Multi-hadron observables from lattice QCD

Maxwell T. Hansen

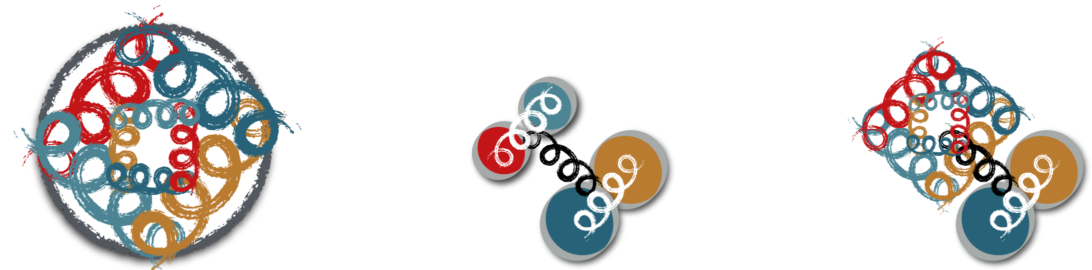
Confinement XIII

August 5th, 2018

Scattering and Spectroscopy

Experiments worldwide are exploring
the exotic resonance spectrum

'exotic' = outside the *quark model*



glueballs tetraquarks hybrids

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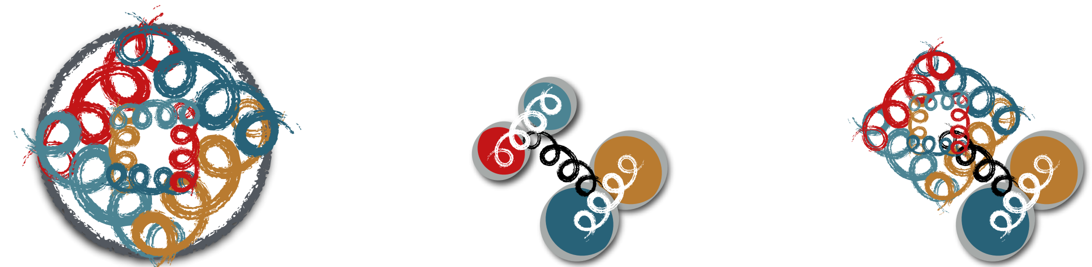
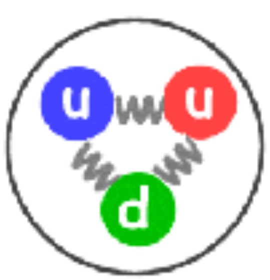
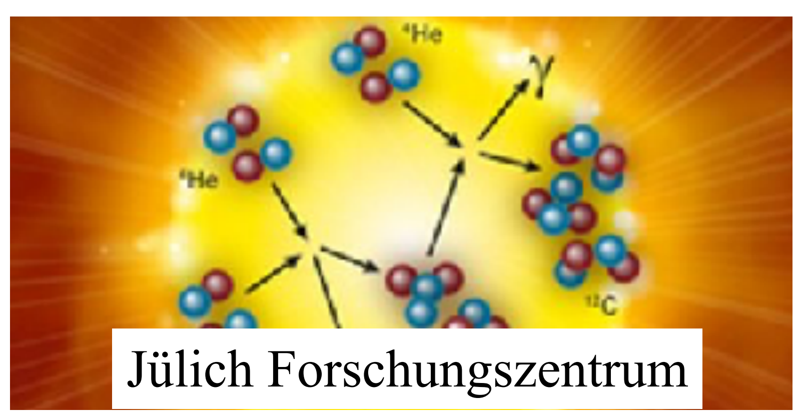


glueballs | tetraquarks | hybrids

Understanding these requires the full machinery of…
Quantum Chromodynamics (QCD)



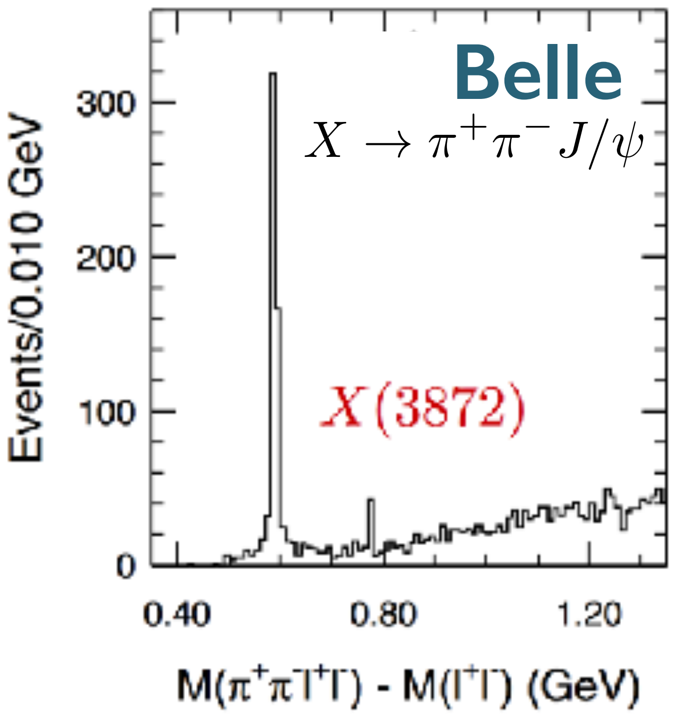
Quarks and gluons
(interactions constrained by symmetries)



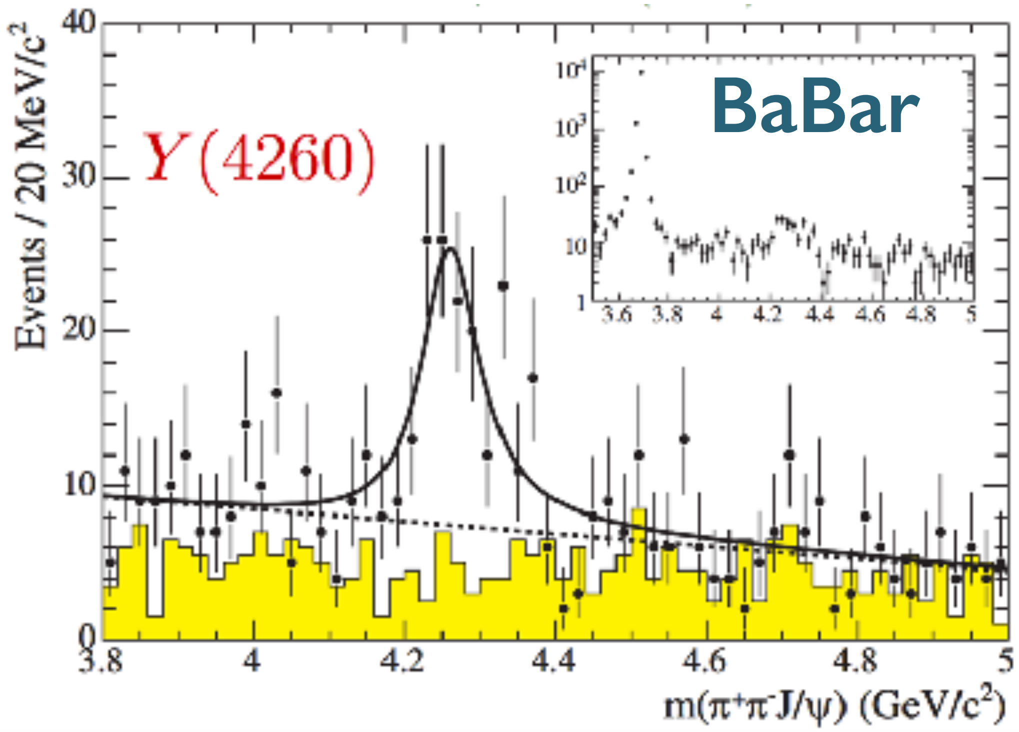
Simple underlying structure leads to a rich variety of phenomena

Difficult to extract predictions from the underlying theory

Many resonances, many questions



PRL 91, 262001 (2003)



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see very nice talk from Ryan Mitchell

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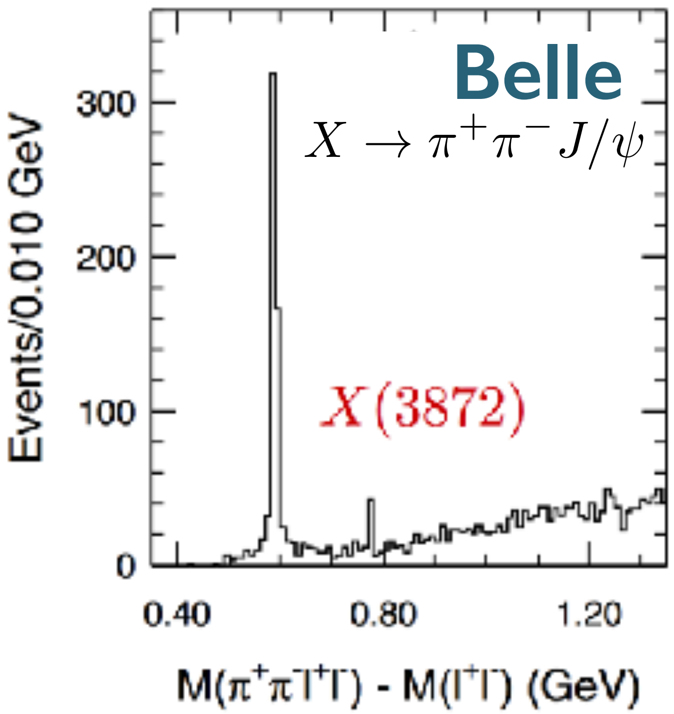

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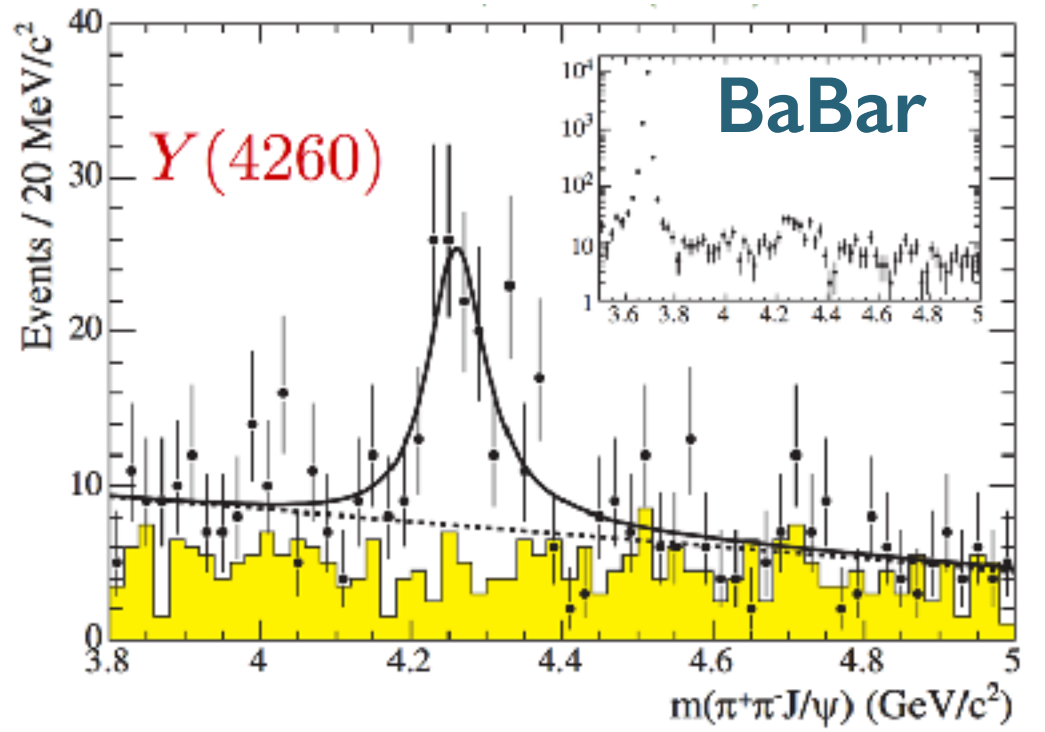

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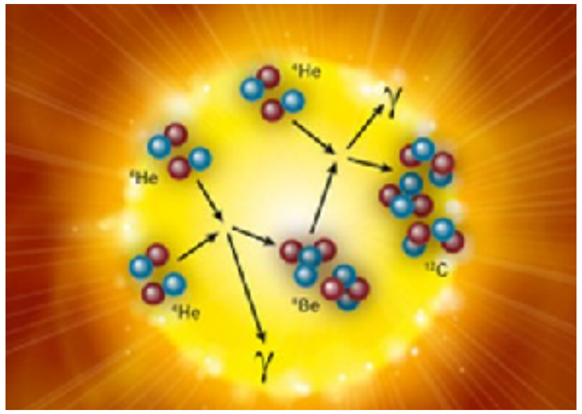
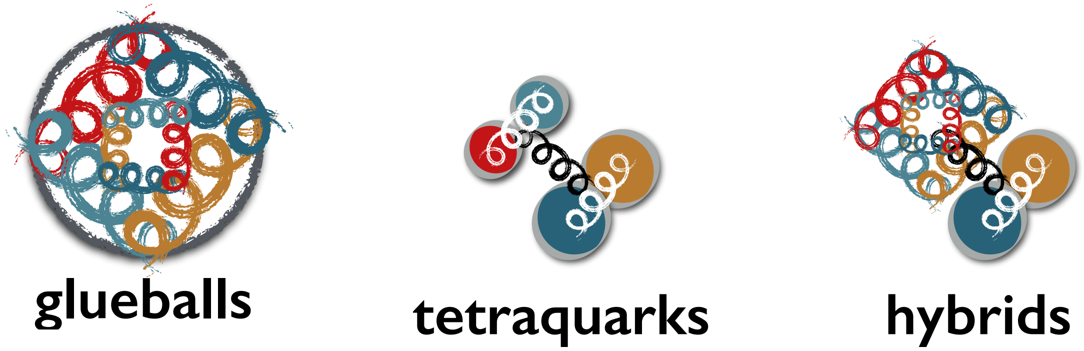
- ☐ How does this rich structure emerge from such a simple underlying theory?
- ☐ How do resonances quantitatively modify scattering and production rates?
- ☐ Why are some states well described by the quark model and others not?
- ☐ How do resonance properties depend on QCD's fundamental parameters?



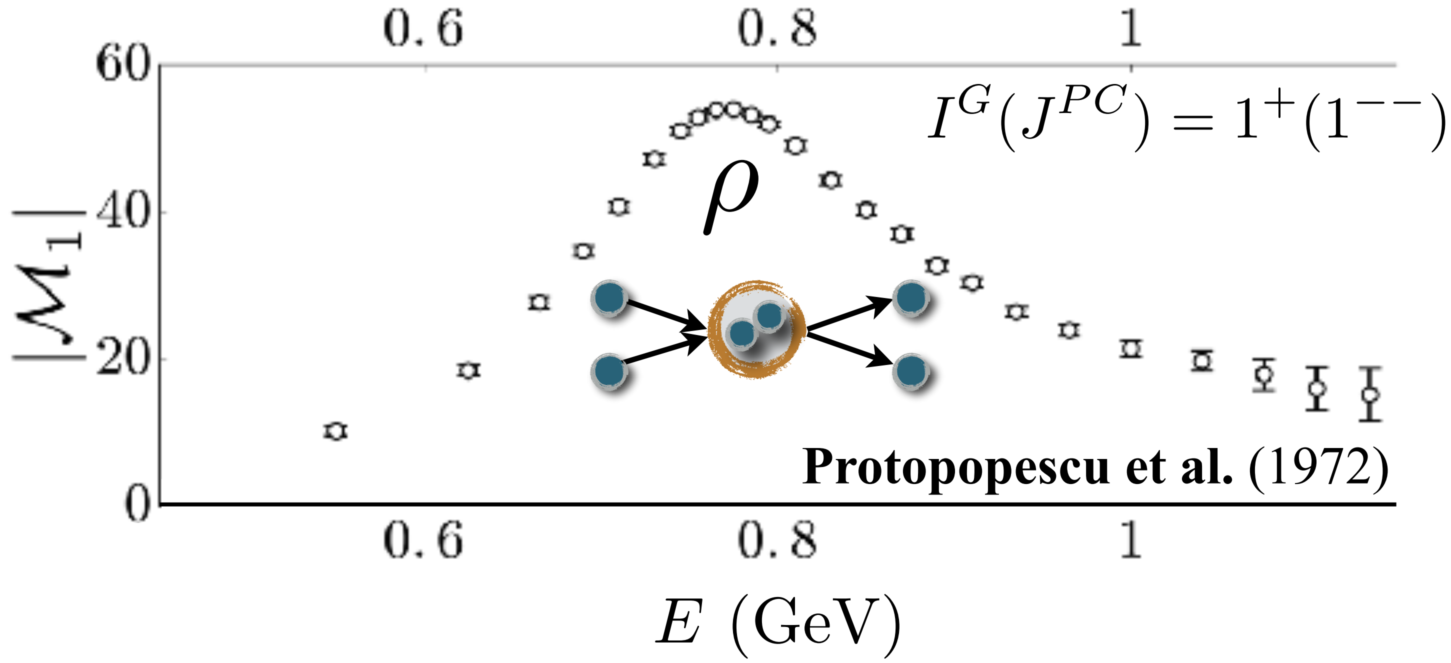


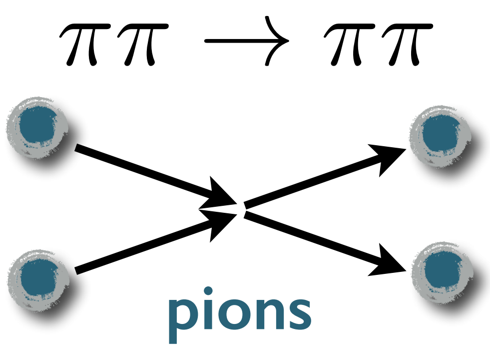
Definition of a resonance

- Roughly speaking, a bump in $|\mathcal{M}(E)|^2 \propto |e^{2i\delta(E)} - 1|^2 \propto \sin^2 \delta(E)$

