Towards a precise determination of the equation of state of QCD at high-temperature

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Introduction

The goal

QCD equation of state

$$s(T),\ p(T),\ \varepsilon(T)$$

Thermodynamics

$$s(T) = \frac{\partial p(T)}{\partial T}$$

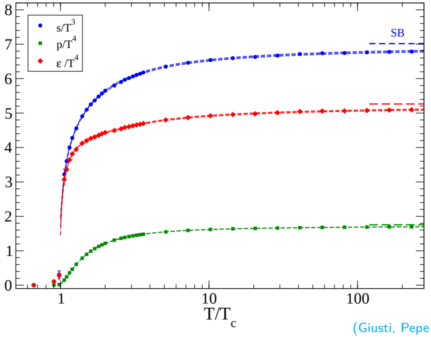
$$Ts(T) = p(T) + \varepsilon(T)$$

Why is this important?

- Fundamental property of QCD
- Heavy-ion collisions
- Cosmology
- etc ...

Ultimately, one would wish for

$$s(T,\mu),\ p(T,\mu),\ \varepsilon(T,\mu) \qquad \mu \equiv \text{chemical potential}$$



(Giusti, Pepe '17)

SU(3) YM – $T_c \approx 300\,\mathrm{MeV}$

Introduction

A non-perturbative problem

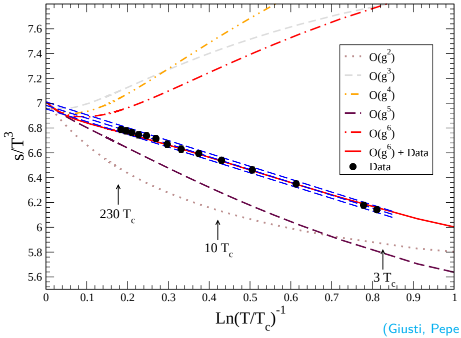
Asymptotic freedom

$$\alpha_s(\mu) \xrightarrow{\mu\to\infty} 0$$

taking $\mu \approx T \Rightarrow$ PTh
should work at high-T

Free quarks & gluons

$$\frac{s_{\rm SB}(T)}{T^3} = \frac{\pi^2}{45}\left(32 + 21 \times N_{\rm f}\right) \overset{N_{\rm f}=0}{\approx} 7.02$$



(Giusti, Pepe '17)

SU(3) YM – $T_c \approx 300\,\text{MeV}$

Problems

- (**Generic**) PTh is only asymptotic!
- (**Specific**) High-T expansion shows very poor convergence!
  - Only works up to a finite, observable dependent order, no matter how **weak** the coupling is! (Lindé '80)
  - Here O(g^6) + Data at $T \approx 230\,T_c \approx 68\,\text{GeV}$ is about **30%** of the total entropy!

Solution

Lattice QCD is the **only** framework for a first principle **non-perturbative** determination

Introduction

A difficult non-perturbative problem

Free energy

$$f = -p = -\frac{T}{V}\ln \mathcal{Z}$$

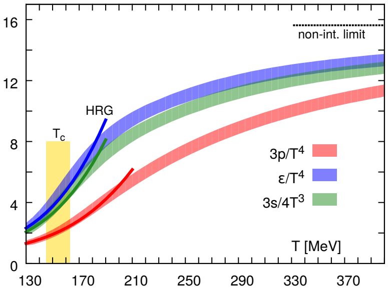
Trace anomaly

(Engels et. al. '81; Umeda et. al. '09; ...)

$$\frac{I(T)}{T^4} \equiv \frac{\varepsilon - 3p}{T^4} = T\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{p}{T^4}\right)$$

Pressure

$$\frac{p(T)}{T^4} = \frac{p(T_0)}{T_0^4} + \int_{T_0}^{T} \mathrm{d}T' \, \frac{I(T')}{T'^5}$$



(HotQCD Bazavov et. al. '14)

Lattice obs.

