Wigner function approach to polarization-vorticity coupling and hydrodynamics with spin

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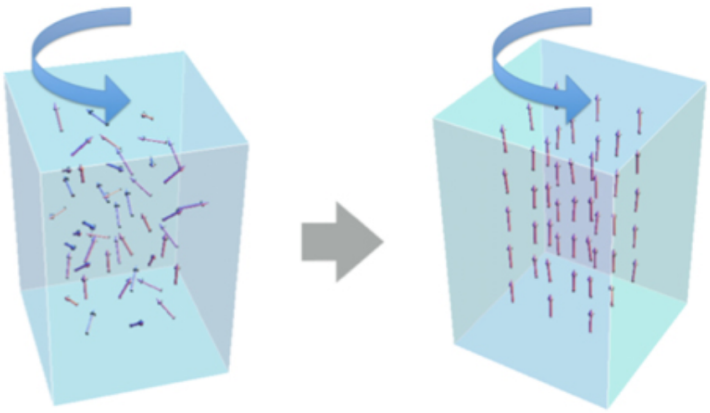
Rotation and Polarization

Barnett Effect

S. J. Barnett, Rev. Mod. Phys. 7, 129 (1935)



$$B_\Omega = \Omega/\gamma$$

Figure: Mechanical rotation of an unmagnetized metallic object induces magnetization, an effective magnetic field emerges.

Einstein-de Haas Effect

A. Einstein and W. de Haas, Deutsche Physikalische Gesellschaft, Verhandlungen 17, 152 (1915)

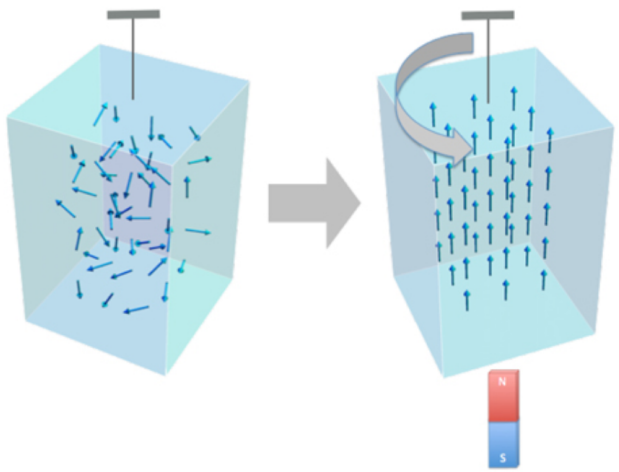


Figure: Application of magnetic field on an unmagnetized metallic object induces magnetization, body start rotating (mechanical angular momentum emerges)

Case of Heavy ion collision experiment

Global angular momentum $J \approx 10^4 \hbar$ (RHIC Au-Au 200 GeV, b=2.5 fm) [arXiv:0711.1253v3 [nucl-th] 18 Feb 2008].

Global rotation of the matter created in the non-central collisions can induce spin polarization, similar to magnetomechanical Barnett effect and Einstein and de Haas effect.

Emerging particle are expected to be globally polarized with their spins on average pointing along the system angular momentum.