scattering rate

unitarity relation





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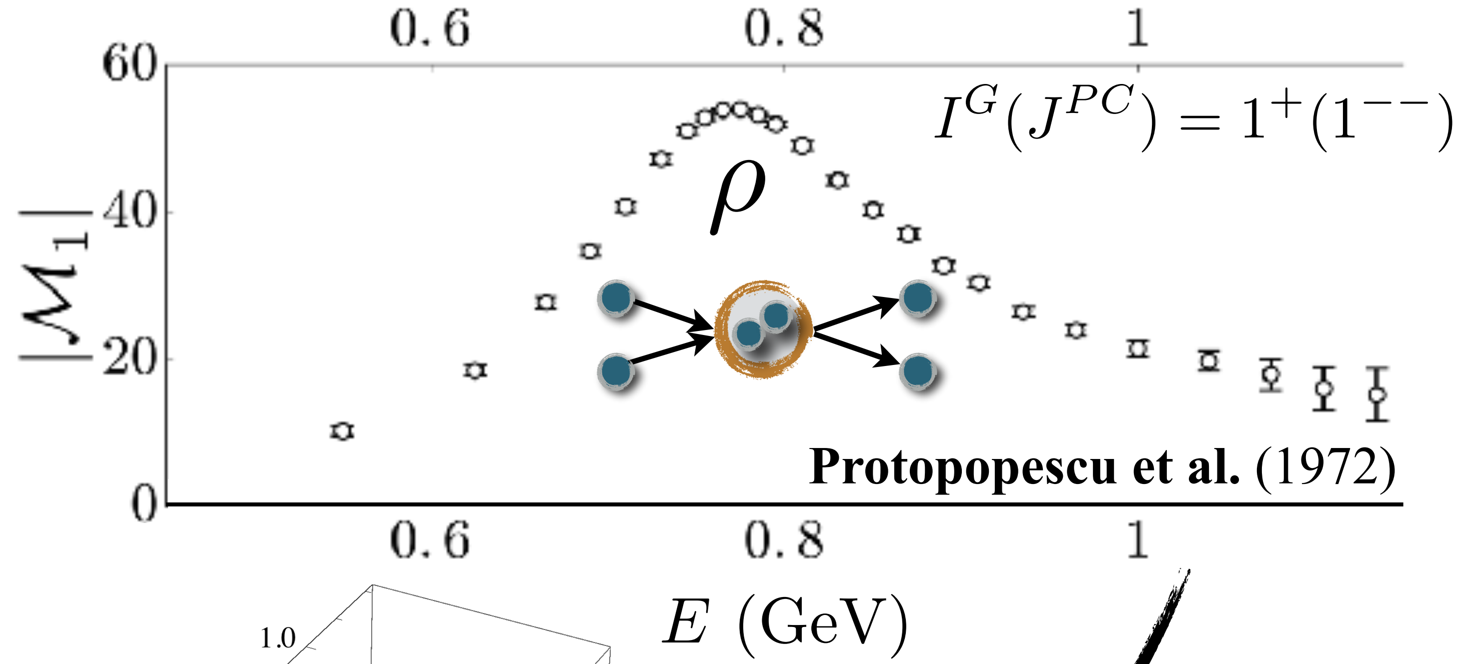

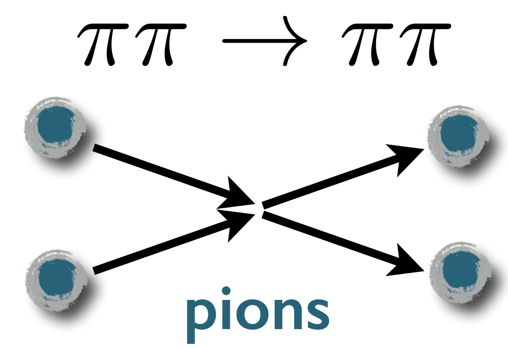

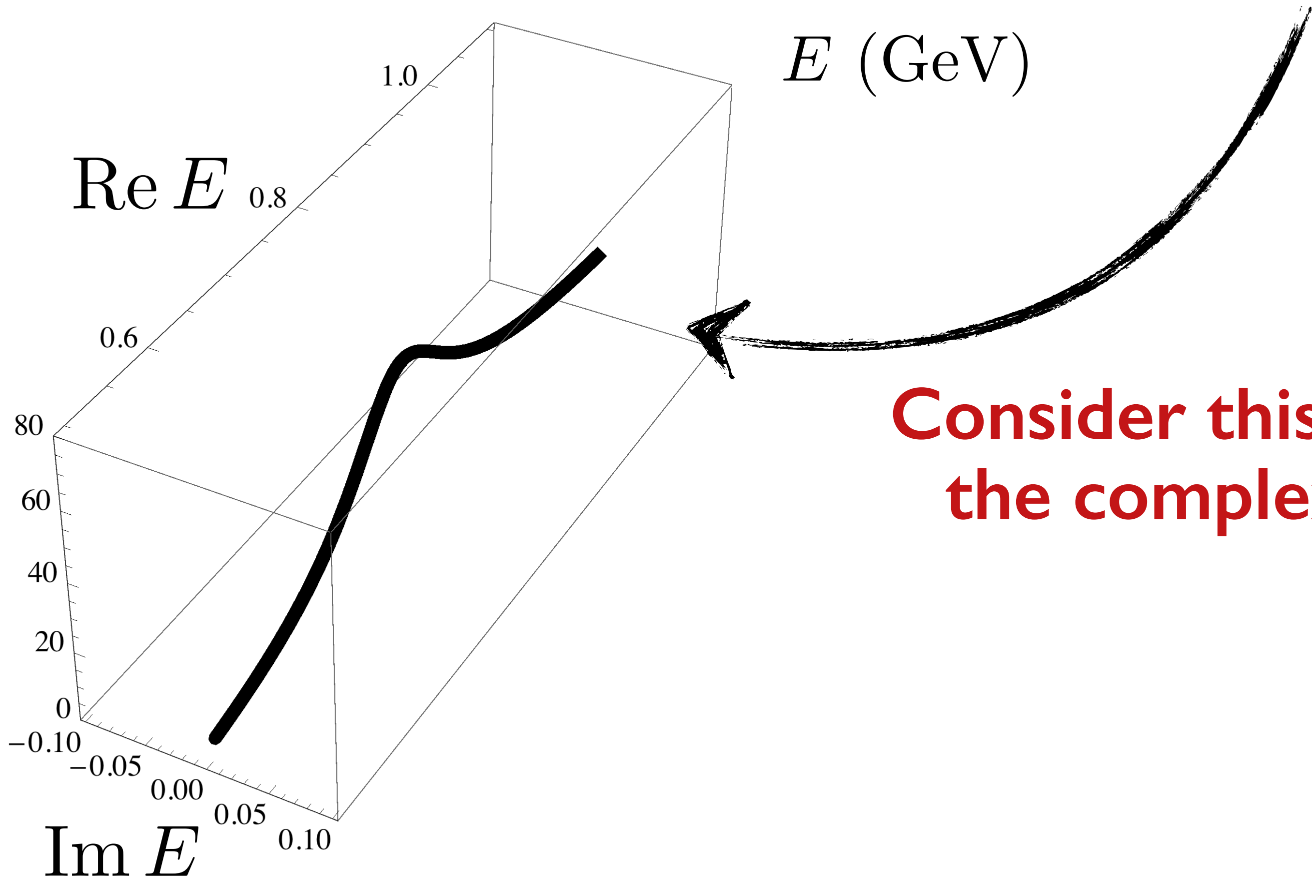

Consider this curve in the complex plane

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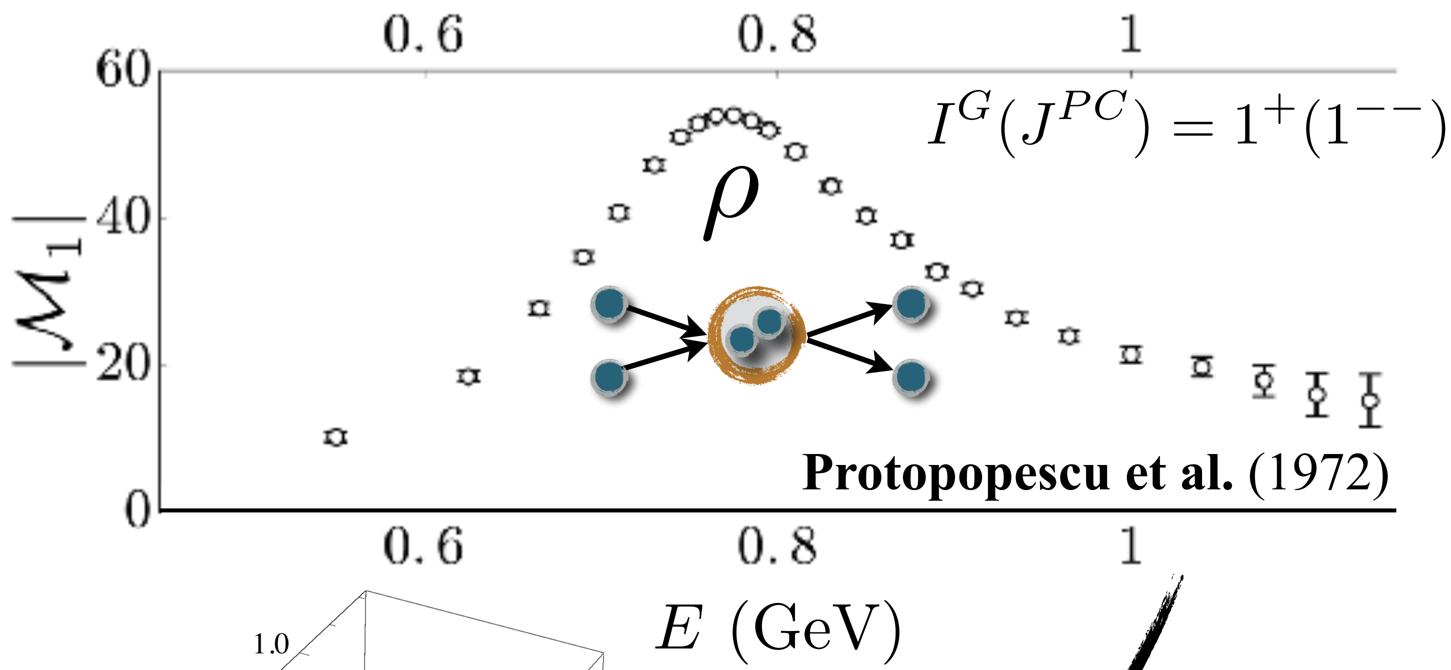

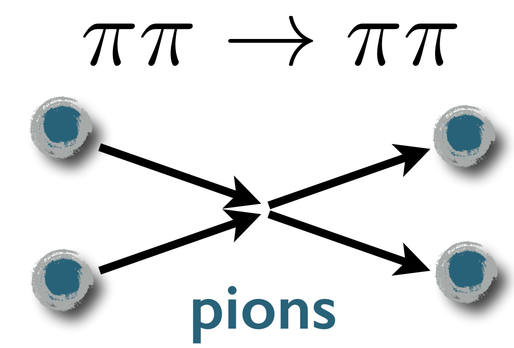

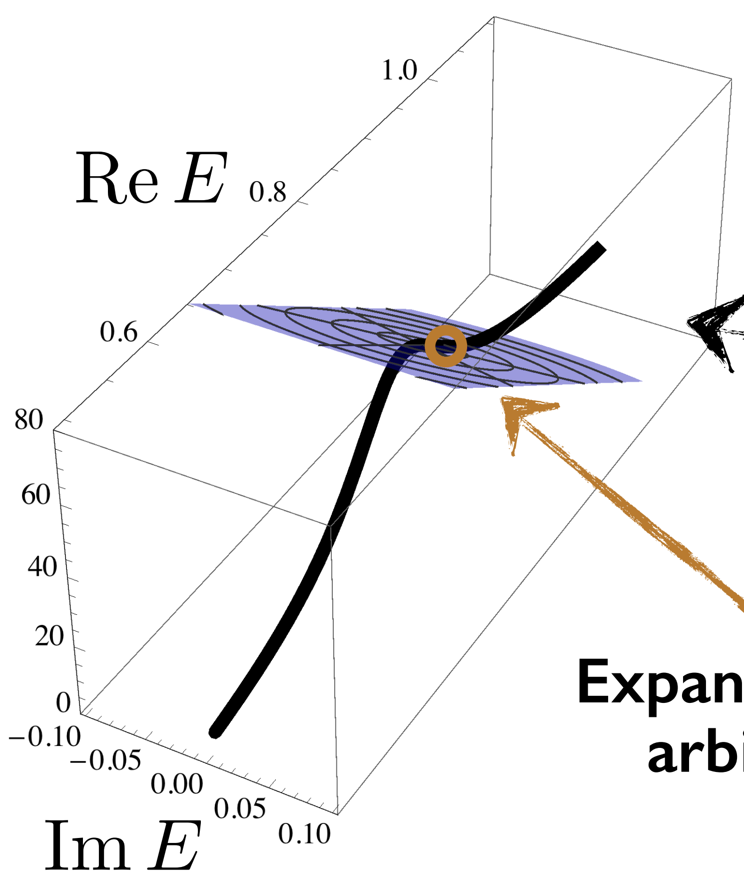

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Expand about some arbitrary point

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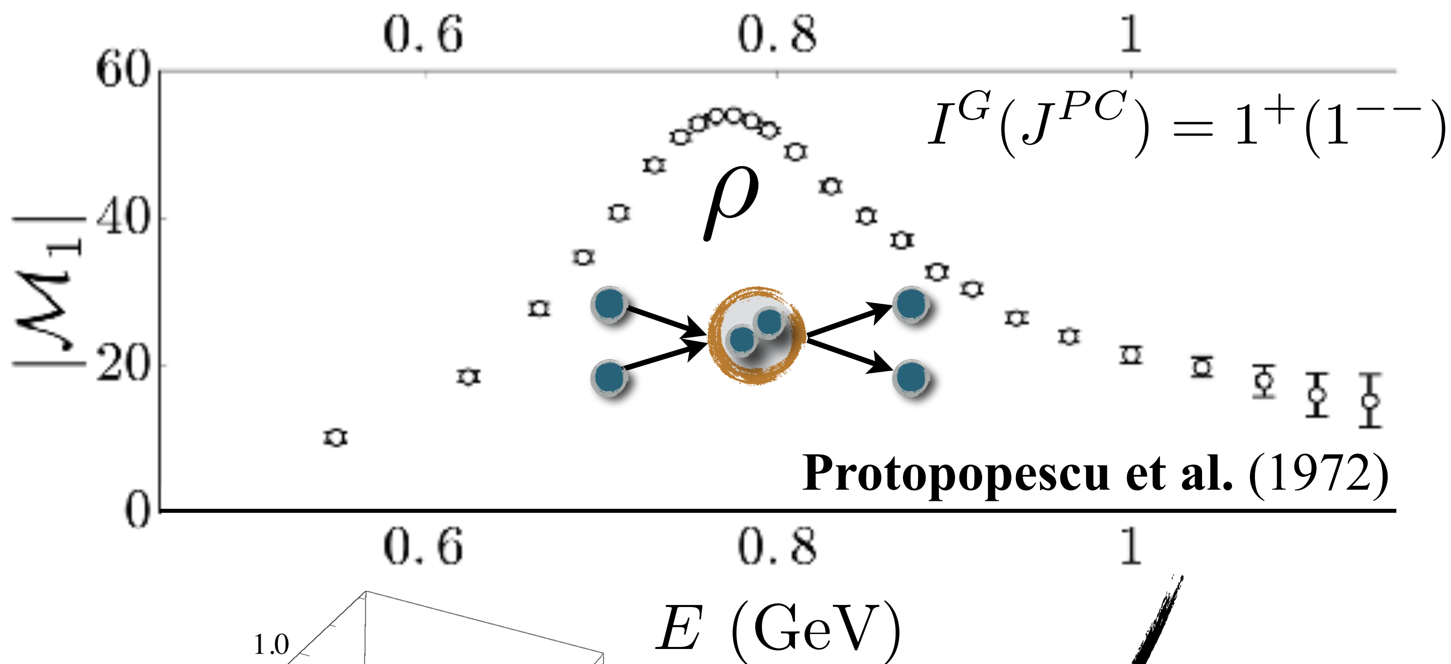

