# Anomalous Transport, Massive Gravity Theories and Holographic Momentum Relaxation


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## Abstract


Quantum anomalies give rise to new transport phenomena. In particular a magnetic field can induce an anomalous electric current via the chiral magnetic effect, and a vortex in a relativistic fluid can also induce an electric current via the chiral vortical effect. We derive the Kubo formulae relevant for anomalous transport, and perform a computation in AdS/CFT of the transport coefficients in a massive gravity model including vorticity and external electromagnetic fields. To arrive at this result we suggest a new definition of the Energy-Momentum tensor (EMT) in presence of the gravitational Chern-Simons coupling. The anomalous conductivities turn out to be independent of the holographic disorder couplings.


## 1. Hydrodynamics of relativistic fluids

- The basic ingredients to study hydrodynamics are the constitutive relations, i.e. expressions of the Energy-Momentum tensor and the charge currents in terms of fluid quantities (charge density, fluid velocity, etc), organized in a derivative expansion, also called hydrodynamic expansion [1]:

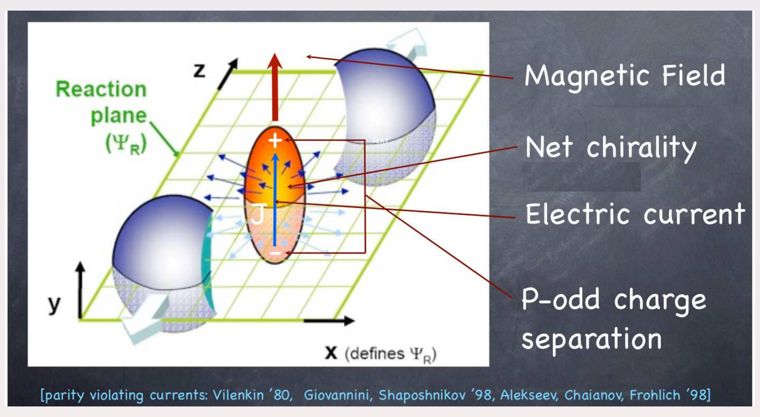
$$\langle T^{\mu\nu}\rangle = \underbrace{(\varepsilon + P)u^\mu u^\nu + P g^{\mu\nu}}_{\text{Ideal Hydro}} + \underbrace{\langle T^{\mu\nu}\rangle_{\text{diss \& anom}}}_{\text{Dissipative \& Anomalous}}, \quad (1)$$

$$\langle J^{\mu}\rangle = \underbrace{n u^\mu}_{\text{Ideal Hydro}} + \underbrace{\langle J^{\mu}\rangle_{\text{diss \& anom}}}_{\text{Dissipative \& Anomalous}}. \quad (2)$$

In addition to the equilibrium contributions, there are extra terms in the constitutive relations which lead to dissipative and anomalous effects, corresponding to derivative corrections.

- The Chiral Magnetic Effect is an example of anomalous transport: a very strong magnetic field produced during a non-central collision of heavy ions induces a parity-odd charge separation, and as a consequence an electric current parallel to the magnetic field is generated [2].

**Chiral Magnetic Effect**




A vortex in the fluid can induce also an electric current. This is the so-called Chiral Vortical Effect [3]. Both phenomena can be summarized with the formula:

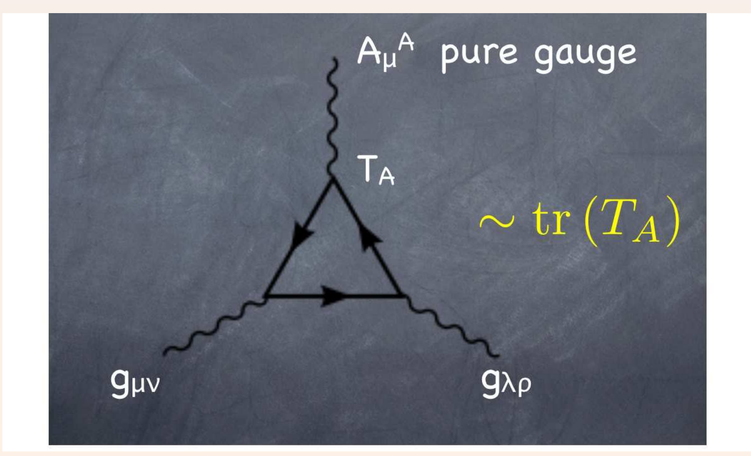
$$\langle J^\mu\rangle = \underbrace{n u^\mu}_{\text{Ideal Hydro}} + \underbrace{\sigma^{\mathcal{B}} B^\mu + \sigma^{\mathcal{V}}\omega^\mu}_{\text{Anomalous}} + \cdots .$$