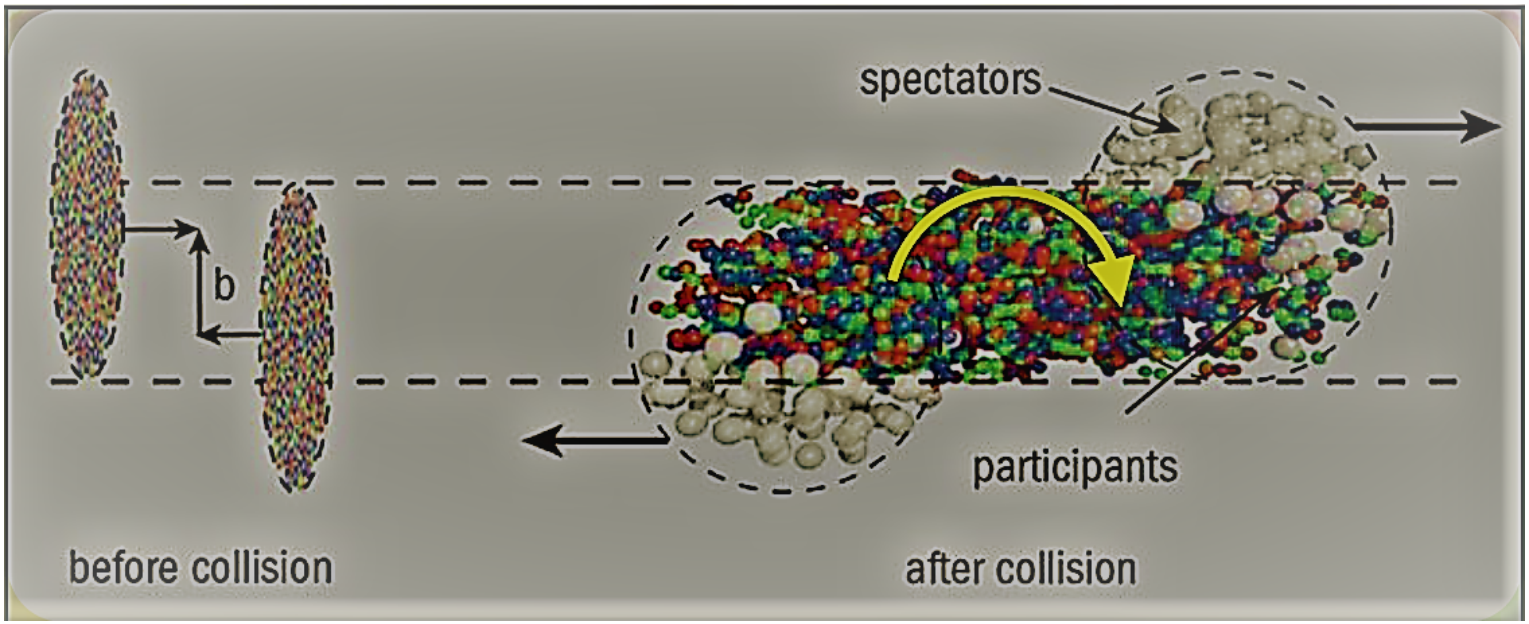


Figure: Geometry of a non-central heavy ion collision

Global Λ-polarization in RHIC experiment

The average polarization $\bar{P}_H$ (where $H = \Lambda$ or $\bar{\Lambda}$) from $20 - 50\%$ central Au+Au collisions [L. Adamczyk *et al.* (STAR), Nature 548 (2017) 62-65, arXiv:1701.06657 [nucl-ex]].

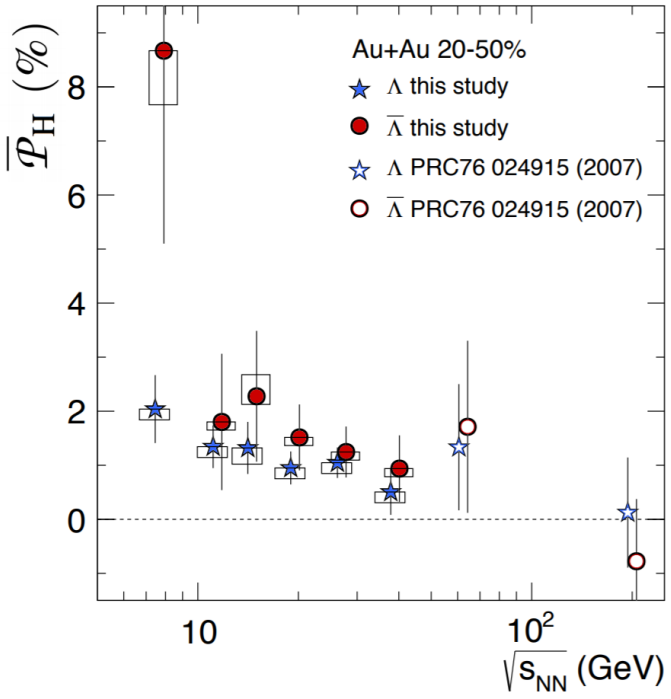


Figure: The average polarization versus collision energy

Present phenomenological prescription used to describe the data:

1. Run any type of hydro code (Ideal or viscous).
2. Find β^μ on the freeze-out hypersurface.
3. Calculate the thermal vorticity and identify it with spin polarization [F. Becattini, I. Karpenko, M. Lisa, I. Upsal, S. Voloshin, Phys. Rev. C 95, 054902 (2017)] .
   Note that the fact that thermal vorticity and spin polarization are same holds in global equilibrium if energy momentum tensor $T^{\mu\nu}$ is asymmetric .
4. Make prediction about spin polarization.

A natural framework for dealing simultaneously with polarization and vorticity would be relativistic hydrodynamics of polarized fluids.
In local equilibrium thermal vorticity and spin polarization tensor are independent.
They may become related if the system reaches global equilibrium.

In this talk

1. Using the equilibrium distributions functions for particles of spin 1/2 as an input to the Wigner function and its semi-classical expansion we show how a kinetic approach can lead us to the fact that the thermal vorticity and spin polarization tensor are constant, however, not necessarily equal (β_μ and $\omega_{\mu\nu}$ are independent).
2. A procedure to construct the hydrodynamic framework that can deal with the spin physics.

Global and local equilibrium – spinless particles

Boltzmann equation

$$p^\mu \partial_\mu f(x,p) = C[f(x,p)]$$

1. Satisfies exactly for free streaming.
2. Satisfied in global equilibrium:

via some constraint equations on the hydrodynamic parameters (μ, T, u_μ) used to specify the form of $feq(x,p)$.

3. Does not vanish in the local thermodynamic equilibrium:

constraints on the hydrodynamic parameters are found by adding further assumptions (for instance momentum moments, $\int dPp_\mu \ldots f(x,p)$, yielding the conservation laws for energy and momentum etc)

For free streamimg $= 0$

Global equilbrium $= 0$

Local equailibrium $= 0$

Global equilibrium and the Killing equation

The equilibrium distribution function has the form

$$f_{\rm eq}(x,p) = exp\left[\xi(x) - \beta_\mu(x)p^\mu\right] \qquad \text{with} \qquad \beta_\mu = u_\mu(x)/T(x) \qquad \xi = \mu(x)/T(x).$$

It satisfies the LHS of Boltzmann equation *i.e.* $p^\mu \partial_\mu f_{\rm eq}(x,p) = 0$ via,

$$p^\mu \partial_\mu \xi + p^\mu p^\nu \partial_\mu \beta_\nu = p^\mu \partial_\mu \xi + \frac{1}{2} p^\mu p^\nu (\partial_\mu \beta_\nu + \partial_\nu \beta_\mu) = 0$$

One arrives at constraints

$$\partial_\mu \xi = 0 \qquad\qquad \partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0 \quad \text{Killing equation}$$

giving

$$\xi = \text{constant}$$
$$\beta_\mu(x) = \beta^0_\mu + \varpi^0_{\mu\nu} x^\nu \qquad \text{with} \qquad \beta^0_\mu = \text{const}, \quad \varpi^0_{\mu\nu} = -\varpi^0_{\nu\mu} = \text{constant}.$$

Thermal vorticity is given by

$$\varpi_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right) \equiv \varpi^0_{\mu\nu}.$$

The treatment of the collisionless kinetic equation for the Wigner function $\mathcal{W}(x,k)$ that includes spin degrees of freedom has many features in common with the simple spinless system discussed above.