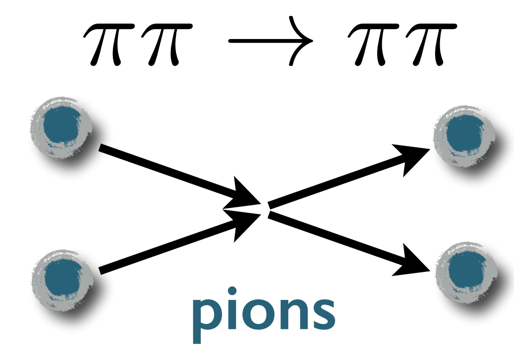

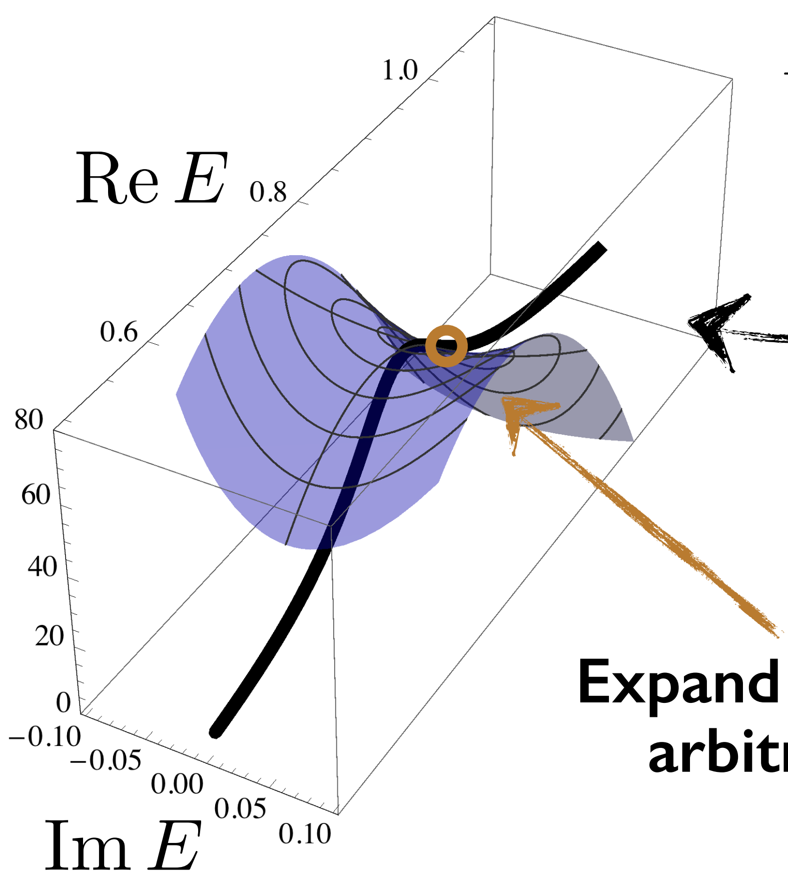

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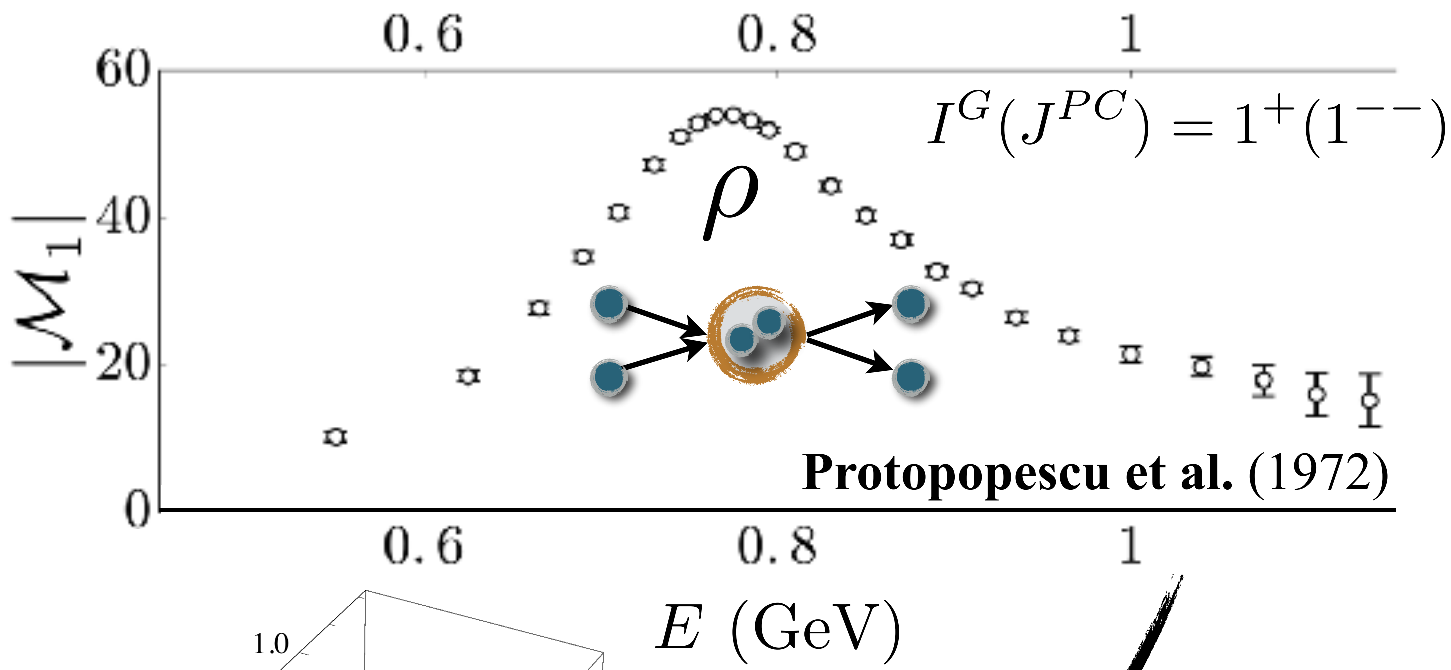

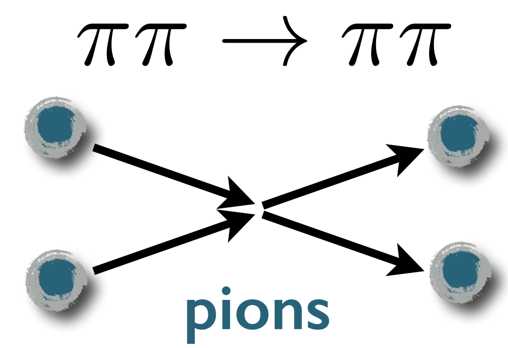

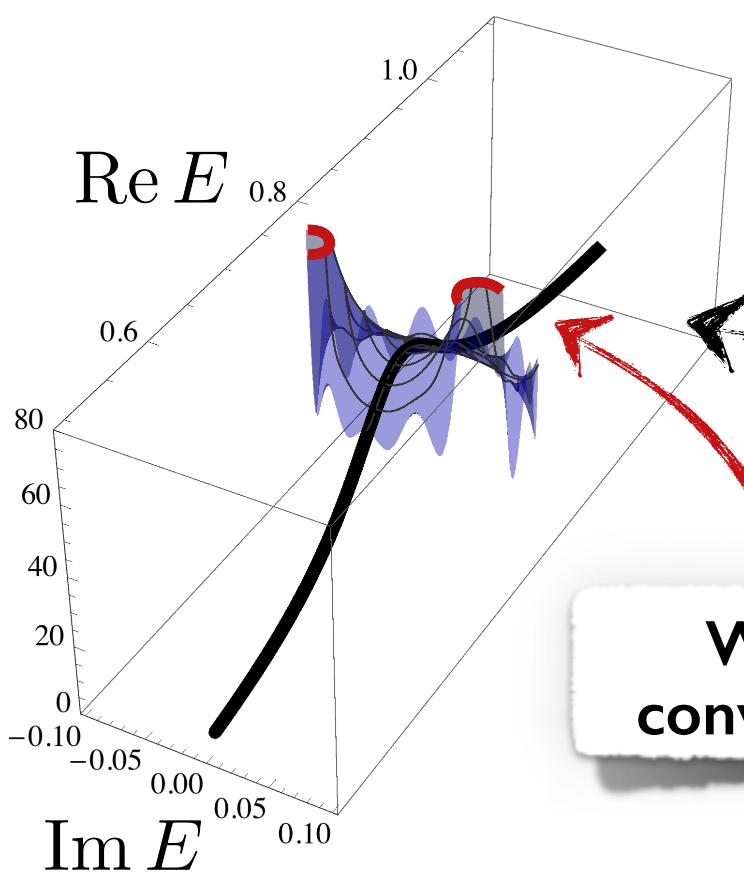

Consider this curve in the complex plane

We see a feature that limits the convergence of the Taylor expansion

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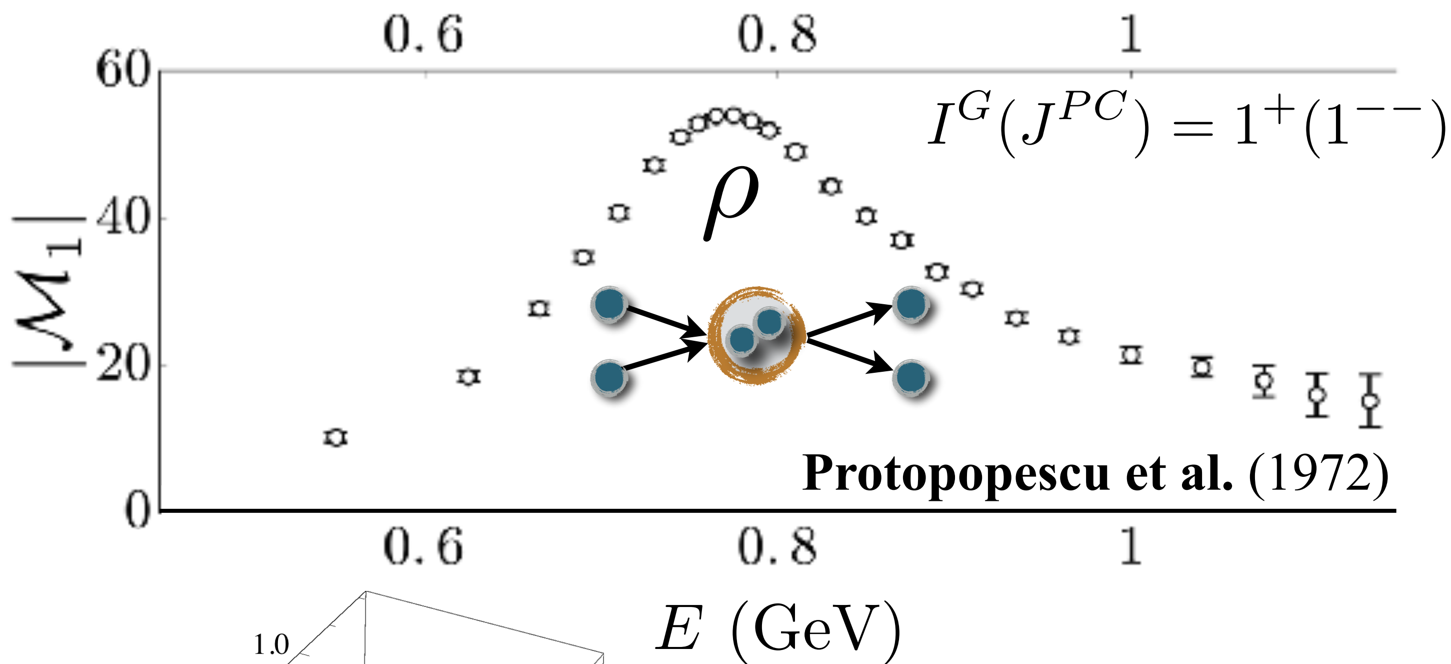
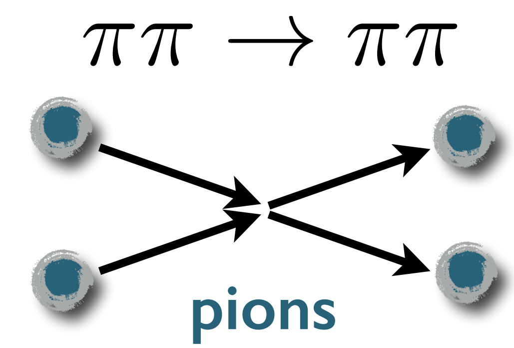
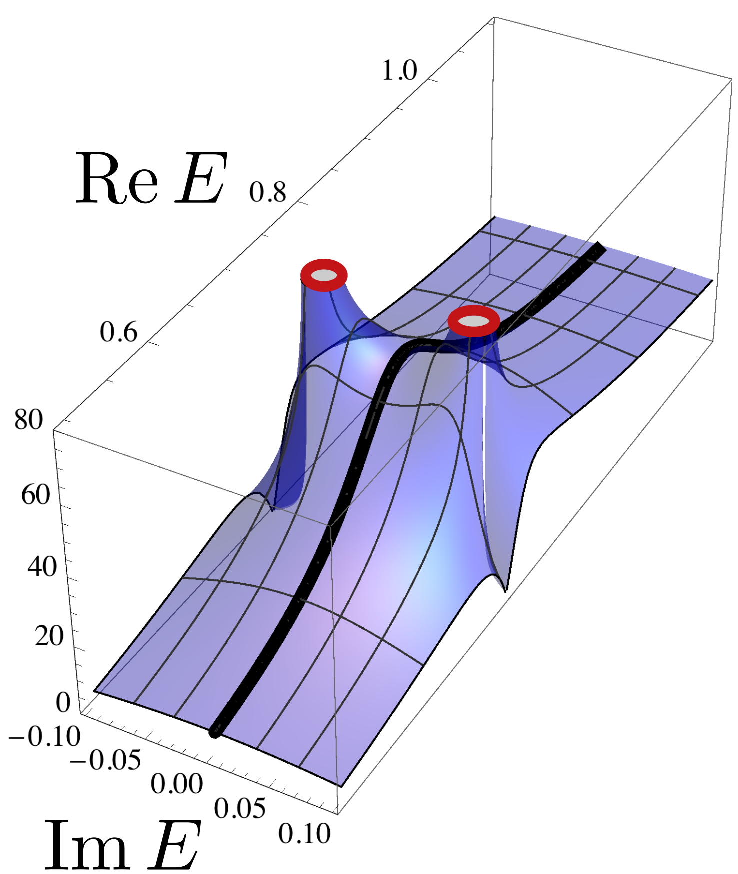
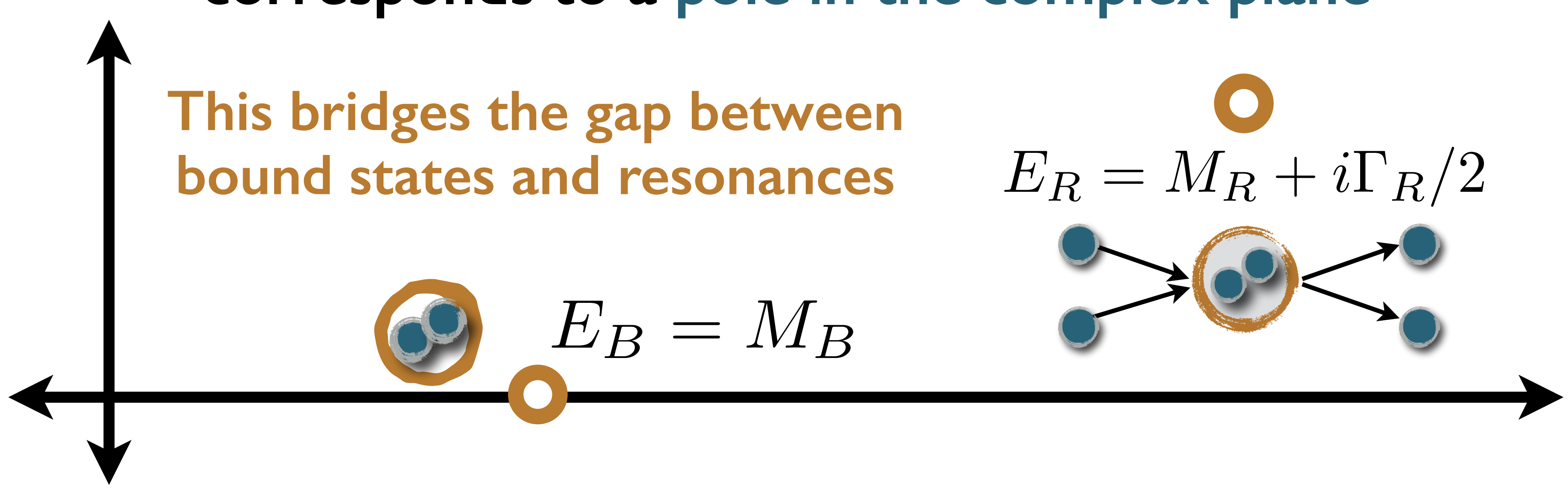






Analytic continuation reveals that the bump corresponds to a pole in the complex plane