QCD with $N_\mathrm{f} = 3$ quarks

$$\widehat{I}(T) = -\frac{T}{V}\frac{\mathrm{d}\ln\widehat{\mathcal{Z}}}{\mathrm{d}\ln a} = \frac{T}{V}\left(a\frac{\mathrm{d}\vec{b}}{\mathrm{d}a}\right)\left\langle \frac{\partial \widehat{S}_{\mathrm{QCD}}}{\partial \vec{b}} \right\rangle_T$$

$$\vec{b} = \{g_0(a), m_{0,f}(a), \ldots\} \Leftarrow \mathrm{LCP}$$

Renormalization

$$I(T) = \lim_{a\to 0} \widehat{I}_R(T) = \lim_{a\to 0}\left[\widehat{I}(T) - \widehat{I}(0)\right]\Big|_{\vec{b}}$$

Problem

The renormalization **unnaturally** ties together two **separate** physical scales ...

$$L^{-1} \ll T \ll a^{-1} \text{ AND } L^{-1} \sim m_\pi \;\Rightarrow\; L/a = \mathrm{O}(100) \text{ for } T = \mathrm{O}(1\,\mathrm{GeV})$$

Thermodynamics in a moving frame

The relativistic liquid or gas **(Minkowski space)**

Energy-momentum tensor (EMT)

$\mathcal{T}_{\mu\nu}$ contains all the information we need

Local rest frame

$$\mathcal{T}_{\mu\nu} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Moving frame

$$\mathcal{T}_{0k} = \frac{p+\varepsilon}{1-\boldsymbol{v}^2}\, v_k \qquad \boldsymbol{v} \equiv \text{velocity}$$

$$\mathcal{T}_{00} = \frac{p+\varepsilon}{1-\boldsymbol{v}^2} - p \qquad \mathcal{T}_{jk} = \frac{p+\varepsilon}{1-\boldsymbol{v}^2}\, v_j v_k + p\,\delta_{jk}$$

Entropy density (using $Ts = p + \varepsilon$)

$$Ts = \frac{\mathcal{T}_{0k}}{\gamma^2 v_k} \qquad \gamma = \frac{1}{\sqrt{1-\boldsymbol{v}^2}}$$

The entropy is a **physical** quantity and thus a **natural** observable to consider!

Thermodynamics in a moving frame

Shifted boundary conditions (Giusti, Meyer '11 '13)

Euclidean partition function

$$\mathcal{Z}(L_0, \boldsymbol{\xi}) = \mathrm{Tr}\left\{e^{-L_0(H - i\boldsymbol{\xi}\cdot\boldsymbol{P})}\right\} \qquad \left[\,\boldsymbol{\xi} = -i\boldsymbol{v}\,\right]$$

Free energy

$$f(L_0, \boldsymbol{\xi}) = -\frac{1}{L_0 V}\ln \mathcal{Z}(L_0, \boldsymbol{\xi}) \qquad f(L_0, \boldsymbol{\xi}) \overset{V\to\infty}{\longrightarrow} f(L_0\sqrt{1+\boldsymbol{\xi}^2}, 0)$$

QCD path integral

$$\mathcal{Z}(L_0, \boldsymbol{\xi}) = \int [DA][D\overline{\psi}][D\psi]\, e^{-S_{\mathrm{QCD}}[A,\overline{\psi},\psi]}$$

$$A_\mu(L_0, \boldsymbol{x}) = A_\mu(0, \boldsymbol{x} - \boldsymbol{\xi}L_0) \qquad \psi(L_0, \boldsymbol{x}) = -\psi(0, \boldsymbol{x} - \boldsymbol{\xi}L_0)$$

"Ward identities"

$$\langle \mathcal{T}_{0k}\rangle_\xi = -\frac{\partial}{\partial \xi_k} f(L_0, \boldsymbol{\xi}) \quad \Rightarrow \quad \frac{\partial}{\partial \xi_k}\langle \mathcal{O}\rangle_\xi = L_0 \langle \overline{\mathcal{T}}_{0k}(x_0)\mathcal{O}\rangle_{\xi,c}$$