Global and local thermodynamic equilibrium; particles with spin

The distribution function depends on β_μ, parameter $\xi = \frac{\mu}{T}$ and spin polarization tensor $\omega_{\mu\nu}$[Wojciech Florkowski, Bengt Friman, Amaresh Jaiswal, Enrico Speranza, Phys. Rev. C 97, 041901 (2018)].

We can distinguish four rather than two different type of equilibrium.

1. global equilibrium — β_μ field is a Killing vector, $\varpi_{\mu\nu} = \omega_{\mu\nu} =$ constt, in addition $\xi =$ constt.

2. extended global equilibrium — β_μ field is a Killing vector, $\varpi_{\mu\nu} =$ constt, $\omega_{\mu\nu} =$ constt, but $\varpi_{\mu\nu} \neq \omega_{\mu\nu}$, in addition $\xi =$ constt.

3. local equilibrium — β_μ field is not a Killing vector but we still have $\omega_{\mu\nu}(x) = \varpi_{\mu\nu}(x)$, $\xi = \xi(x)$,

4. extended local equilibrium — β_μ field is not a Killing vector and $\omega_{\mu\nu}(x) \neq \varpi_{\mu\nu}(x)$, $\xi = \xi(x)$.

Conservation laws

"Conservation laws of the currents are associated with the microscopic symmetries of the system (Noether's theorem)"

Internal symmetries:

Conservation of charge (baryon number, electric charge)

$$\partial_\mu \hat{N}^\mu(x) = 0, \qquad \text{1 equation}$$

Poincaré symmetry:

Conservation of energy and momentum

$$\partial_\mu \hat{T}^{\mu\nu}(x) = 0, \qquad \text{4 equations}$$

Conservation of total angular momentum

$$\partial_\mu \hat{J}^{\mu,\alpha\beta}(x) = 0, \qquad \hat{J}^{\mu,\alpha\beta}(x) = -\hat{J}^{\mu,\beta\alpha}(x) \qquad \text{6 equations}$$

Total angular momentum is the sum of orbital and spin parts:

$$\hat{J}^{\mu,\alpha\beta}(x) = \hat{L}^{\mu,\alpha\beta}(x) + \hat{S}^{\mu,\alpha\beta}(x),$$

$$\hat{L}^{\mu,\alpha\beta}(x) = x^\alpha \hat{T}^{\mu\beta}(x) - x^\beta \hat{T}^{\mu\alpha},$$

Conservation of energy momentum and total angular momentum implies

$$\partial_\mu T^{\mu\nu}(x) = 0, \quad \partial_\lambda J^{\lambda,\mu\nu}(x) = 0, \Rightarrow \partial_\lambda S^{\lambda,\mu\nu}(x) = T^{\mu\nu}(x) - T^{\nu\mu}(x) \neq 0.$$

Thus spin tensor $\hat{S}^{\mu,\alpha\beta}(x)$ is in general not conserved.

Equilibrium Wigner Functions

We start with the equilibrium Wigner functions [S. de Groot, W. van Leeuwen, and C. van Weert, Relativistic Kinetic Theory: Principles and Applications (1980)]

$$\mathcal{W}^{+}_{\rm eq}(x,k) = \frac{1}{2}\sum_{r,s=1}^{2}\int dP\,\delta^{(4)}(k-p)u^{r}(p)\bar{u}^{s}(p)f^{+}_{rs}(x,p),$$

$$\mathcal{W}^{-}_{\rm eq}(x,k) = -\frac{1}{2}\sum_{r,s=1}^{2}\int dP\,\delta^{(4)}(k+p)v^{s}(p)\bar{v}^{r}(p)f^{-}_{rs}(x,p).$$

We take $f^{+}_{rs}(x,p)$ and $f^{-}_{rs}(x,p)$ [F. Becattini et al. Annals Phys. 338 (2013) 32]

$$f^{+}_{rs}(x,p) = \frac{1}{2m}\bar{u}_{r}(p)X^{+}u_{s}(p), \qquad f^{-}_{rs}(x,p) = -\frac{1}{2m}\bar{v}_{s}(p)X^{-}v_{r}(p).$$

m is the (anti)particle mass, while $u_r(p)$ and $v_r(p)$ are Dirac bispinors.

$$X^{\pm} = \exp\left[\pm\xi(x) - \beta_{\mu}(x)p^{\mu}\right]M^{\pm}, \qquad M^{\pm} = \exp\left[\pm\frac{1}{2}\omega_{\mu\nu}(x)\Sigma^{\mu\nu}\right].$$

- $\beta^{\mu}(x) = u^{\mu}(x)/T(x)$ and $\xi(x) = \mu(x)/T(x)$, with $\mu(x)$ being the chemical potential.
- The quantity $\omega_{\mu\nu}(x)$ is the spin polarization tensor, while $\Sigma^{\mu\nu} = (i/4)[\gamma^{\mu},\gamma^{\nu}]$.