This bridges the gap between bound states and resonances

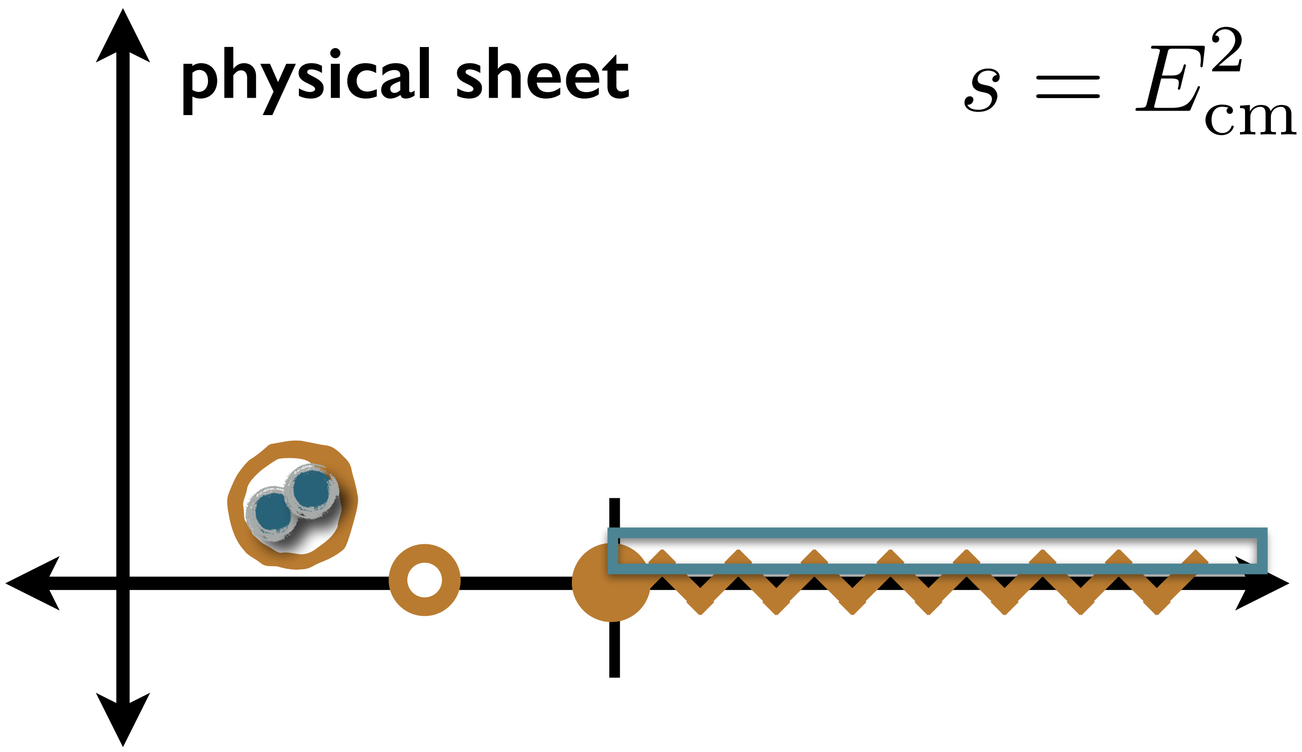


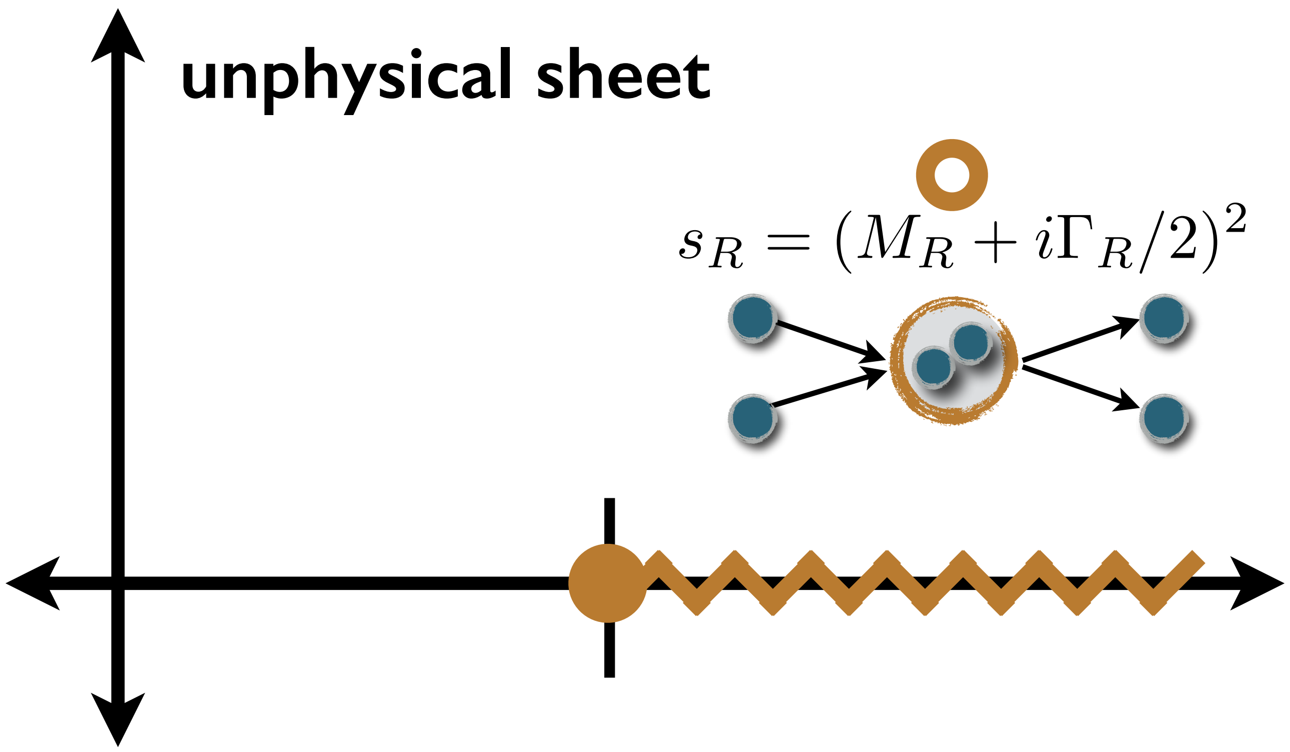
Multiple Riemann sheets

- It is more useful to analytically continue the scattering amplitude

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} + \rho(s)} \qquad \rho(s) \propto i\sqrt{s - (2M_\pi)^2}$$

- Each two-particle channel generates a squre-root branchcut and doubles the number of Riemann sheets

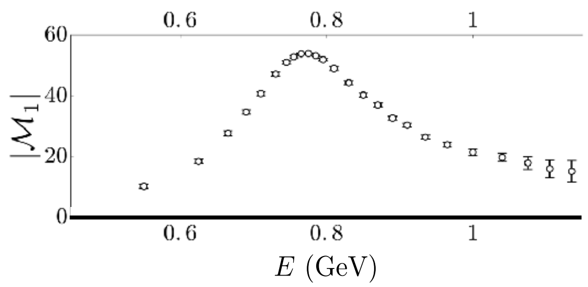


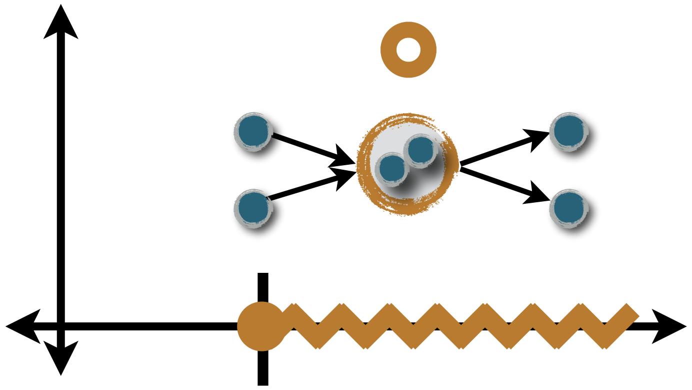


Towards a detailed, first-principles understanding of resonances

Energies and decay channels

- Locate complex poles in scattering amplitudes
- The residues at the poles measure couplings to multi-particle decay products





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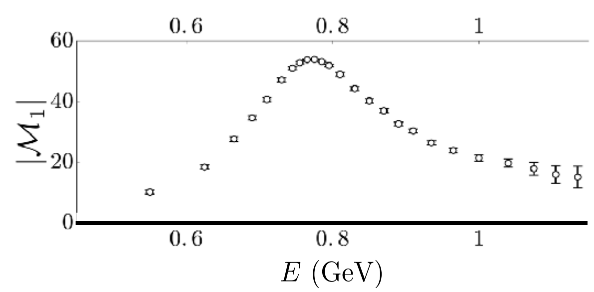
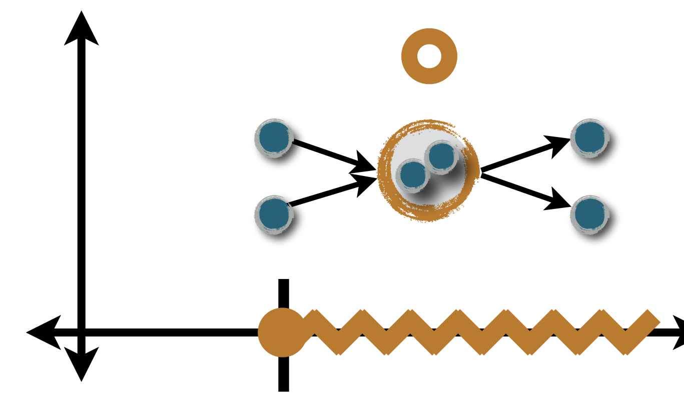
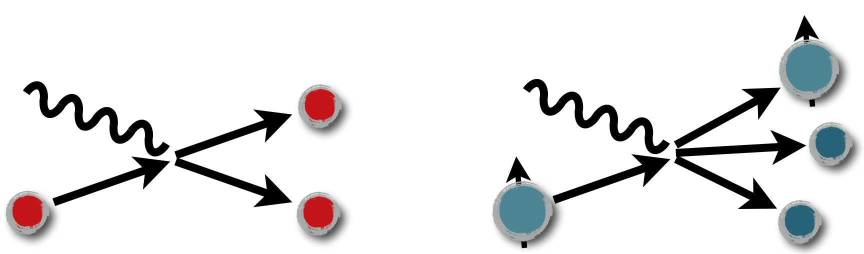




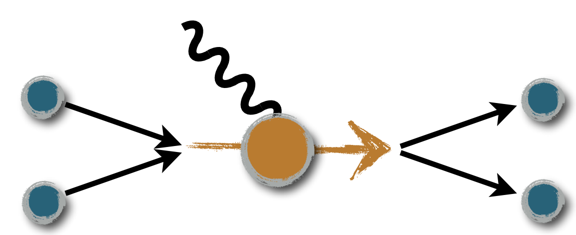
Transition amplitudes

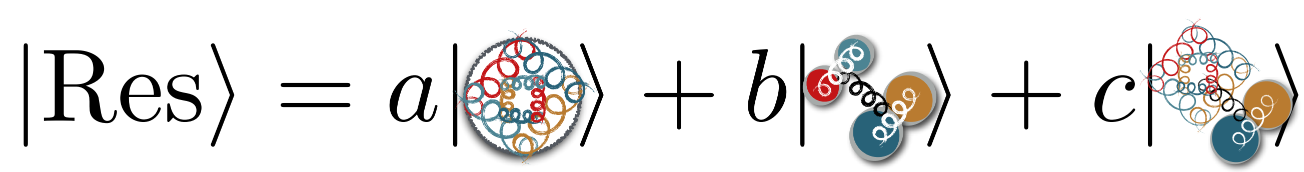
- ❑ Measure how photons and other currents mediate exotic resonance production



Resonant form factors

- ❑ Predict how currents couple to the resonance





Lattice QCD is a powerful tool for extracting QCD predictions

LQCD = evaluating a difficult integral numerically

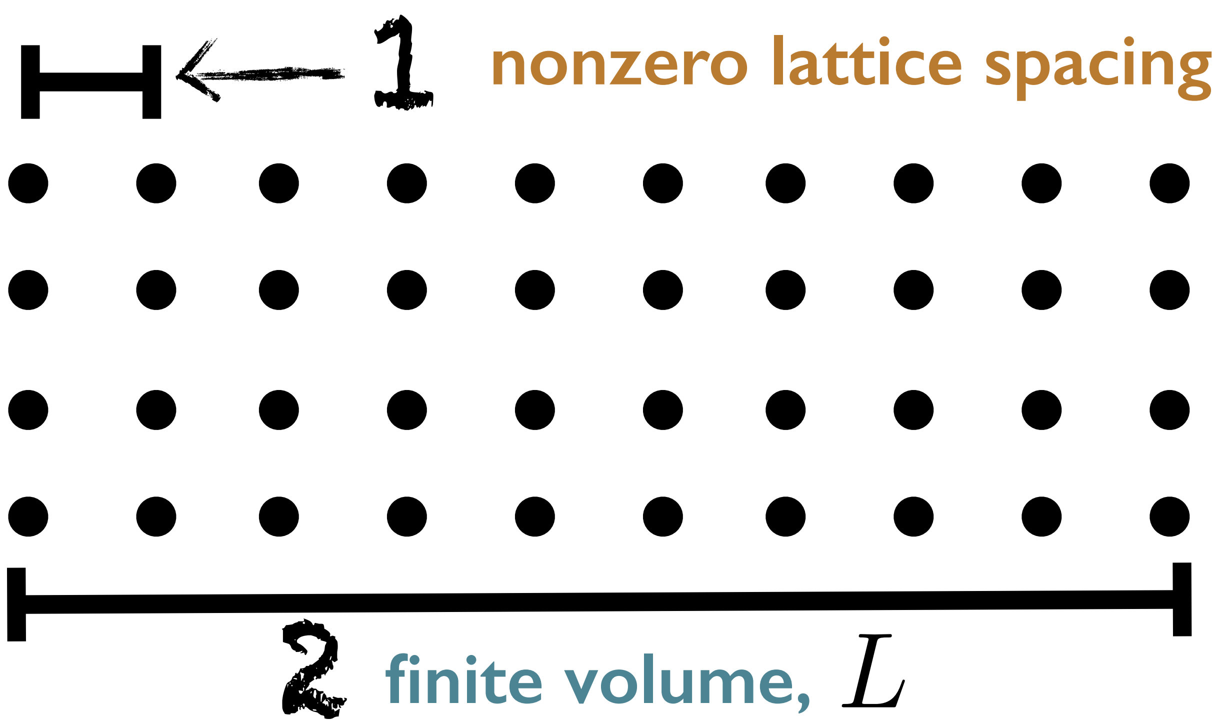
$$\text{observable} = \int \mathcal{D}\phi \; e^{iS} \left[\begin{array}{c} \text{interpolator} \\ \text{for observable} \end{array} \right]$$

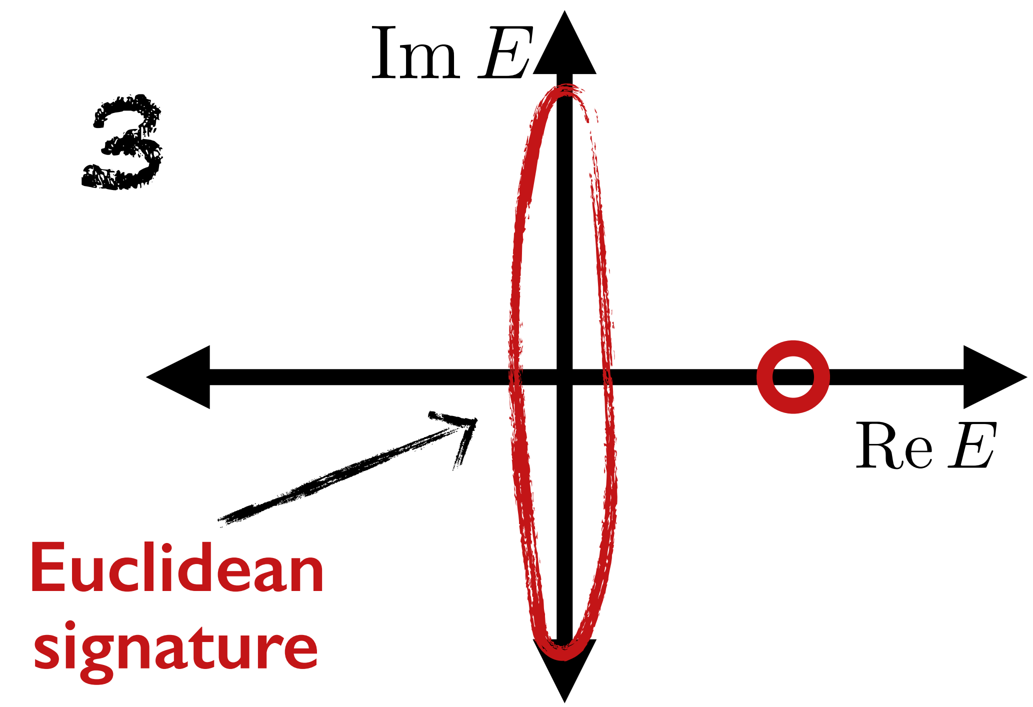
Lattice QCD is a powerful tool for extracting QCD predictions

LQCD = evaluating a difficult integral numerically

$$\left(\begin{array}{c}\textbf{observable?}\\ \text{discretized, finite volume,}\\ \text{Euclidean, heavy pions}\end{array}\right) = \int \prod_{i}^{N} d\phi_i \; e^{-S} \left[\begin{array}{c}\text{interpolator}\\ \text{for observable}\end{array}\right]$$