Entropy

$$Ts(T) = -\frac{(1+\boldsymbol{\xi}^2)}{\xi_k}\langle \mathcal{T}_{0k}\rangle_\xi \qquad T = \frac{1}{L_0\sqrt{1+\boldsymbol{\xi}^2}}$$

The energy-momentum tensor

Renormalization on and off the lattice

Continuum EMT

(Callan, Coleman, Jackiw '71; ...)

$$\mathcal{T}^R_{\mu\nu} = \mathcal{T}_{\mu\nu} = \mathcal{T}^F_{\mu\nu} + \mathcal{T}^G_{\mu\nu}$$

$$\mathcal{T}^F_{\mu\nu} = \frac{1}{4}\left\{\overline{\psi}\gamma_\mu \overleftrightarrow{D}_\nu \psi + \overline{\psi}\gamma_\nu \overleftrightarrow{D}_\mu \psi\right\} - \delta_{\mu\nu}\mathcal{L}^F \qquad \mathcal{T}^G_{\mu\nu} = \frac{1}{g_0^2} F^a_{\mu\alpha}F^a_{\nu\alpha} - \delta_{\mu\nu}\mathcal{L}^G$$

On the lattice

(Caracciolo et. al. '90 '91 '92)

- The lattice regulator explicitly **breaks** Poincaré symmetry
  $\Rightarrow$ The EMT requires renormalization
- Poincaré symmetry is however **recovered** for $a \to 0$
  $\Rightarrow$ The renormalization is scale-independent

Lattice EMT

- $\widehat{\mathcal{T}}^R_{0k} = Z^F_T\, \widehat{\mathcal{T}}^F_{0k} + Z^G_T\, \widehat{\mathcal{T}}^G_{0k}$ with $Z^{F,G}_T \overset{g_0 \to 0}{=} 1 + c_0^{F,G}\, g_0^2 + \ldots$
- $\langle \widehat{\mathcal{T}}_{\mu\mu}\rangle_T / T^4 \overset{a\to 0}{\propto} 1/(aT)^4 \quad \Rightarrow \quad \widehat{\mathcal{T}}_{\mu\mu}$ requires power-subtractions!

Renormalization conds.

Ex.: $\langle \widehat{\mathcal{T}}^R_{0k}\rangle_\xi \overset{!}{=} -\frac{\partial}{\partial \xi_k}\widehat{f}(L_0, \boldsymbol{\xi})$

Towards the EoS of QCD

Preparing the set-up (MDB, Giusti, Pepe '17)

Master formula

$$\frac{s(T)}{T^3} = \lim_{a\to 0} \frac{\widehat{s}(T)}{T^3} \qquad \frac{\widehat{s}(T)}{T^3} = -\frac{L_0^4(1+\boldsymbol{\xi}^2)^3}{\xi_k}\langle \widehat{\mathcal{T}}_{0k}^R \rangle_{\xi} \qquad T = \frac{1}{L_0\sqrt{1+\boldsymbol{\xi}^2}}$$

First steps

1. Choose the lattice set-up

   $N_{\rm f} = 2+1$ O(a)-improved Wilson fermions

2. Choose a lattice regularization $\widehat{\mathcal{T}}_{0k}$ of $\mathcal{T}_{0k}$

3. Determine sensible sets of kinematical parameters

   $L_0/a,\ L/a,\ \boldsymbol{\xi},\ \ldots$

4. Estimate the CPU effort for precise determinations of the bare expectation values

   $\langle \widehat{\mathcal{T}}_{0k}^F \rangle_{\xi}, \quad \langle \widehat{\mathcal{T}}_{0k}^G \rangle_{\xi}$