- A convenient way to compute the anomalous conductivities are the Kubo formulae [2, 4]

$$\sigma^{\mathcal{B}} = \lim_{k_c\to 0}\frac{i}{2k_c}\sum_{a,b}\epsilon_{abc}\langle J^a J^b\rangle|_{\omega=0}, \qquad \sigma^{\mathcal{V}} = \lim_{k_c\to 0}\frac{i}{2k_c}\sum_{a,b}\epsilon_{abc}\langle J^a T^{0b}\rangle|_{\omega=0}. \quad (3)$$

A 1 loop calculation of the Chiral Vortical Conductivity for a theory of free chiral fermions leads to contributions of the chiral anomaly, and also of the mixed gauge-gravitational anomaly [5]:

$$(\sigma^{\mathcal{V}})_A = \underbrace{\frac{1}{8\pi^2}\sum_{B,C} d_{ABC}\,\mu^B\mu^C}_{\text{Chiral Anomaly}} + \underbrace{\frac{T^2}{24}\text{tr}\,(T_A)}_{\text{Gauge-Gravitational Anomaly}}$$




The same conclusion is obtained in the strongly coupled regime with a holographic model in 5 dim in which the anomalies are implemented through Chern-Simons terms in the action, cf. Eq. (8) below [6]. This suggests the existence of non-renormalization theorems for anomalous transport.

## 2. Massive gravity and holographic momentum relaxation

- Massive gravity theories have been proposed to study disorder effects in condensed matter (see Refs. [7, 8] for a recent holographic implementation in 4 dim). The model in 5 dim writes:

$$S = \int d^5x\sqrt{-g}\left[R + 12 - \frac{1}{2}\partial^\mu X^I\partial_\mu X^I - \frac{1}{4}F^2 - \frac{\mathcal{J}}{4}\partial_\mu X^I\partial_\nu X^I F^\nu{}_\lambda F^{\lambda\mu}\right] + S_{\text{GH}} \quad (4)$$

with scalars $X^I = k\,\delta^I_i\,x^i$ that break translational invariance. $k = 0 \Longrightarrow$ massless gravity limit.

- We consider as background the charged black hole solution with AdS asymptotics

$$ds^2 = \frac{1}{u}\left(-f(u)dt^2 + dx^2 + dy^2 + dz^2\right) + \frac{du^2}{4u^2 f(u)}, \qquad A_t = \phi(u). \quad (5)$$

The temperature is

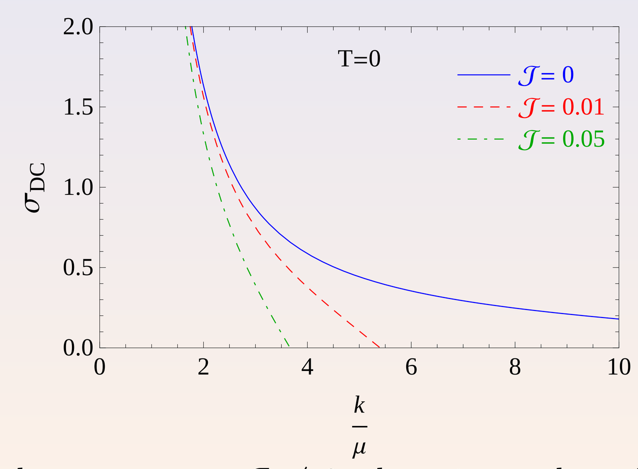
$$T = \frac{1}{\pi}\left(1 - \frac{k^2}{8} - \frac{\mu^2}{6}\right), \quad (6)$$

and it does not depend on the charge disorder coupling $\mathcal{J}$.

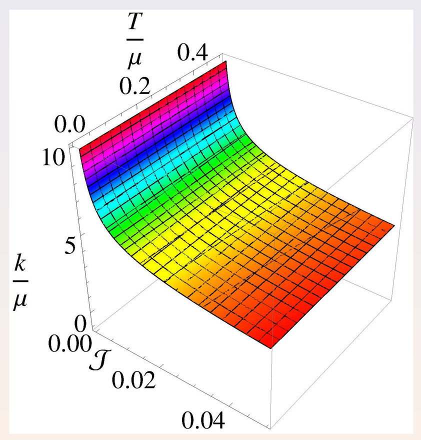
- A computation of the electric DC conductivity leads to [9, 10]

$$\sigma_{DC} = \frac{J_x}{E} = r_h\left(1 - \frac{\mathcal{J}k^2}{2r_h^2}\right)\left[1 + \left(1 - \frac{\mathcal{J}k^2}{2r_h^2}\right)\frac{4\mu^2}{k^2 M(r_h)}\right] \quad \text{with} \quad M(r_h) = 1 + \frac{2\mathcal{J}\rho^2}{r_h^6}. \quad (7)$$

**$\sigma_{DC}$ at $T = 0$**




**Region where $\sigma_{DC}$ vanishes**




When the parameter $\mathcal{J} \neq 0$, there is a value of $k = k_{\text{max}}$ for which $\sigma_{DC} = 0$ ⟶ In this regime the system behaves as an insulator.