To do so we have to make three compromises

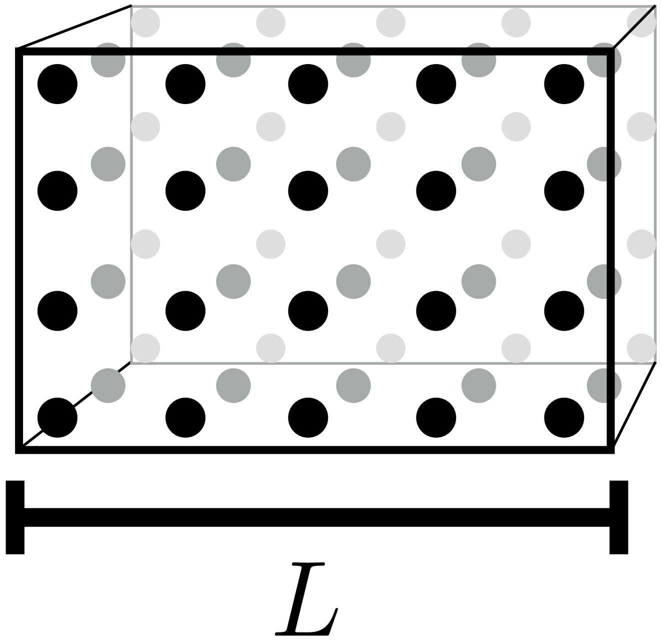




Also… Unphysical pion masses $M_{\pi,\mathrm{lattice}} > M_{\pi,\mathrm{our\ universe}}$

But calculations at the physical pion are becoming common

Difficulties for scattering



- The most important modification for scattering is the **finite volume**…
  - Discretizes the spectrum
  - Eliminates the branch cuts
  - Removes the second Riemann sheet
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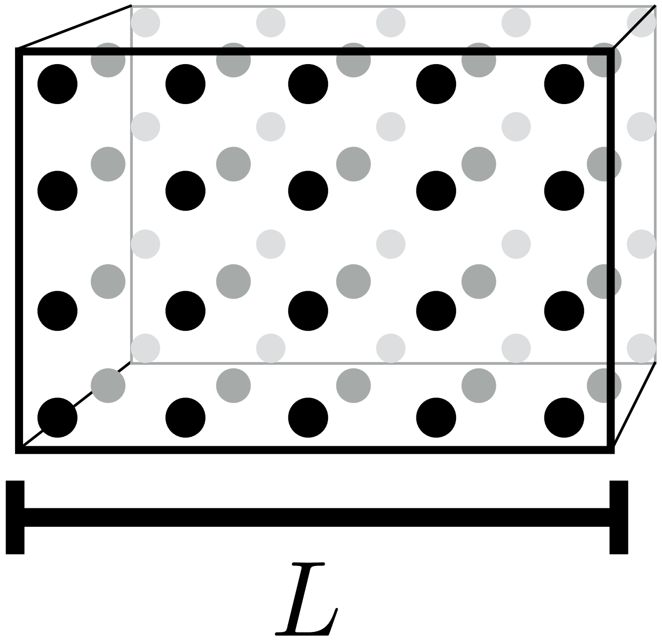

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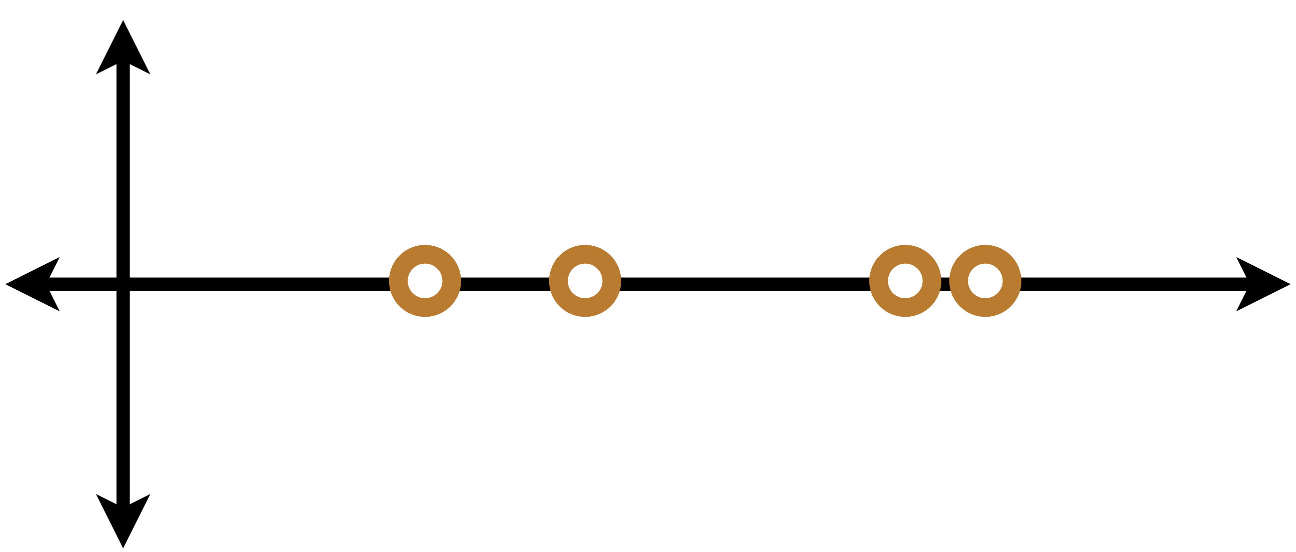
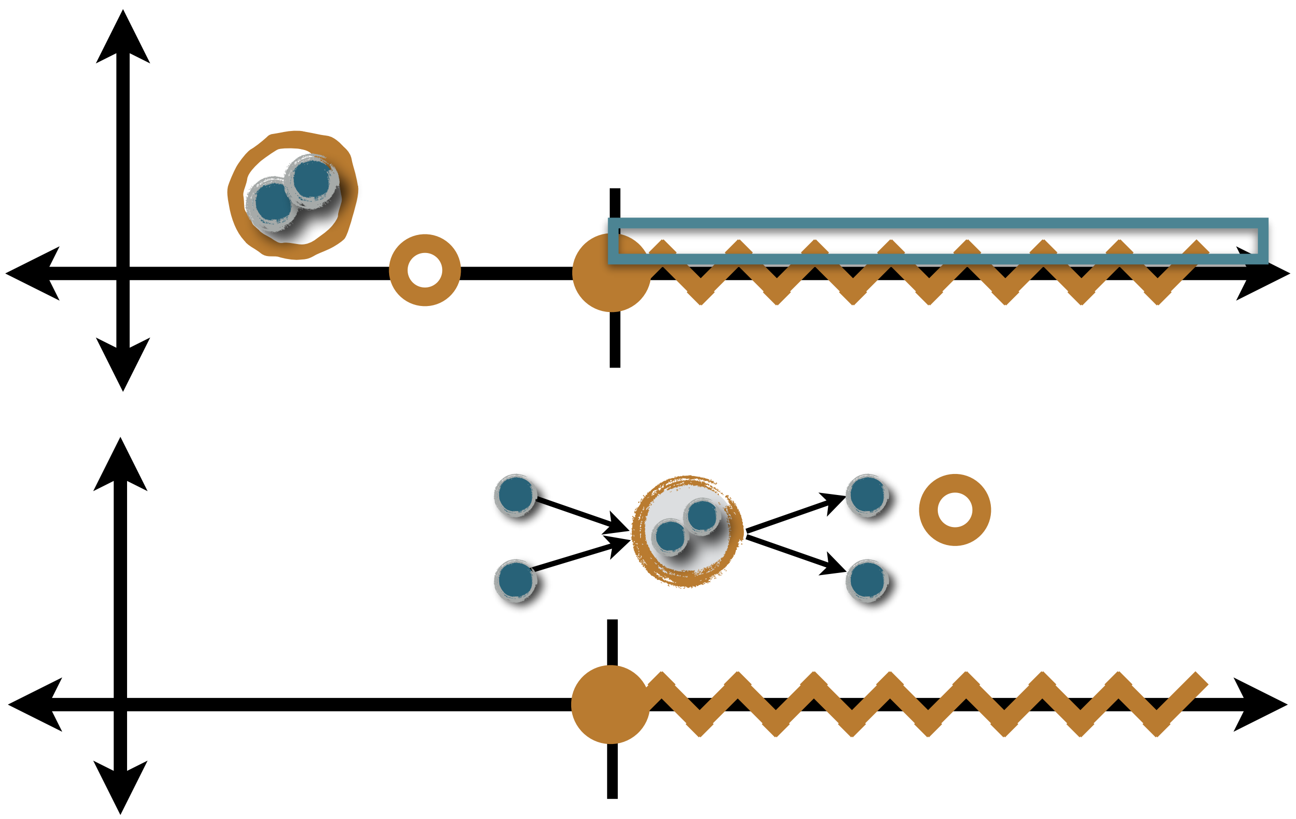
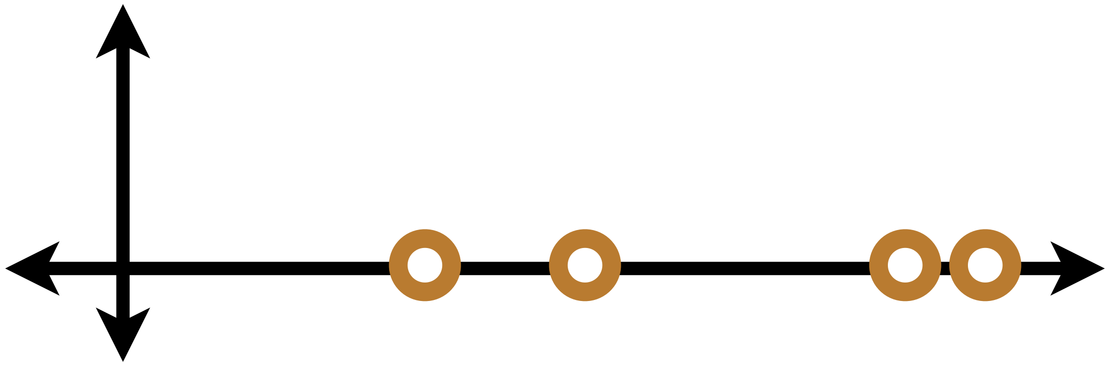
Finite-volume analytic structure



Infinite-volume analytic structure



Observables available in LQCD



'On the lattice' one can calculate finite-volume energies and matrix elements

$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

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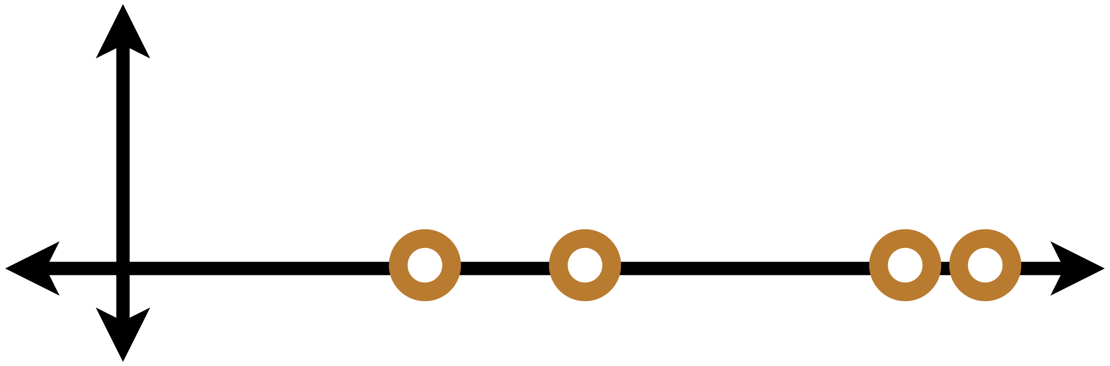

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A key technical breakthrough: Distillation **M. Peardon et al. 2009**

the extraction of the essential meaning or most important aspects of something

Construct smeared quark propagators via Laplacian eigenvectors

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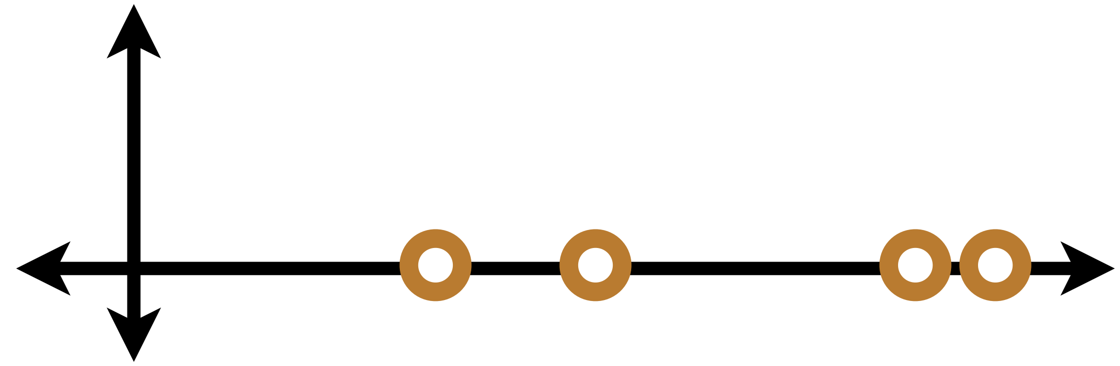

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the extraction of the essential meaning or most important aspects of something

Construct smeared quark propagators via Laplacian eigenvectors

Can determine optimized operators by 'diagonalizing' the correlator matrix (GEVP)

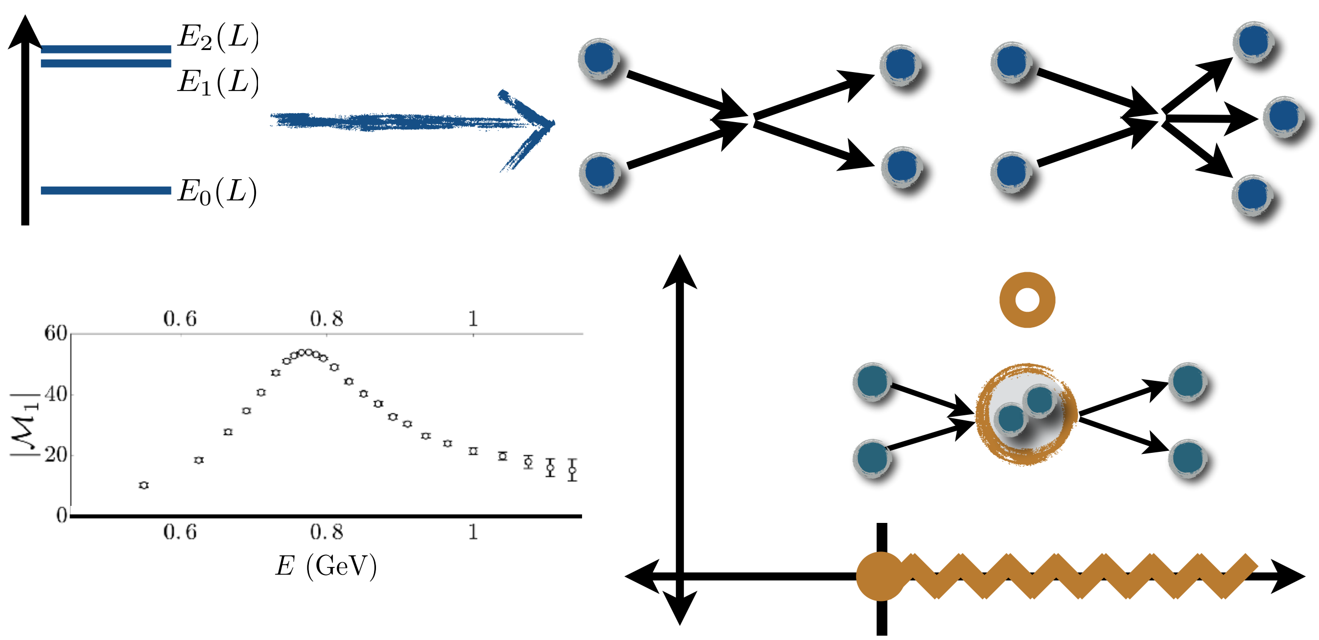
This gives a method to determine energies and matrix elements

$$\big\langle \Omega_m(\tau)\Omega_m^\dagger(0)\big\rangle \sim e^{-E_m(L)\tau} + \cdots$$

$$\big\langle \Omega_{m'}(\tau)\ \mathcal{J}(0)\ \Omega_m^\dagger(-\tau)\big\rangle \sim e^{-E_{m'}\tau}\ e^{-E_m\tau}\ \langle E_{m'}|\mathcal{J}(0)|E_m\rangle + \cdots$$

Scattering in LQCD relies on relations between finite- and infinite-volume quantities