5. Find convenient renormalization conditions to fix

   $Z_T^F, \quad Z_T^G$

6. Apply for CPU time ...

On the discretization of the EMT

Basic and O(a)-improved definition

Basic

(Caracciolo et. al. '90 '91 '92)

$$\widehat{\mathcal{T}}^R_{0k} = Z^F_T(g_0)\,\widehat{\mathcal{T}}^F_{0k} + Z^G_T(g_0)\,\widehat{\mathcal{T}}^G_{0k}$$

$$\widehat{\mathcal{T}}^F_{0k} = \frac{1}{4}\left\{\overline{\psi}\gamma_k \overleftrightarrow{\nabla}_0 \psi + \overline{\psi}\gamma_0 \overleftrightarrow{\nabla}_k \psi\right\} \qquad \widehat{\mathcal{T}}^G_{0k} = \frac{1}{g_0^2}\widehat{F}^a_{0\alpha}\widehat{F}^a_{k\alpha} \qquad \overleftrightarrow{\nabla}_\mu \overset{a\to 0}{=} \overleftrightarrow{D}_\mu + \mathrm{O}(a^2)$$

O(a)-improved (mass-degenerate quarks)

$$\widehat{\mathcal{T}}^R_{I,0k} = Z^F_T(\widetilde{g}_0)\widehat{\mathcal{T}}^F_{I,0k} + Z^G_T(\widetilde{g}_0)\,\widehat{\mathcal{T}}^G_{I,0k} \qquad \left[\widetilde{g}_0^2 = g_0^2\left(1 + b_g(g_0)am_q\right)\right]$$

(Sommer, Sint '96)

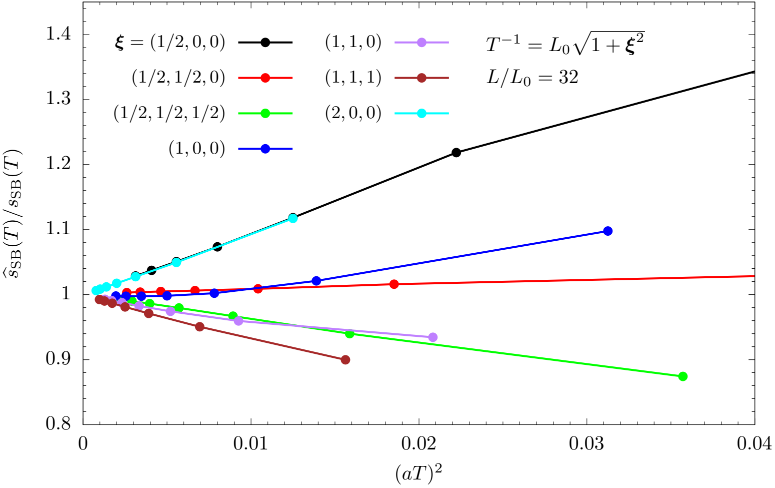
- $\widehat{\mathcal{T}}^G_{I,0k} = \left(1 + b^G_T(g_0)am_q\right)\widehat{\mathcal{T}}^G_{0k}$
- $\widehat{\mathcal{T}}^F_{I,0k} = \left(1 + b^F_T(g_0)am_q\right)\left\{\widehat{\mathcal{T}}^F_{0k} + c^F_T(g_0)a\delta\widehat{\mathcal{T}}^F_{0k}\right\}$
- $\delta\widehat{\mathcal{T}}^F_{0k} = \overline{\psi}\,i\left[\sigma_{0\rho}\,\widehat{F}_{k\rho} + \sigma_{k\rho}\,\widehat{F}_{0\rho}\right]\psi \;\Leftarrow$ **breaks chiral symmetry!**

Observations

- In PTh $b^F_T = 1 + \mathrm{O}(g_0^2)$ while b^G_T and b_g are $\mathrm{O}(g_0^2)$
  - For mass non-degenerate quarks one more counterterm of $\mathrm{O}(g_0^4)$
  - If $aT \ll 1$, at large T: $am_q \ll m_q/T \ll 1 \Rightarrow$ negligible effects!
- In the chiral limit for $T > T_c$ chiral symmetry is **restored** $\Rightarrow$ automatic O(a)-improvement of massless QCD is **recovered**
- For $T \lesssim T_c$ one may need to determine c^F_T non-perturbatively!