Equilibrium Wigner functions

Spin polarization tensor $\omega_{\mu\nu}$ satisfies the two conditions [Wojciech Florkowski, Bengt Friman, Amaresh Jaiswal, Enrico Speranza, Phys. Rev. D 97, 116017 (2018)]

$$\omega_{\mu\nu}\omega^{\mu\nu} \geq 0, \quad \omega_{\mu\nu}\tilde{\omega}^{\mu\nu} = 0, \quad \text{where } \tilde{\omega}^{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\omega^{\alpha\beta}$$

$$M^{\pm} = \cosh(\zeta) \pm \frac{\sinh(\zeta)}{2\zeta}\,\omega_{\mu\nu}\Sigma^{\mu\nu}.$$

ζ is defined by the expression

$$\zeta = \frac{\Omega}{T} = \frac{1}{2}\sqrt{\frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu}}.$$

Ω plays a role of the spin chemical potential.
The total Wigner function

$$\mathcal{W}_{\rm eq}(x,k) = \mathcal{W}^{+}_{\rm eq}(x,k) + \mathcal{W}^{-}_{\rm eq}(x,k).$$

Spinor decomposition of the equilibrium Wigner function

The Wigner functions $\mathcal{W}^{\pm}_{\rm eq}(x,k)$, being four-by-four matrices satisfying the relations $\mathcal{W}^{\pm}_{\rm eq}(x,k) = \gamma_0 \mathcal{W}^{\pm}_{\rm eq}(x,k)^{\dagger}\gamma_0$, can always be expanded in terms of the 16 independent generators of the Clifford algebra

$$\begin{aligned}\mathcal{W}^{\pm}_{\rm eq}(x,k) &= \frac{1}{4}\left[\mathcal{F}^{\pm}_{\rm eq}(x,k) + i\gamma_5\mathcal{P}^{\pm}_{\rm eq}(x,k) + \gamma^{\mu}\mathcal{V}^{\pm}_{{\rm eq},\mu}(x,k)\right.\\ &\qquad \left. + \gamma_5\gamma^{\mu}\mathcal{A}^{\pm}_{{\rm eq},\mu}(x,k) + \Sigma^{\mu\nu}\mathcal{S}^{\pm}_{{\rm eq},\mu\nu}(x,k)\right].\end{aligned}$$

The coefficient functions in the equilibrium Wigner function expansion can be obtained by the following traces:

$$\begin{aligned}
\mathcal{F}^{\pm}_{\rm eq}(x,k) &= {\rm tr}\left[\mathcal{W}^{\pm}_{\rm eq}(x,k)\right] = 2m\cosh(\zeta)\int dP\; e^{-\beta\cdot p\pm\xi}\;\delta^{(4)}(k\mp p),\\
\mathcal{P}^{\pm}_{\rm eq}(x,k) &= -i\,{\rm tr}\left[\gamma^5\mathcal{W}^{\pm}_{\rm eq}(x,k)\right] = 0,\\
\mathcal{V}^{\pm}_{{\rm eq},\mu}(x,k) &= {\rm tr}\left[\gamma_\mu\mathcal{W}^{\pm}_{\rm eq}(x,k)\right] = \pm\,2\cosh(\zeta)\int dP\, e^{-\beta\cdot p\pm\xi}\;\delta^{(4)}(k\mp p)\,p_\mu,\\
\mathcal{A}^{\pm}_{{\rm eq},\mu}(x,k) &= {\rm tr}\left[\gamma_\mu\gamma^5\mathcal{W}^{\pm}_{\rm eq}(x,k)\right] = -\frac{\sinh(\zeta)}{\zeta}\int dP\, e^{-\beta\cdot p\pm\xi}\;\delta^{(4)}(k\mp p)\,\tilde{\omega}_{\mu\nu}\,p^{\nu},\\
\mathcal{S}^{\pm}_{{\rm eq},\mu\nu}(x,k) &= 2\,{\rm tr}\left[\Sigma^{\mu\nu}\mathcal{W}^{\pm}_{\rm eq}(x,k)\right] = \pm\frac{\sinh(\zeta)}{m\zeta}\int dP\, e^{-\beta\cdot p\pm\xi}\;\delta^{(4)}(k\mp p)\left[\left(p_\mu\omega_{\nu\alpha} - p_\nu\omega_{\mu\alpha}\right)p^{\alpha} + m^2\omega_{\mu\nu}\right].
\end{aligned}$$

Relations between equilibrium coefficient functions

One can verify that the equilibrium coefficient functions satisfy the following set of constraints:

$$k^\mu \, \mathcal{V}^{\pm}_{\mathrm{eq},\mu}(x,k) - m\,\mathcal{F}^{\pm}_{\mathrm{eq}}(x,k) = 0,$$

$$k_\mu \, \mathcal{F}^{\pm}_{\mathrm{eq}}(x,k) - m\,\mathcal{V}^{\pm}_{\mathrm{eq},\mu}(x,k) = 0,$$

$$\mathcal{P}^{\pm}_{\mathrm{eq}}(x,k) = 0,$$

$$k^\mu \, \mathcal{A}^{\pm}_{\mathrm{eq},\,\mu}(x,k) = 0,$$

$$k_\mu \, \mathcal{V}^{\pm}_{\mathrm{eq},\nu}(x,k) - k_\nu \, \mathcal{V}^{\pm}_{\mathrm{eq},\mu}(x,k) = 0,$$

$$k^\mu \, \mathcal{S}^{\pm}_{\mathrm{eq},\,\mu\nu}(x,k) = 0,$$

$$k^\beta \, \tilde{\mathcal{S}}^{\pm}_{\mathrm{eq},\mu\beta}(x,k) + m\,\mathcal{A}^{\pm}_{\mathrm{eq},\,\mu}(x,k) = 0,$$

$$\epsilon_{\mu\nu\alpha\beta} \, k^\alpha \, \mathcal{A}^{\pm}_{\mathrm{eq}}{}^{\beta}(x,k) + m\,\mathcal{S}^{\pm}_{\mathrm{eq},\,\mu\nu}(x,k) = 0.$$

The above relationship holds for any form of $\beta_\mu(x)$, $\xi(x)$ and $\omega_{\mu\nu}(x)$.
This leads us to the fact that only two equilibrium coefficient functions $\mathcal{F}^{\pm}_{\mathrm{eq}}$ and $\mathcal{A}^{\pm}_{\mathrm{eq}}$ can be treated as the basic independent one.