## 3. Anomalous transport in massive gravity theories

- We can study the anomalous transport in the massive gravity theories by adding Chern-Simons terms to the action, $S_{\text{tot}} = S + S_{\text{CS}} + S_{\text{CSK}}$, with $S_{\text{CSK}}$ a convenient counterterm [6, 10] and

$$S_{\text{CS}} = \int d^5x\sqrt{-g}\,\epsilon^{\mu\nu\rho\lambda\sigma}A_\mu\left(\frac{\alpha}{3}F_{\nu\rho}F_{\lambda\sigma} + \lambda R^\alpha{}_{\beta\nu\rho}R^\beta{}_{\alpha\lambda\sigma}\right). \quad (8)$$

- In the standard Fefferman-Graham coordinates, $ds^2 = dr^2 + \gamma_{ij}dx^i dx^j$, the variation of the on-shell action with a timelike hypersurface at a fixed $r$ is

$$\delta S_{\text{tot}} = \frac{1}{2}\int_\partial\sqrt{-\gamma}\left(t^{ij}\delta\gamma_{ij} + u^{ij}\delta K_{ij}\right) + \delta S_{\text{matter}}. \quad (9)$$

We define the holographic Energy-Momentum tensor as [10]

$$\Theta^i{}_j = t^i{}_j + u^{il}K_{lj}, \qquad \text{with} \qquad t^{ij} = t_0^{ij} + t_\lambda^{ij}, \quad (10)$$

where $t_0^{ij} = -2\sqrt{-\gamma}(K^{ij} - K\gamma^{ij})$ is the standard Brown-York contribution, and

$$t_\lambda^{ij} = -8\lambda\sqrt{-\gamma}\epsilon^{mnp(i}\left(2D_nK_p^{j)}F_{rm} + \gamma^{j)l}\dot{K}_{ln}F_{pm} - F_{pm}K_l^{j)}K_n^l\right), \quad (11)$$

$$u^{ij} = 8\lambda\sqrt{-\gamma}\epsilon^{mnp(i}F_{mn}K_p^{j)}. \quad (12)$$

$(t_0)^i_j$ is divergent and needs to be regularized by the standard counterterms [11]. In contrast, the contributions $(t_\lambda)^i_j$ and $u^{il}K_{lj}$ are already finite before the holographic renormalization.

- Using the Kubo formulae, a computation of the anomalous conductivities leads to [10]:

$$\begin{aligned}
\sigma^{\mathcal{B}} &\simeq \langle J^x J^y\rangle = \alpha 4\mu, \\
\sigma^{\mathcal{V}} &\simeq \langle J^x T^{0y}\rangle = \alpha 2\mu^2 + \lambda 16\pi^2 T^2, \\
\sigma^{\mathcal{B}}_\epsilon &\simeq \langle T^{0x} J^y\rangle = \alpha 2\mu^2 + \lambda 16\pi^2 T^2, \\
\sigma^{\mathcal{V}}_\epsilon &\simeq \langle T^{0x} T^{0y}\rangle = \alpha\frac{4}{3}\mu^3 + \lambda 32\pi^2 T^2\mu.
\end{aligned} \quad (13)$$

- Chiral Magnetic ($\sigma^{\mathcal{B}}$, $\sigma^{\mathcal{B}}_\epsilon$) and Vortical ($\sigma^{\mathcal{V}}$, $\sigma^{\mathcal{V}}_\epsilon$) conductivities are the same as in massless gravity ⟶ No dependence on the holographic disorder couplings ($k$, $\mathcal{J}$).
- There is a regime in which $\sigma_{DC} = 0$ and $\sigma_{\text{anomalous}} \neq 0$.
- The fact that $\sigma^{\mathcal{B}}_\epsilon = \sigma^{\mathcal{V}}$ follows non-trivially from the definition of the EMT in Eq. (10). The term $u^{il}K_{lj}$ induces a contribution which cancels exactly a dependence $\sigma^{\mathcal{B}}_\epsilon \propto \lambda 2k^2$ [10].

## 4. Conclusions

- We have studied the effects of *external electromagnetic fields* and *vortices* in relativistic fluids.
- Mixed gauge-gravitational anomaly contributes even at 1st order in hydrodynamic expansion.
- Results *at zero frequency* agree using two different methods and regimes:
  i) Field Theory - Weak Coupling. ii) Holography - Strong Coupling.
- We have studied anomalous transport in massive gravity theories. We find a regime in which: i) electric DC conductivity vanishes: $\sigma_{DC} = 0$; and ii) anomalous conductivities do not vanish: $\sigma_{\text{anomalous}} \neq 0$. This allows to study anomalous effects in these systems in a clean way.
- The mixed gauge-gravitational Chern-Simons term modifies the definition of the holographic EMT. Only by using this corrected form we find the expected result that the anomalous transport is insensitive to the holographic disorder. This solves the puzzle found in Ref. [9].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