Energies and decay channels



Finite-volume energies related to scattering

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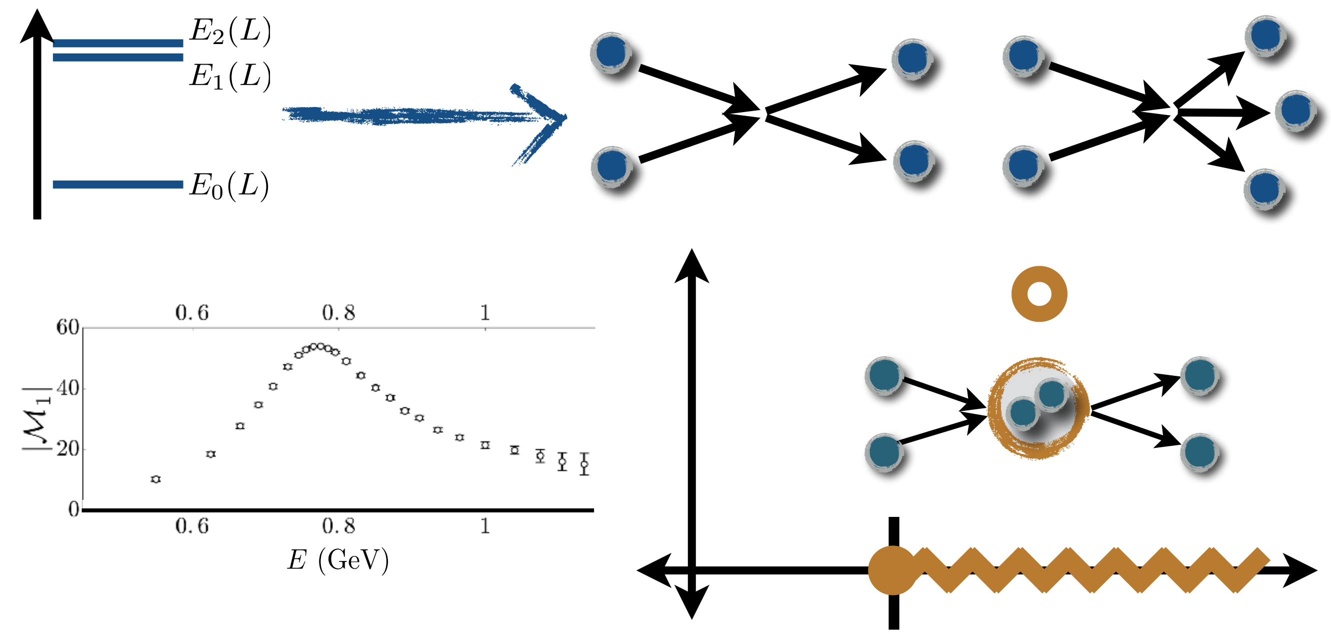
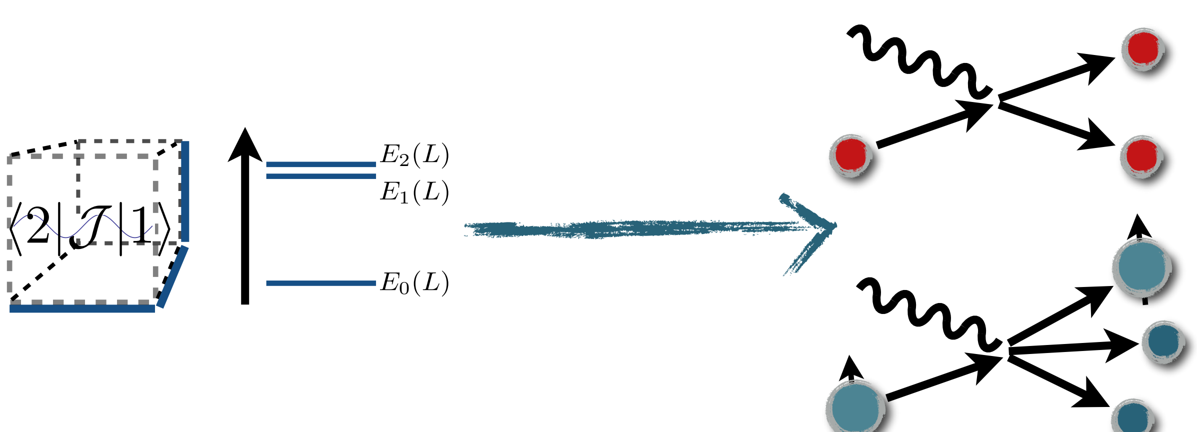


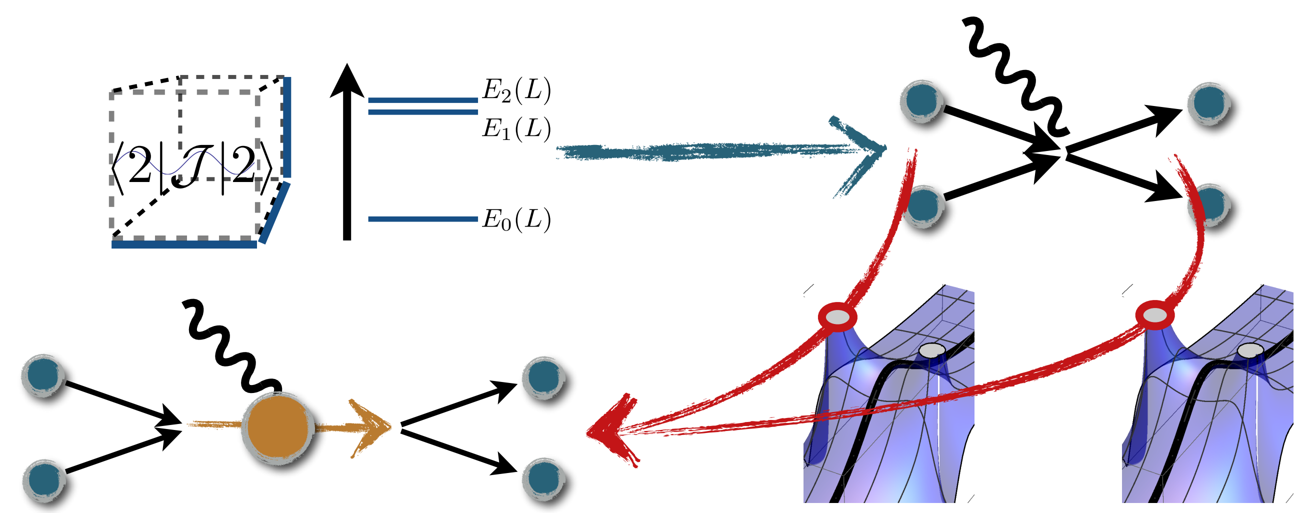
Finite-volume energies related to scattering

Transition amplitudes

Energies and finite-volume matrix elements are related to transitions



Resonant form factors



See talk by Raúl Briceño

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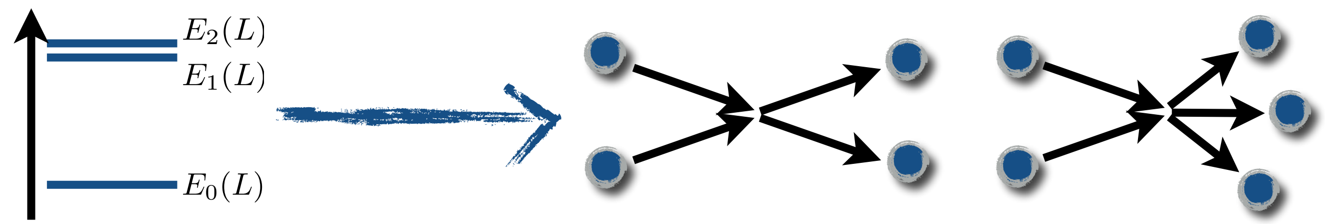

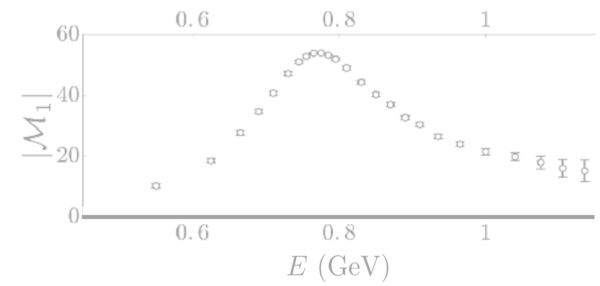

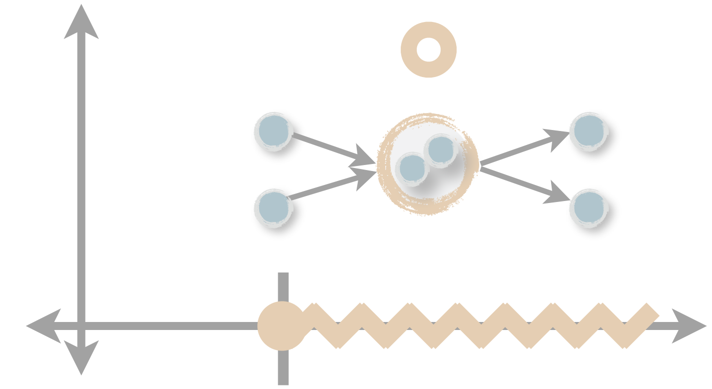

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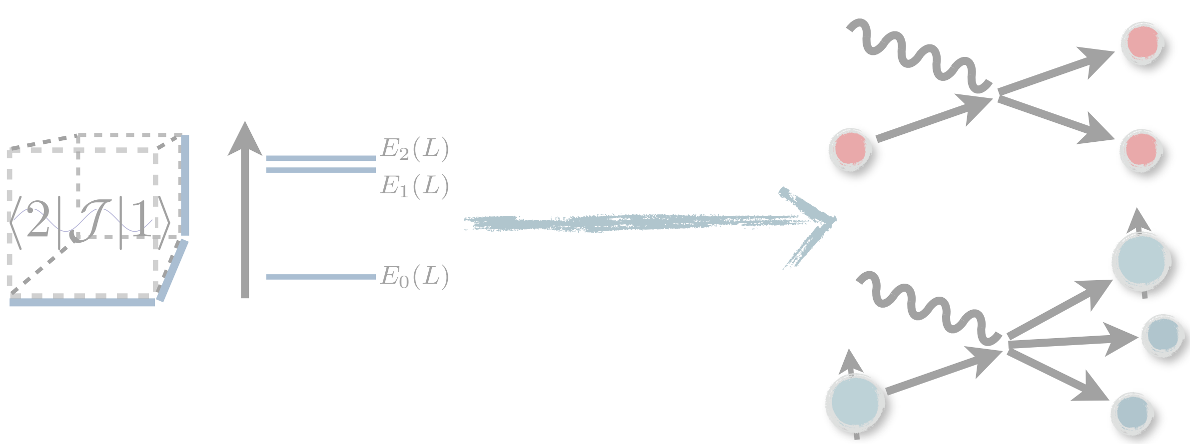

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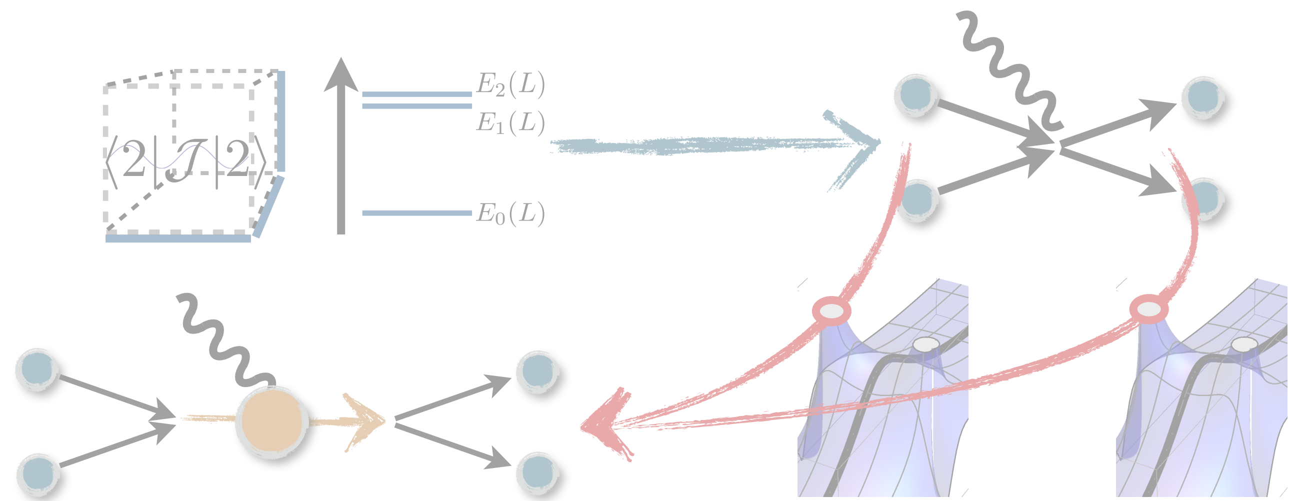

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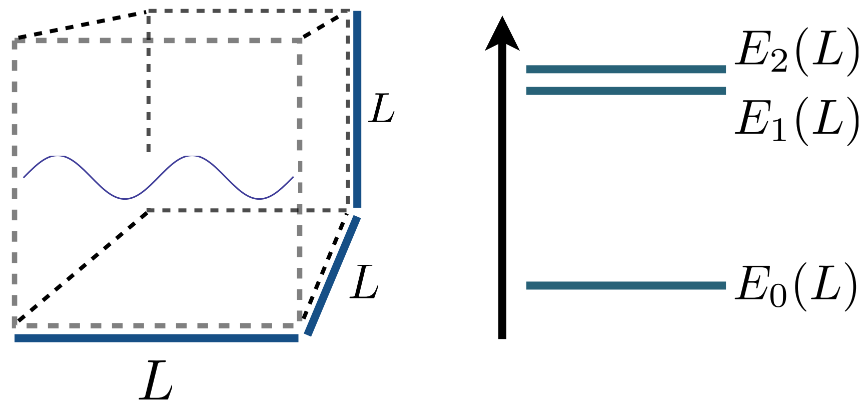
Basic idea

- Finite-volume set-up



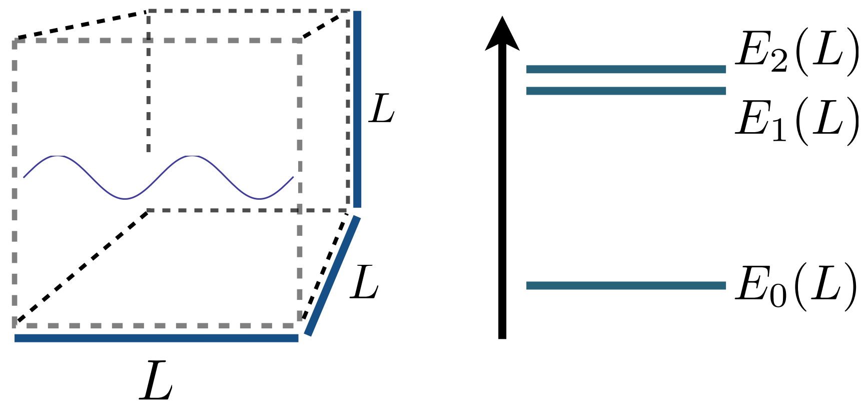
- cubic, spatial volume (extent *L*)
- periodic boundary conditions

$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \quad \vec{n} \in \mathbb{Z}^3$$

- *L* is large enough to neglect $e^{-M_\pi L}$

Basic idea

- Finite-volume set-up



- cubic, spatial volume (extent *L*)
- periodic boundary conditions

$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \quad \vec{n} \in \mathbb{Z}^3$$

- *L* is large enough to neglect $e^{-M_\pi L}$

- Scattering observables leave an imprint on finite-volume quantities

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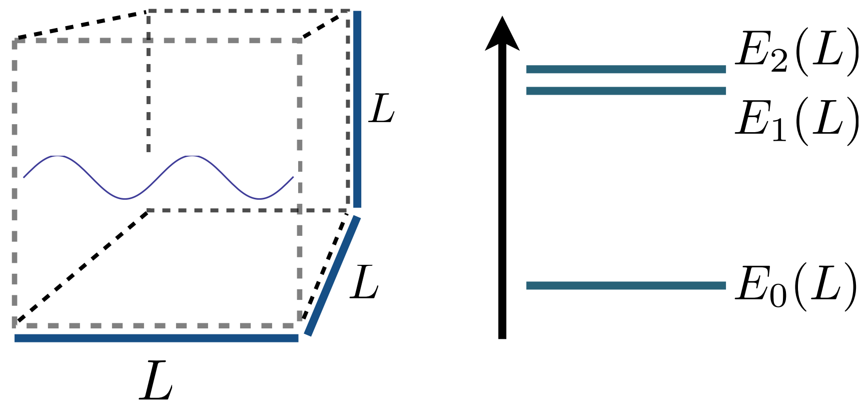

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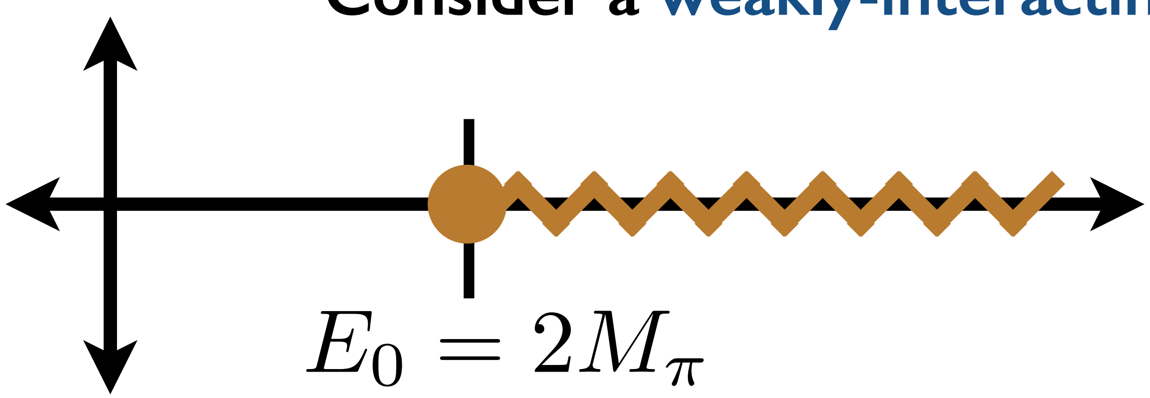
Consider a weakly-interacting, two-body system with no bound states



$$E_0 = 2M_\pi$$

Infinite-volume ground state

$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

Information is in the scattering amplitude

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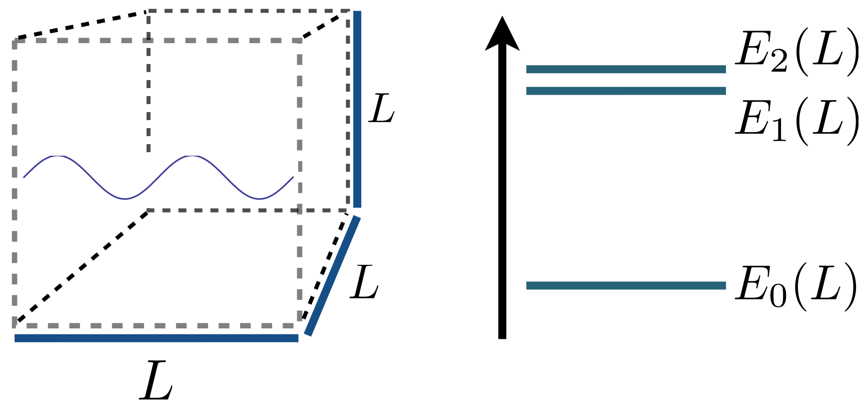