Semi-classical expansion

The general form of Wigner function

$$\mathcal{W}(x,k) = \frac{1}{4}\left[\mathcal{F}(x,k) + i\gamma_5\mathcal{P}(x,k) + \gamma^\mu\mathcal{V}_\mu(x,k) + \gamma_5\gamma^\mu\mathcal{A}_\mu(x,k) + \Sigma^{\mu\nu}\mathcal{S}_{\mu\nu}(x,k)\right].$$

The Wigner function satisfies the equation of the form

$$(\gamma_\mu K^\mu - m)\,\mathcal{W}(x,k) = 0; \qquad K^\mu = k^\mu + \frac{i\hbar}{2}\partial^\mu.$$

The above equation holds in global equilibrium (similar to the case of spin-less particles where $p^\mu\partial_\mu f_{eq} = 0$) and should give the constraint on hydrodynamic variables μ, T, u^μ and $\omega_{\mu\nu}$.

The real parts:

$k^\mu\mathcal{V}_\mu - m\mathcal{F} = 0,$

$\frac{\hbar}{2}\partial^\mu\mathcal{A}_\mu + m\mathcal{P} = 0,$

$k_\mu\mathcal{F} - \frac{\hbar}{2}\partial^\nu\mathcal{S}_{\nu\mu} - m\mathcal{V}_\mu = 0,$

$-\frac{\hbar}{2}\partial_\mu\mathcal{P} + k^\beta\tilde{\mathcal{S}}_{\mu\beta} + m\mathcal{A}_\mu = 0,$

$\frac{\hbar}{2}\left(\partial_\mu\mathcal{V}_\nu - \partial_\nu\mathcal{V}_\mu\right) - \epsilon_{\mu\nu\alpha\beta}k^\alpha\mathcal{A}^\beta - m\mathcal{S}_{\mu\nu} = 0,$

The imaginary parts:

$\hbar\partial^\mu\mathcal{V}_\mu = 0,$

$k^\mu\mathcal{A}_\mu = 0,$

$\frac{\hbar}{2}\partial_\mu\mathcal{F} + k^\nu\mathcal{S}_{\nu\mu} = 0,$

$k_\mu\mathcal{P} + \frac{\hbar}{2}\partial^\beta\tilde{\mathcal{S}}_{\mu\beta} = 0,$

$(k_\mu\mathcal{V}_\nu - k_\nu\mathcal{V}_\mu) + \frac{\hbar}{2}\epsilon_{\mu\nu\alpha\beta}\partial^\alpha\mathcal{A}^\beta = 0.$

Solutions in the form

$$\mathcal{X} = \mathcal{X}^{(0)} + \hbar\mathcal{X}^{(1)} + \hbar^2\mathcal{X}^{(2)} + \cdots. \qquad \mathcal{X} \in \{\mathcal{F}, \mathcal{P}, \mathcal{V}_\mu, \mathcal{A}_\mu, \mathcal{S}_{\nu\mu}\}$$

Kinetic equations for coefficient functions

By solving the coupled equations for zeroth and first order one get the following equations for the coefficient functions

$$k^\mu \partial_\mu \mathcal{F}_{(0)}(x,k) = 0.$$

$$k^\mu \partial_\mu \mathcal{A}^\nu_{(0)}(x,k) = 0$$

To get the equation for the first order coefficient functions one has to go beyond first order (upto second order) if one does so one can easily show,

$$k^\mu \partial_\mu \mathcal{F}_{(1)}(x,k) = 0,$$

$$k^\mu \partial_\mu \mathcal{A}^\nu_{(1)}(x,k) = 0.$$

One can check the algebraic structure of the equilibrium coefficient functions χ_{eq} is consistent with the zeroth-order equations obtained from the semi-classical expansion of the Wigner function. Therefore one can take,

$$\begin{aligned}
\mathcal{F}^{(0)} &= \mathcal{F}_{\rm eq}, \\
\mathcal{P}^{(0)} &= 0, \\
\mathcal{V}^{(0)}_\mu &= \mathcal{V}_{{\rm eq},\mu}, \\
\mathcal{A}^{(0)}_\mu &= \mathcal{A}_{{\rm eq},\mu}, \\
\mathcal{S}^{(0)}_{\mu\nu} &= \mathcal{S}_{{\rm eq},\mu\nu}.
\end{aligned}$$

Extended global equilibrium

In this way one can get:

$$k^\mu \partial_\mu \mathcal{F}_{\rm eq}(x,k) = 0,$$

$$k^\mu \partial_\mu \mathcal{A}^\nu_{\rm eq}(x,k) = 0.$$

These equations will be exactly fulfilled if,

$$\begin{aligned} \partial_\mu \beta_\nu(x) + \partial_\nu \beta_\mu(x) &= 0 \qquad \text{(Killing equation)} \\ \omega_{\mu\nu} &= \text{constant} \\ \xi = \frac{\mu}{T} &= \text{constant.} \end{aligned}$$

Solution to the Killing equation

$\beta_\mu(x) = \beta^0_\mu + \varpi^0_{\mu\nu} x^\nu, \qquad \beta^0_\mu$=constant, $\varpi^0_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right) = \varpi_{\mu\nu}$ =constant

• It does not constrain that the spin polarization tensor $\omega_{\mu\nu}$ is equal to thermal vorticity $\varpi_{\mu\nu}$.

