $c_{3,1} \neq 0$.
- Two sources of logs:
  - Corrections to the one-loop ring sum that require non-HTL corrections ($c_{3,1}$ only)
  - Two-loop resummations analogous to the NNLO case ($c_{3,1}$ and $c_{3,2}$, focus on these)
We can again split between naively loop-expanded hard contributions and resummed soft contributions, and we choose to compute the logarithm as the UV-limit of the soft contribution like in the NNNLO case.

There are four classes of IR-sensitive two-loop diagrams. They must be resummed by dressing the gluonic lines:
Each logarithm arises from an integral $\int \frac{dP}{P}$, hence each resummed gluon line can yield at most a single log.

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For dimensional reasons, in fact only the "double-blob insertions" pictured above contribute.
The remaining resummed diagrams contribute only at $\mathcal{O}(\alpha_s^3 \log \alpha_s)$:
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They, and the subleading logs from the gluonic diagrams, can be handled with a similar expansion with the following prescription:

- Gluons IR-sensitive $\implies$ resummed, but HTL-approximated
- Fermions IR-insensitive $\implies$ not resummed, but loop-corrected
Wrap-up of the NNNLO Logarithms

- Double log $c_{3,2}$ was reasonably simple to extract
- One needs two resummed gluon lines, so only the first two diagram classes contribute
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- Double log $c_{3,2}$ was reasonably simple to extract
- One needs two resummed gluon lines, so only the first two diagram classes contribute
- The HTL corrections to the $ggg$ and $gggg$ vertices are required
- For the leading log only one-loop (HTL) self-energies and vertices are required...
  - ...But for the subleading log $c_{3,1}$ this will not be enough
  - ...And more diagram classes need to be considered
  - There is some effort involved in making sure that double counting etc. is avoided
Our Calculation of $c_{3,2}$

- Relevant HTL diagrams appear in the literature \(^8\)
- One can extract the coefficient of the double-log (but not the constant under the log) from the semisoft regime
- We ended up with lengthy angular integrals and obtained the $\mathcal{O}(\alpha_s^3 \log^2 \alpha_s)$-term...

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\(^8\) Andersen et al., hep-ph/0205085
The Result

...Which is surprisingly simple:

\[
c_{3,2} \alpha_s^3 \log^2 \alpha_s = -\frac{11}{48} \frac{N_c d_A}{(2\pi)^3} \alpha_s m_\infty^4 \log^2 \alpha_s, \quad m_\infty = \frac{N_f}{9\pi} \mu_B^2 \alpha_s
\]
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- No renormalisation scale dependence \( \Rightarrow \) error bands still large ...

- But nothing blows up \( \Rightarrow \) seems like pQCD still works!
$m_\infty$-TRICK

- Physical intuition: In the semisoft region transverse gluons massive with mass $m^2_\infty$, longitudinal gluons massless

- Makes sense, because this is the UV limit of HTL, and the semisoft regime is the UV limit of the soft contribution (or the IR limit of the hard contribution)
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This is a sufficient approximation for the $\log^2 \alpha_s$! Can remove complicated $\Pi_{HTL}$ leaving just a constant mass

Much easier to compute, we verified that it gives the same $\log^2 \alpha_s$
Conclusions
**Conclusions**

- Resummation in the IR-sector allowed us to go beyond the well-understood NNLO order of dense pQCD
- $O(\alpha_s^3 \log^2 \alpha_s)$ done!
- $O(\alpha_s^3 \log \alpha_s)$ in progress, will require eg. higher order HTL-corrections, but might help constrain the error bands more
- The full $O(\alpha_s^3)$ might also be possible (??)
- Resummation schemes, small-T corrections etc. also under consideration...