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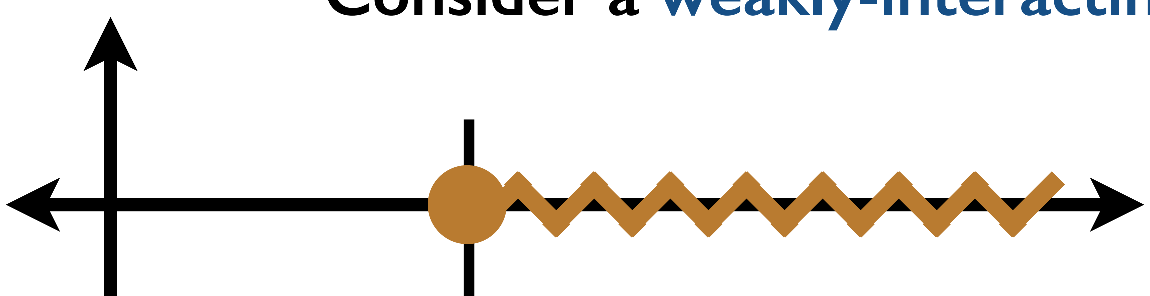

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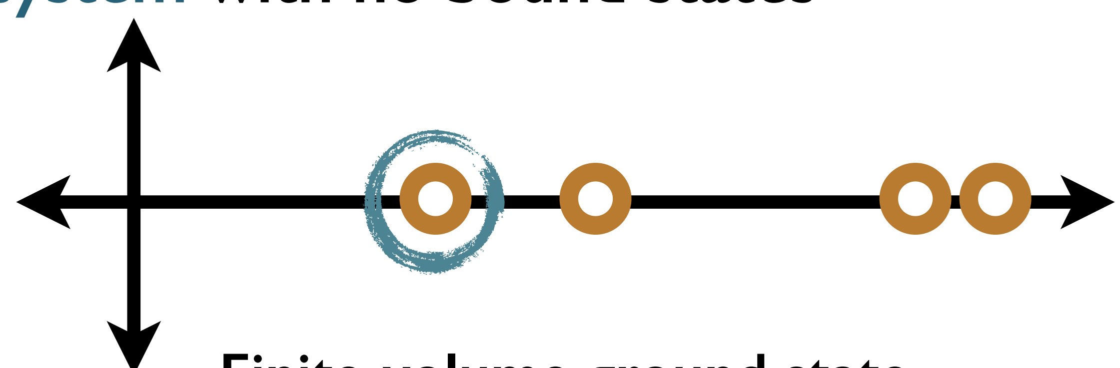

Finite-volume ground state

$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

Huang, Yang (1958)

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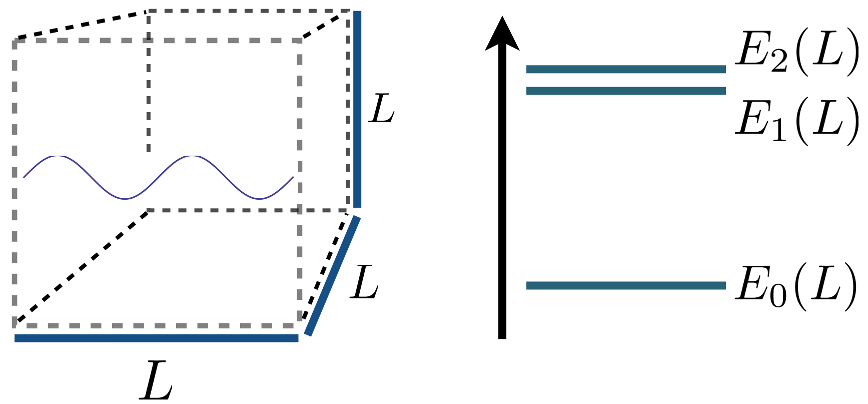

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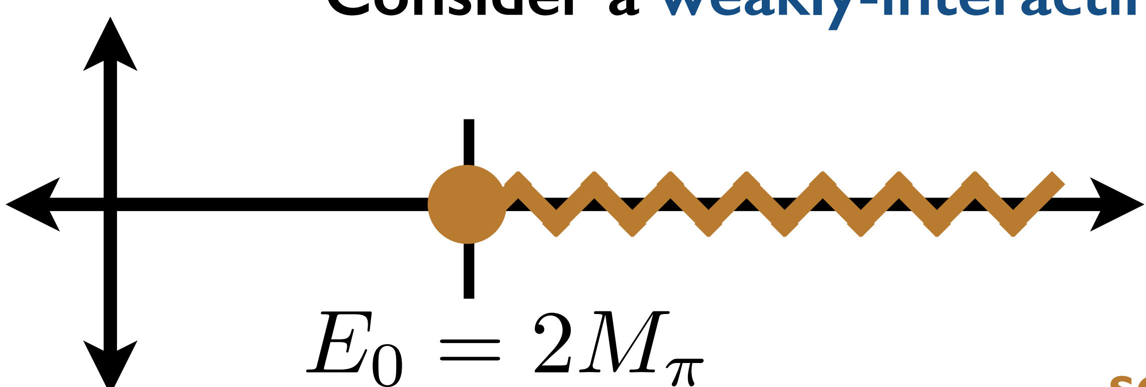

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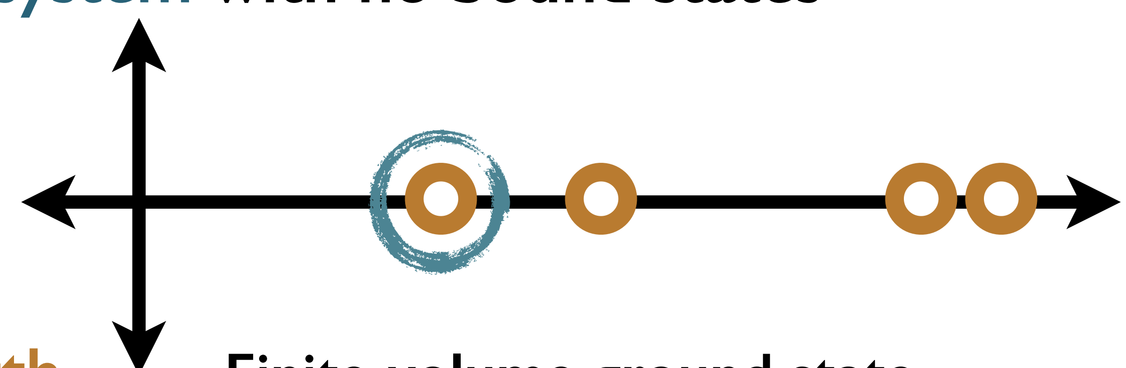

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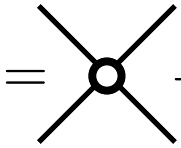
Hint of the derivation

- In the infinite-volume world…

$\mathcal{M}_{\ell=0}(E_{\rm cm}) =$  $+ \cdots = -\lambda + \mathcal{O}(\lambda^2)$

Lines represent low-energy degrees of freedom (e.g. pions in QCD)

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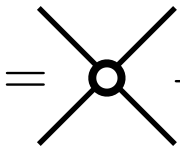

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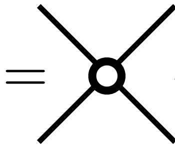
$\mathcal{M}_{\ell=0}(E_{\rm cm}) =$  $+ \cdots = -\lambda + \mathcal{O}(\lambda^2) \longrightarrow a = \dfrac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2)$

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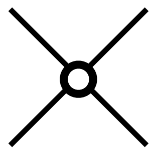
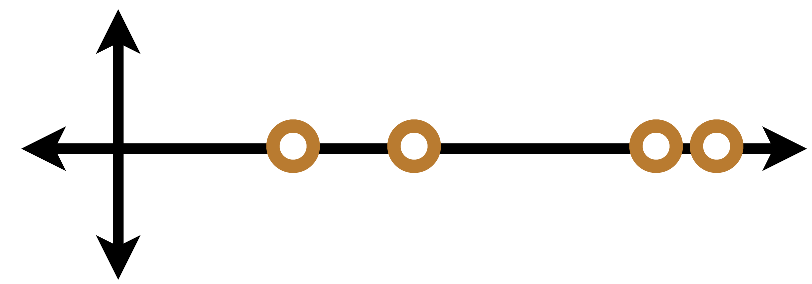
$\mathcal{M}_L(E_{\rm cm}) =$  $+ \cdots$

$= -\lambda + \cdots$

At leading order, the finite-volume amplitude has no poles



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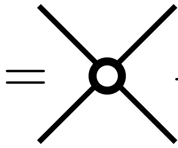
$$\mathcal{M}_{\ell=0}(E_{\rm cm}) =$$  $$+ \cdots = -\lambda + \mathcal{O}(\lambda^2) \longrightarrow a = \frac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2)$$

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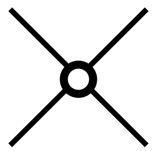
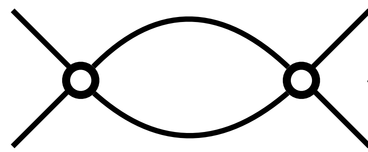
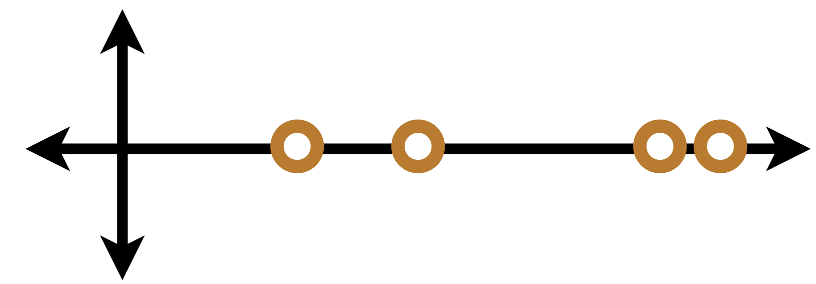
$$\mathcal{M}_L(E_{\rm cm}) =$$  $$+$$  $$+ \cdots$$

$$= -\lambda - \lambda \frac{1}{2}\frac{1}{L^3}\sum_{\mathbf{k}} \frac{1}{(2\omega_{\mathbf{k}})^2(E_{\rm cm} - 2\omega_{\mathbf{k}})}\lambda + \cdots$$



At next-to-leading order, we see poles of two non-interacting particles

$$\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + M_\pi^2} \quad \textbf{where} \quad \mathbf{k} = \frac{2\pi}{L}\mathbf{n}\,, \quad \mathbf{n} \in \mathbb{Z}^3$$

Hint of the derivation

- In the infinite-volume world…

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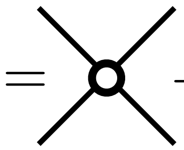
$$\mathcal{M}_{\ell=0}(E_{\rm cm}) =$$  $$+ \cdots = -\lambda + \mathcal{O}(\lambda^2) \longrightarrow a = \frac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2)$$

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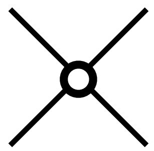
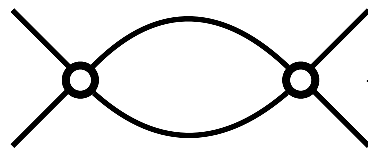
$$\mathcal{M}_L(E_{\rm cm}) =$$  $$+$$  $$+ \cdots$$

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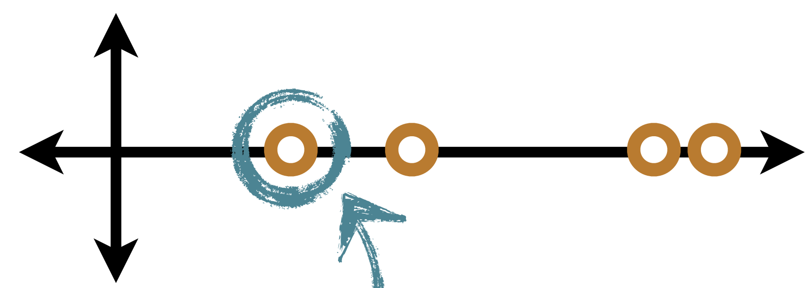


zoom-in on the lowest-lying pole

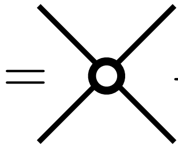
pole from two zero-momentum pions

The truncated series is failing because we are interested in $E_{\rm cm} - 2M_\pi = \mathcal{O}(\lambda)$

Hint of the derivation

- **In the infinite-volume world…**

scattering length

$\mathcal{M}_{\ell=0}(E_{\rm cm}) =$  $+ \cdots = -\lambda + \mathcal{O}(\lambda^2) \longrightarrow a = \dfrac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2)$

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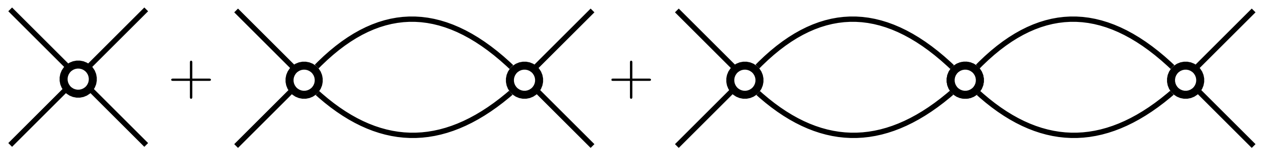
$\mathcal{M}_L(E_{\rm cm}) =$  $+ \cdots$

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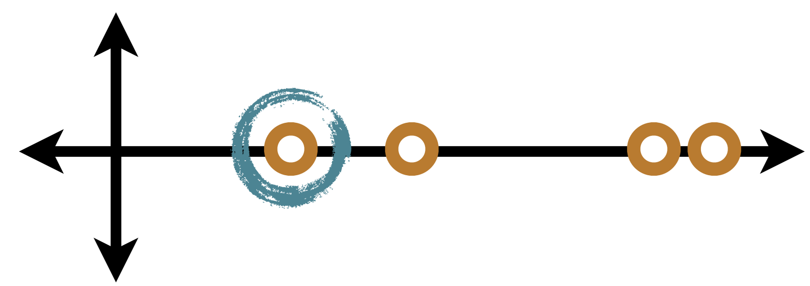

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$$= -\lambda \sum_{n=0}^{\infty}[f(E_{\rm cm}, L)\ \lambda]^n = \frac{1}{-1/\lambda + f(E_{\rm cm}, L)}$$

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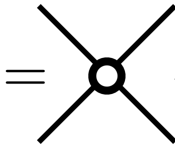
$$\mathcal{M}_{\ell=0}(E_{\rm cm}) =$$  $$+ \cdots = -\lambda + \mathcal{O}(\lambda^2) \longrightarrow a = \frac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2)$$

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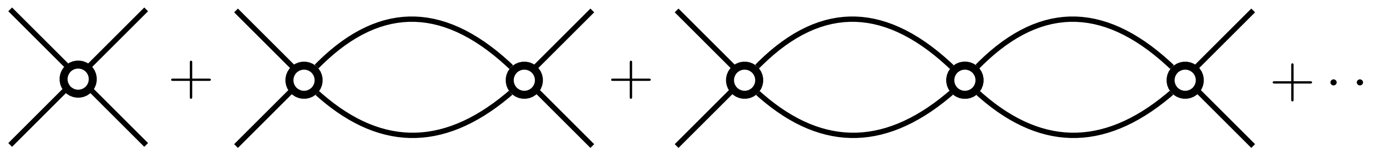
$$\mathcal{M}_L(E_{\rm cm}) =$$ 

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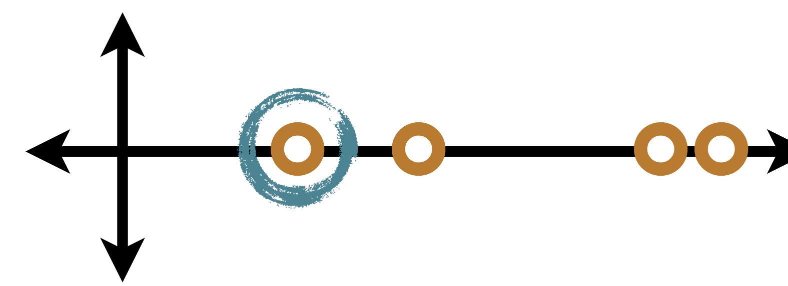