Charge current

We use the definition of the charge current [S. de Groot, W. van Leeuwen, and C. van Weert, Relativistic Kinetic Theory: Principles and Applications (1980)]

$$N^{\alpha}_{\rm eq}(x) = {\rm tr}\int d^4k\,\gamma^{\alpha}\,\mathcal{W}_{\rm eq}(x,k) = \int d^4k\,\mathcal{V}^{\alpha}_{\rm eq}(x,k) = \frac{1}{m}\int d^4k\,k^{\alpha}\mathcal{F}_{\rm eq}(x,k).$$

$$N^{\alpha}_{\rm eq} = \int \frac{d^3p}{2(2\pi)^3E_p}p^{\alpha}\left[{\rm tr}_4(X^+) - {\rm tr}_4(X^-)\right] = 4\cosh(\zeta)\sinh(\xi)\int\frac{d^3p}{(2\pi)^3E_p}\,p^{\alpha}\,e^{-\beta\cdot p} = n\,u^{\alpha}$$

where

$$n = 4\,\cosh(\zeta)\sinh(\xi)\,n_{(0)}(T)$$

$n_{(0)}(T) = \langle(u\cdot p)\rangle_0$ is the number density of spin-0, neutral Boltzmann particles, $\langle\cdots\rangle_0 \equiv \int\frac{d^3p}{(2\pi)^3E_p}(\cdots)\,e^{-\beta\cdot p}$.

The charge current should be conserved *i.e.*

$$\partial_{\alpha}N^{\alpha}_{\rm eq}(x) = 0.$$

Automatically holds in (extended) global equilibrium.
Gives condition in (extended) local equlibrium.

Energy-momentum tensor

The energy-momentum tensor as defined in [S. de Groot, W. van Leeuwen, and C. van Weert, Relativistic Kinetic Theory: Principles and Applications (1980)]

$$T^{\mu\nu}_{\rm eq}(x) = \frac{1}{m}\,{\rm tr}\int d^4k\, k^\mu\, k^\nu\, \mathcal{W}_{\rm eq}(x,k) = \frac{1}{m}\int d^4k\, k^\mu\, k^\nu\, \mathcal{F}_{\rm eq}(x,k).$$

$$T^{\mu\nu}_{\rm eq} = \int \frac{d^3p}{2(2\pi)^3 E_p} p^\mu p^\nu \left[{\rm tr}_4(X^+) + {\rm tr}_4(X^-)\right] = 4\cosh(\zeta)\cosh(\xi)\int \frac{d^3p}{(2\pi)^3 E_p} p^\mu p^\nu e^{-\beta\cdot p} = (\varepsilon + P)u^\mu u^\nu - P g^{\mu\nu}.$$

where, the energy density and pressure are given by the expressions

$$\varepsilon = 4\,\cosh(\zeta)\cosh(\xi)\,\varepsilon_{(0)}(T), \quad P = 4\,\cosh(\zeta)\cosh(\xi)\,P_{(0)}(T),$$

$\varepsilon_{(0)}(T) = \langle (u\cdot p)^2\rangle_0$,
$P_{(0)}(T) = -(1/3)\langle \left[p\cdot p - (u\cdot p)^2\right]\rangle_0$.
The energy-momentum tensor should be conserved,

$$\partial_\alpha T^{\alpha\beta}_{\rm eq}(x) = 0.$$

Automatically holds in (extended) global equilibrium.
Gives condition in (extended) local equlibrium.

1. Canonical form

The canonical form of spin tensor is [F. Becattini, V. Chandra, L. Del Zanna, and E. Grossi, Annals Phys. 338 (2013) 32–49].

$$
\begin{aligned}
S^{\lambda,\mu\nu}_{\rm can}(x) &= \frac{1}{4}\int d^4k\,{\rm tr}\left[\left\{\sigma^{\mu\nu},\gamma^\lambda\right\}\mathcal{W}_{\rm eq}(x,k)\right] = \frac{1}{2}\epsilon^{\kappa\lambda\mu\nu}\int d^4k\,\mathcal{A}_{{\rm eq},\kappa}(x,k) \\
&= \frac{\sinh(\zeta)\cosh(\xi)}{\zeta}\int dP\,e^{-\beta\cdot p}\left(\omega^{\mu\nu}p^\lambda+\omega^{\nu\lambda}p^\mu+\omega^{\lambda\mu}p^\nu\right) \\
&= \frac{w}{4\zeta}\left(u^\lambda\omega^{\mu\nu}+u^\mu\omega^{\nu\lambda}+u^\nu\omega^{\lambda\mu}\right),
\end{aligned}
$$

where we have introduced the spin density w defined by the expression

$$w = 4\sinh(\zeta)\cosh(\xi)n_{(0)}(T).$$

We expect

$$\partial_\lambda S^{\lambda,\mu\nu}_{\rm can}(x) = 0.$$

But the above equation does not hold for global equilibrium.

2. Phenomenological form

A phenomenological form of the spin tensor is [F. Becattini and L. Tinti, Annals Phys. 325 (2010) 1566–1594],

$$S^{\lambda,\mu\nu}_{\rm ph}(x) = \frac{1}{2}\int dP\, e^{-\beta\cdot p}\, p^\lambda \mathrm{tr}[(X^+ - X^-)\Sigma^{\mu\nu}].$$

Carrying out the trace calculation we get

$$S^{\lambda,\mu\nu}_{\rm ph}(x) = \frac{\sinh(\zeta)\cosh(\xi)}{\zeta}\int dP\, e^{-\beta\cdot p}\, p^\lambda\, \omega^{\mu\nu} = \frac{w}{4\zeta} u^\lambda \omega^{\mu\nu}.$$

Note that $S^{\lambda,\mu\nu}_{\rm ph}(x)$ is equal to the first term of $S^{\lambda,\mu\nu}_{\rm can}(x)$.