Towards the EoS of QCD

Discretization effects in the entropy for free lattice quarks and gluons (MDB, Giusti, Pepe '17)



$$\frac{\widehat{s}_{\rm SB}(T)}{T^3} \overset{a\to 0}{\longrightarrow} \frac{s_{\rm SB}(T)}{T^3} = \frac{\pi^2}{45}\left(32 + 21 \times N_{\rm f}\right) \overset{N_{\rm f}=3}{\approx} 20.84$$

Towards the EoS of QCD

Determination of the EMT expectation values (MDB, Giusti, Pepe '17)

Parameters

$$L_0/a = 6,\ L/a = 96,\ \boldsymbol{\xi} = (1,0,0)\ \Rightarrow\ TL \approx 11$$

Finite size effects

(Giusti, Meyer '11 '13)

$$s(T)|_L = s(T)|_{L=\infty} + \mathrm{O}(e^{-ML}) \qquad M = \mathrm{O}(T)\ \leftarrow\ \text{lightest screening mass}$$

Results

T (GeV)	$\langle\widehat{\mathcal{T}}^F_{0k}\rangle_\xi / \widehat{\mathcal{T}}^F_{0k,\mathrm{SB}}$	$\langle\widehat{\mathcal{T}}^G_{0k}\rangle_\xi / \widehat{\mathcal{T}}^G_{0k,\mathrm{SB}}$	N_{ms}	Time/N_{ms}
$T_1 \approx 0.36$	0.6859(56)	0.3775(67)	704	1.8 kch
$T_2 \approx 4$	0.8298(47)	0.6971(119)	548	0.5 kch
$T_3 \approx 45$	0.8711(41)	0.7712(131)	654	0.3 kch

† T_1: m_{ud}, m_s s.t. $m_\pi \approx 340\,\mathrm{MeV}$ and $m_K \approx 440\,\mathrm{MeV}$ @ $T=0$
$T_{2,3}$: $m_{ud} = m_s \approx 0$

- Very good precision with very **modest** CPU effort:

$$\delta\langle\widehat{\mathcal{T}}^F_{0k}\rangle_\xi \approx 0.5 - 1\% \quad \delta\langle\widehat{\mathcal{T}}^G_{0k}\rangle_\xi \approx 2\% \quad @ \quad N_{\mathrm{ms}} \approx 600$$

N.B.: In the free case we have that $\langle\widehat{\mathcal{T}}^F_{0k}\rangle \approx 2\langle\widehat{\mathcal{T}}^G_{0k}\rangle$

- Simulations become **easier** and accuracy **increases** as the temperature **increases** $\Rightarrow$ Easier to obtain **precise** results!

Conclusions & outlook

Conclusions

- The framework of shifted boundary conditions allows to **solve** the window problem of standard lattice methods for the computation of the EoS
- Precise results for the (bare) entropy density can be obtained over a **wide** range of temperatures with **modest** effort

  Here we scanned **two orders of magnitude in T**!

Outlook

- Physical results require **renormalization**
- We recently completed a one-loop study of the EMT with SBC. This allows us to:
  - devise a good renormalization strategy
  - decide on the best lattice set-up to adopt
  - obtain estimates for the O(a)-improvement coefficients and perturbatively improve continuum limit extrapolations
- Interesting to study the **(non-)convergence** of PTh at high-T and the approach to the Stefan-Boltzmann limit

  Using the one-loop estimates for $Z_T^{F,G}$ and our lattice results at $T \approx 40\,\mathrm{GeV}$ we find e.g. $\widehat{s}(T)/s_{\mathrm{SB}}(T) \approx 0.96$

Thank you for your attention
and good night!