One expects,

$$\partial_\lambda S^{\lambda,\mu\nu}_{\rm ph}(x) = 0$$

One can easily check that the above equation holds in global equilibrium.

3. de Groot - van Leeuwen - van Weert formulation

The spin tensor introduced by de Groot, van Leeuwen, and van Weert has the form [S. de Groot, W. van Leeuwen, and C. van Weert, Relativistic Kinetic Theory: Principles and Applications (1980)]

$$S^{\lambda,\mu\nu}_{\rm GLW} = \frac{1}{4}\int d^4k\,{\rm tr}\left[\left(\left\{\sigma^{\mu\nu},\gamma^\lambda\right\} + \frac{2i}{m}\left(\gamma^{[\mu}k^{\nu]}\gamma^\lambda - \gamma^\lambda\gamma^{[\mu}k^{\nu]}\right)\right)\mathcal{W}_{\rm eq}(x,k)\right].$$

Performing the traces, and then carrying out the integration over k we get

$$S^{\lambda,\mu\nu}_{\rm GLW} = \frac{\sinh(\zeta)\cosh(\xi)}{m^2\zeta}\int dP\, e^{-\beta\cdot p}p^\lambda\left(m^2\omega^{\mu\nu} + 2p^\alpha p^{[\mu}\omega^{\nu]}{}_\alpha\right) = \frac{w}{4\zeta}u^\lambda\omega^{\mu\nu} + \frac{2\sinh(\zeta)\cosh(\xi)}{m^2\zeta}s^{\lambda,\mu\nu}_{\rm GLW}$$

where

$$s^{\lambda,\mu\nu}_{\rm GLW} = Au^\lambda u^\alpha u^{[\mu}\omega^{\nu]}{}_\alpha + B\left(\Delta^{\lambda\alpha}u^{[\mu}\omega^{\nu]}{}_\alpha + u^\lambda\Delta^{\alpha[\mu}\omega^{\nu]}{}_\alpha + u^\alpha\Delta^{\lambda[\mu}\omega^{\nu]}{}_\alpha\right)$$

and

$$B = -\frac{1}{\beta}\left(\epsilon_{(0)} + P_{(0)}\right), \quad A = \frac{1}{\beta}\left[3\epsilon_{(0)} + \left(3 + \frac{m^2}{T^2}\right)P_{(0)}\right] = -3B + \frac{m^2}{T}P_{(0)}.$$

We expect

$$\partial_\lambda S^{\lambda,\mu\nu}_{\rm GLW}(x) = 0$$

Obtaining the conservation laws from the kinetic equations

Charge conservation:

$$\int d^4k k^\mu \partial_\mu \mathcal{F}_{\rm eq}(x,k) = 0 \qquad \Rightarrow \partial_\alpha N^\alpha_{\rm eq}(x) = 0.$$

Energy-momentum conservation:

$$\int d^4k k^\mu k^\nu \partial_\mu \mathcal{F}_{\rm eq}(x,k) = 0 \qquad \Rightarrow \partial_\alpha T^{\alpha\beta}_{\rm eq}(x) = 0.$$

Spin conservation:

$$k^\mu \partial_\mu \mathcal{A}^\nu_{\rm eq}(x,k) = 0,$$

$$\Downarrow$$

$$k^\alpha \partial_\alpha \int dP\, e^{-\beta\cdot p} \frac{\sinh(\zeta)}{\zeta} \left[\delta^{(4)}(k-p)e^{\xi} + \delta^{(4)}(k+p)e^{-\xi}\right] \tilde{\omega}_{\mu\nu}\, p^\nu = 0 \tag{1}$$

$$k^\nu \partial_\alpha \int dP\, e^{-\beta\cdot p} \frac{\sinh(\zeta)}{\zeta} \left[\delta^{(4)}(k-p)e^{\xi} + \delta^{(4)}(k+p)e^{-\xi}\right] \tilde{\omega}_{\mu\nu}\, p^\alpha = 0$$

$$\frac{k^\nu}{2}\epsilon_{\mu\nu\rho\sigma}\left\{\partial_\alpha \int dP\, e^{-\beta\cdot p} \frac{\sinh(\zeta)}{\zeta} \left[\delta^{(4)}(k-p)e^{\xi} + \delta^{(4)}(k+p)e^{-\xi}\right] p^\alpha\, \omega^{\rho\sigma}\right\} = 0. \tag{2}$$

1. If we take the expression in the curly bracket of equation (2) and take the moment over k we will get conservation law for phenomenological tensor $\partial_\lambda S^{\lambda,\mu\nu}_{\rm ph}(x) = 0$.
2. If we multiply equation (1) by $k_\eta \epsilon^{\mu\gamma\lambda\eta}$ and then take the moment over k. We will get the conservation of GLW spin tensor *i.e.* $\partial_\lambda S^{\lambda,\mu\nu}_{\rm GLW}(x) = 0$.

Thus GLW spin tensor, in fact is a more natural choice.

Summary

We have discussed about the Wigner function (constructed from the local equilibrium phase space distribution functions for spin-1/2) and it's spinor decomposition.

We have analyzed in more detail the case where the Wigner function satisfy kinetic equation with a vanishing collision term.

We have found, in contrast to many earlier claims found in the literature, Wigner function approach does not imply a direct relation between the thermal vorticity and spin polarization, except for the fact that the two should be constant in global equilibrium.

We have also outlined procedures to formulate hydrodynamics with spin from the kinetic equations derived from Wigner function.

We have found that it would be useful to construct the hydrodynamics with spin with the help of the spin tensors derived by de Groot, van Leeuwen, and van Weert.

Future Plan: Our next task is to incorporate electromagnetic fields and collision term in the present framework.

THANK YOU

Global and local thermodynamic equilibrium; particle with spin

The density operator [D. Zubarev, Nonequilibrium Statistical Thermodynamics (Springer, 1974); F. Becattini, Phys. Rev. Lett. 108, 244502 (2012)],

$$\hat{\rho}(t) = \exp\left[-\int d^3\Sigma_\mu(x)\left(\hat{T}^{\mu\nu}(x)b_\nu(x) - \frac{1}{2}\hat{J}^{\mu,\alpha\beta}(x)\omega_{\alpha\beta}(x) - \xi(x)\hat{N}^\mu\right)\right].$$

Here $d^3\Sigma_\mu$ is an element of a space-like, three-dimensional hypersurface Σ_μ. We can take it as, $d^3\Sigma_\mu = (dV, 0, 0, 0)$.
The operators $\hat{T}^{\mu\nu}(x)$, $\hat{J}^{\mu,\alpha\beta}(x)$ and $\hat{N}^\mu(x)$ are the energy-momentum, angular momentum and charge operators respectively.
In global thermodynamic equilibrium the operator $\hat{\rho}(t)$ should be independent of time.

$$\partial_\mu\left(\hat{T}^{\mu\nu}(x)b_\nu(x) - \frac{1}{2}\hat{J}^{\mu,\alpha\beta}(x)\omega_{\alpha\beta}(x) - \xi(x)\hat{N}^\mu\right)$$
$$= \hat{T}^{\mu\nu}(x)\left(\partial_\mu b_\nu(x)\right) - \frac{1}{2}\hat{J}^{\mu,\alpha\beta}(x)\left(\partial_\mu\omega_{\alpha\beta}(x)\right) - \partial_\mu\xi(x) = 0.$$

For asymmetric energy momentum tensor, $b_\nu = b^0_\nu$, $\omega_{\alpha\beta} = \omega^0_{\alpha\beta}$, $\xi = \xi^0$.
For symmetric energy momentum tensor, $b_\nu = b^0_\nu + \delta\omega^0_{\nu\rho}x^\rho$, $\omega_{\alpha\beta} = \omega^0_{\alpha\beta}$, $\xi = \xi^0$.

Global and local thermodynamic equilibrium; particle with spin

$$\hat{J}^{\mu,\alpha\beta}(x) = \hat{L}^{\mu,\alpha\beta}(x) + \hat{S}^{\mu,\alpha\beta}(x).$$

Using above equation, we can write two cases discussed above can be expressed by a single form of the density operator

$$\hat{\rho}_{\rm EQ} \quad = \quad \exp\left[-\int d^3\Sigma_\mu(x)\left(\hat{T}^{\mu\nu}(x)\beta_\nu(x) - \frac{1}{2}\hat{S}^{\mu,\alpha\beta}(x)\omega^0_{\alpha\beta} - \xi^0\hat{N}^\mu\right)\right].$$

For asymmetric energy-momentum tensor $\beta_\mu(x) = b^0_\mu + \omega^0_{\mu\gamma}x^\gamma$.
$\beta_\mu(x)$ is a Killing vector, $\omega_{\mu\gamma} = \omega^0_{\mu\gamma}$.
For symmetric energy-momentum tensor $\beta_\mu(x) = b^0_\mu + (\delta\omega^0_{\mu\gamma} + \omega^0_{\mu\gamma})x^\gamma$.
$\beta_\mu(x)$ is again a Killing vector, $\omega_{\mu\gamma} \neq \omega^0_{\mu\gamma}$.

1. **global equilibrium —** β_μ field is a Killing vector, $\varpi_{\mu\nu} = \omega_{\mu\nu} =$ constt, in addition $\xi =$ constt.
2. **extended global equilibrium —** β_μ field is a Killing vector, $\varpi_{\mu\nu} =$ constt, $\omega_{\mu\nu} =$ constt but $\varpi_{\mu\nu} \neq \omega_{\mu\nu}$, in addition $\xi =$ constt.
3. **local equilibrium —** β_μ field is not a Killing vector but we still have $\omega_{\mu\nu}(x) = \varpi_{\mu\nu}(x)$, $\xi = \xi(x)$,
4. **extended local equilibrium —** β_μ field is not a Killing vector and $\omega_{\mu\nu}(x) \neq \varpi_{\mu\nu}(x)$, $\xi = \xi(x)$.

1. Pseudo-gauge transformation

defined with the help of arbitrary tensors $\Phi^{\lambda,\,\mu\nu}$ and $Z^{\alpha\lambda,\mu\nu}$

$$T'^{\,\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda\left(\Phi^{\lambda,\,\mu\nu} + \Phi^{\mu,\,\nu\lambda} + \Phi^{\nu,\,\mu\lambda}\right) \equiv T^{\mu\nu} + \frac{1}{2}\partial_\lambda G^{\lambda,\,\mu\nu}$$

$$S'^{\,\lambda,\mu\nu} = S^{\lambda,\mu\nu} - \Phi^{\lambda,\,\mu\nu} + \partial_\alpha Z^{\alpha\lambda,\mu\nu}$$

does not change global charges, new tensors, $T'^{\,\mu\nu}$ and $J'^{\,\lambda,\mu\nu}$, are also conserved

Bellinfante prescription: $\Phi^{\lambda,\,\mu\nu} = S^{\lambda,\mu\nu} \quad \rightarrow \quad S'^{\,\lambda,\mu\nu} = 0, \quad T'^{\,\mu\nu} = T'^{\,\nu\mu}$