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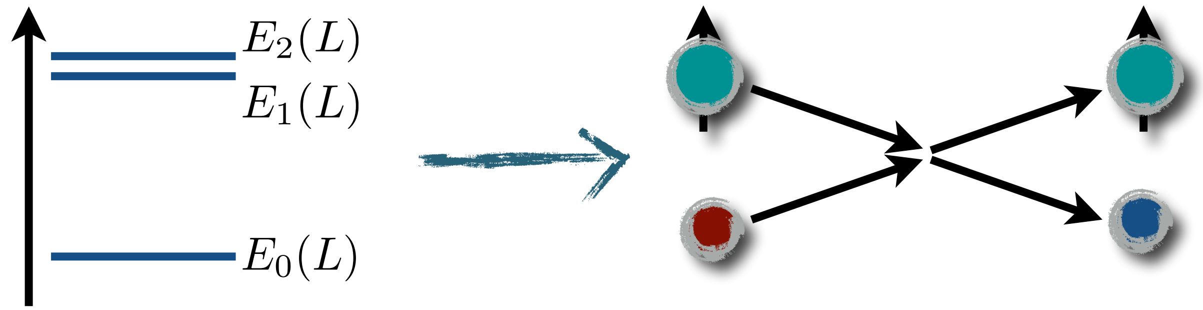
Summing this singular contribution to all orders gives the final expression…

$$-1/\lambda + f(E_{\rm cm}, L) = 0 \implies E_{\rm cm} = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

- This result can be generalized dramatically…

Two-to-two scattering

- Lüscher's formalism + extensions give a general mapping



- All results are contained in a generalized quantization condition

$$\det\left[\mathcal{M}_2^{-1}(E_n^*) + F(E_n, \vec{P}, L)\right] = 0$$

scattering amplitude known geometric function

Matrices in angular momentum, spin and channel space

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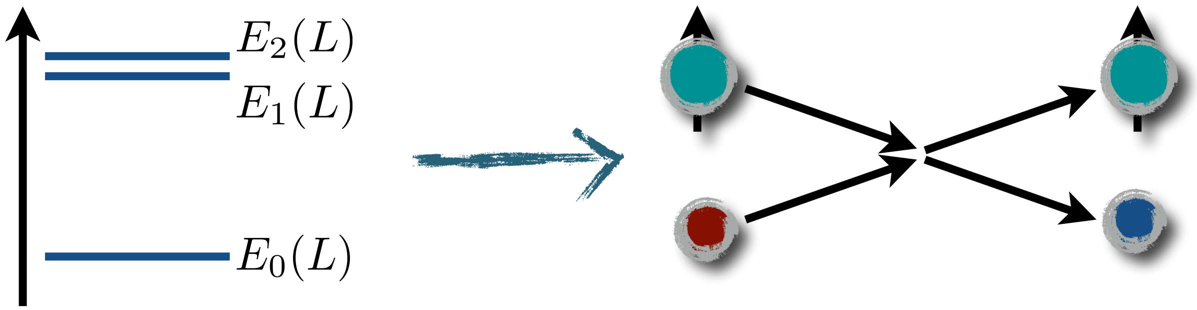

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Matrices in angular momentum, spin and channel space

- Varying $E, \vec{P}$ gives more constraints on functions of $E^{*2} = E^2 - \vec{P}^2$
- Derivation ignores (drops) suppressed volume effects ($e^{-M_\pi L}$)

Huang, Yang (1958) ○ **Lüscher** (1986, 1991) ○ **Rummukainen, Gottlieb** (1995)
Kim, Sachrajda, Sharpe (2005) ○ **Christ, Kim, Yamazaki** (2005) ○ **He, Feng, Liu** (2005)
Beane, Detmold, Savage (2007) ○ **Tan** (2008) ○ **Leskovec, Prelovsek** (2012) ○ **Bernard *et. al.*** (2012)
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- Simplest case is a single channel
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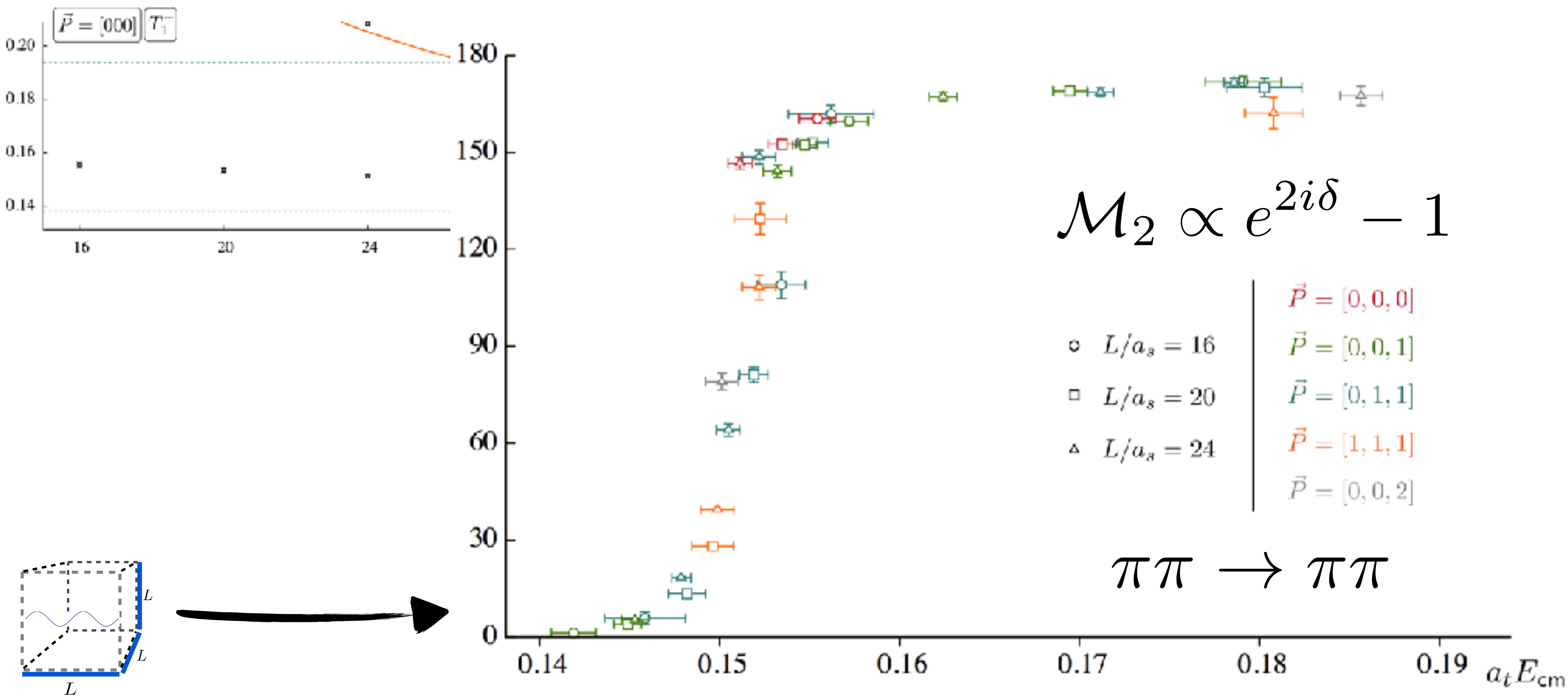

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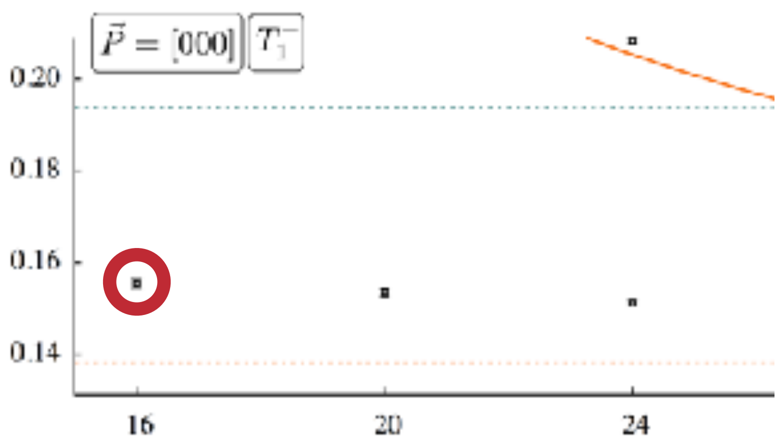

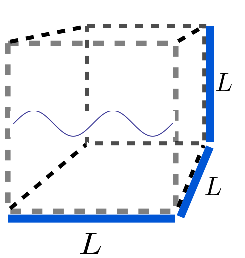

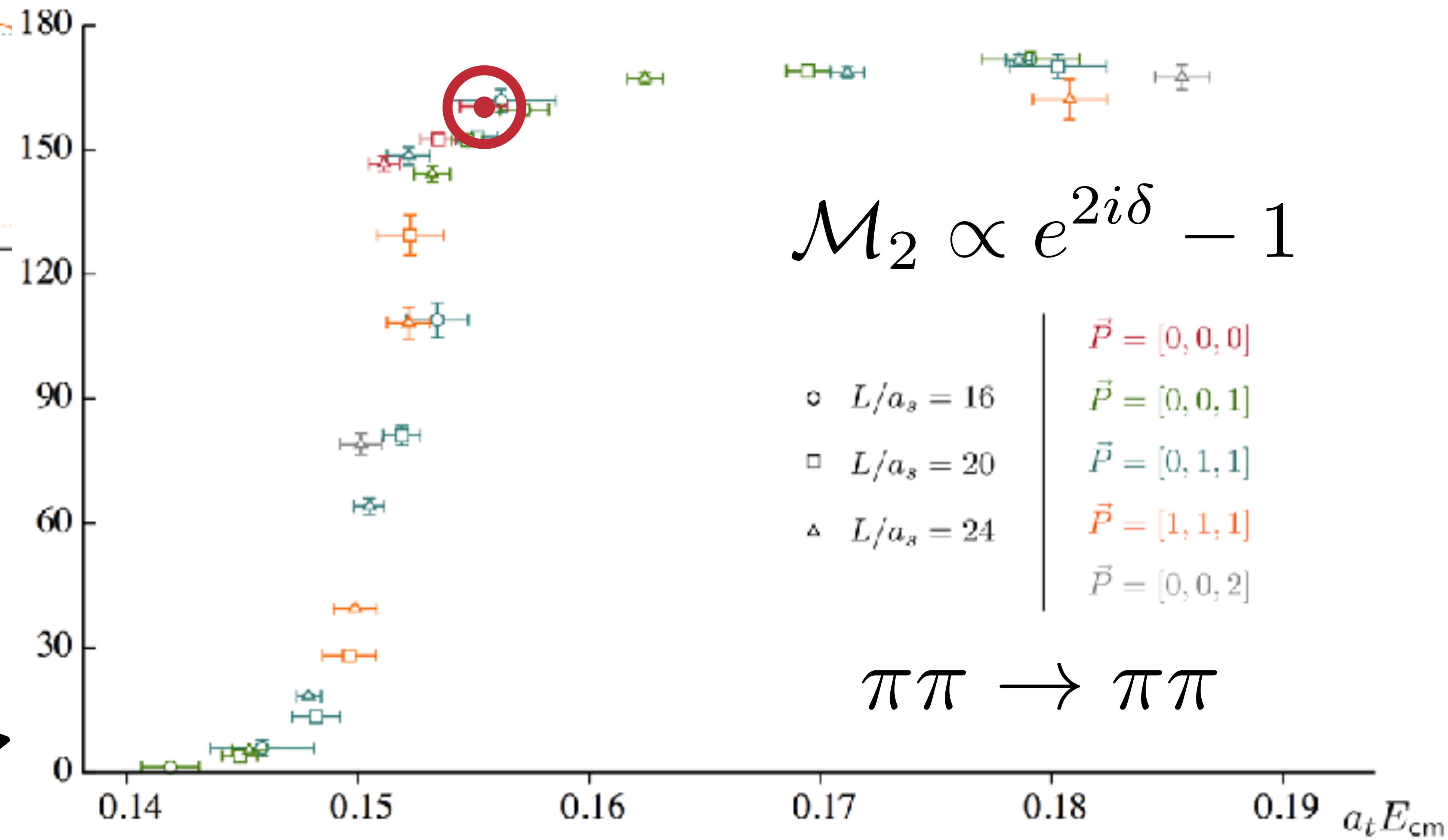

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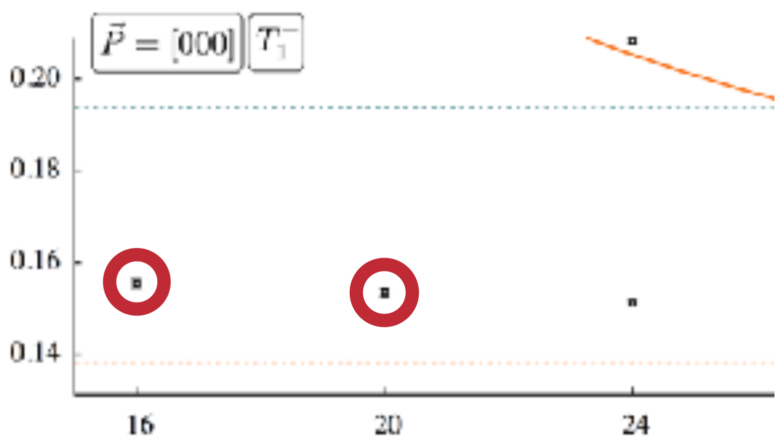

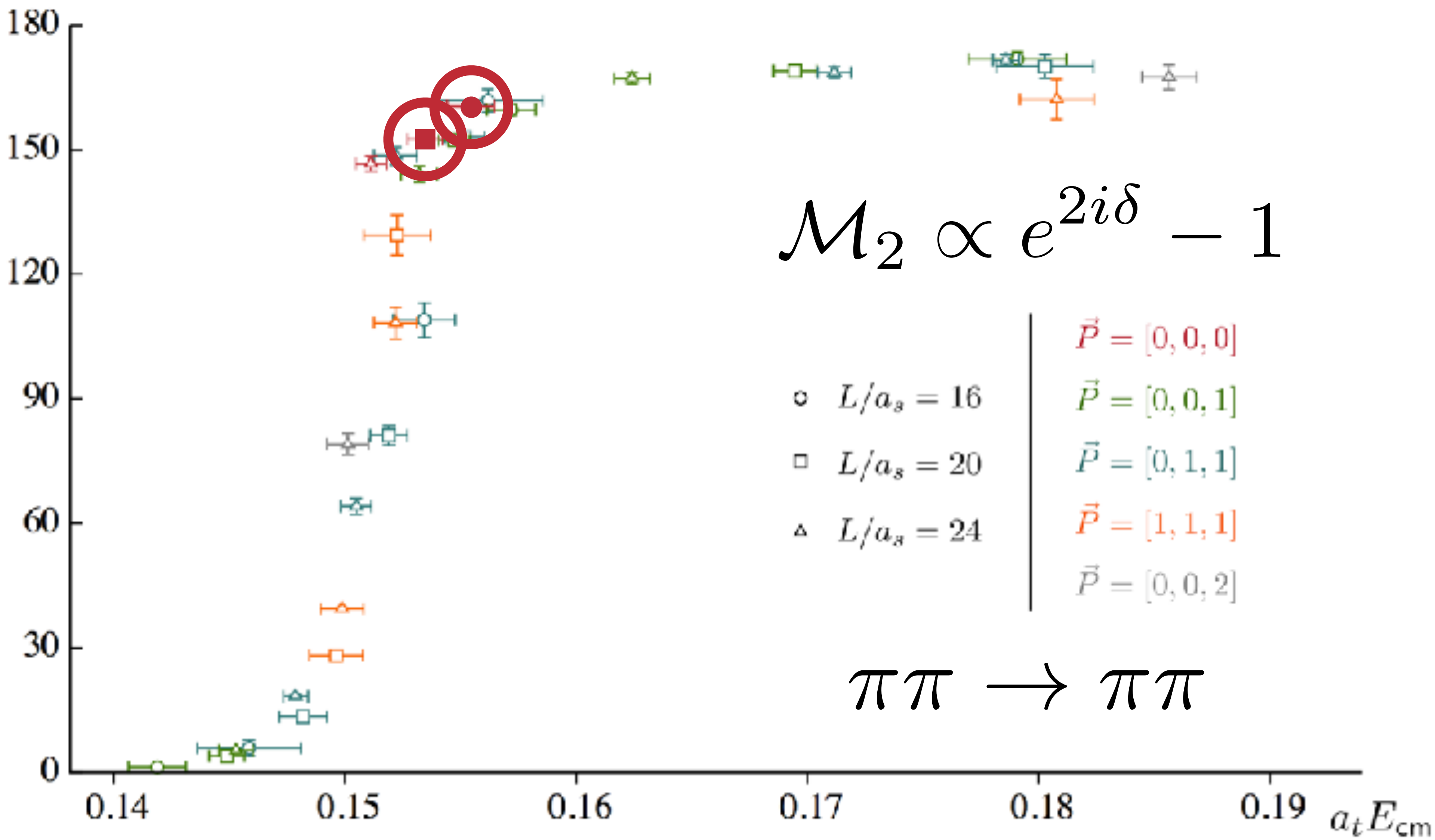

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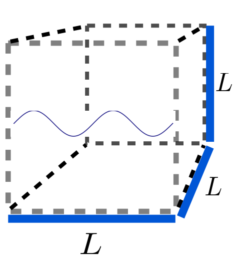

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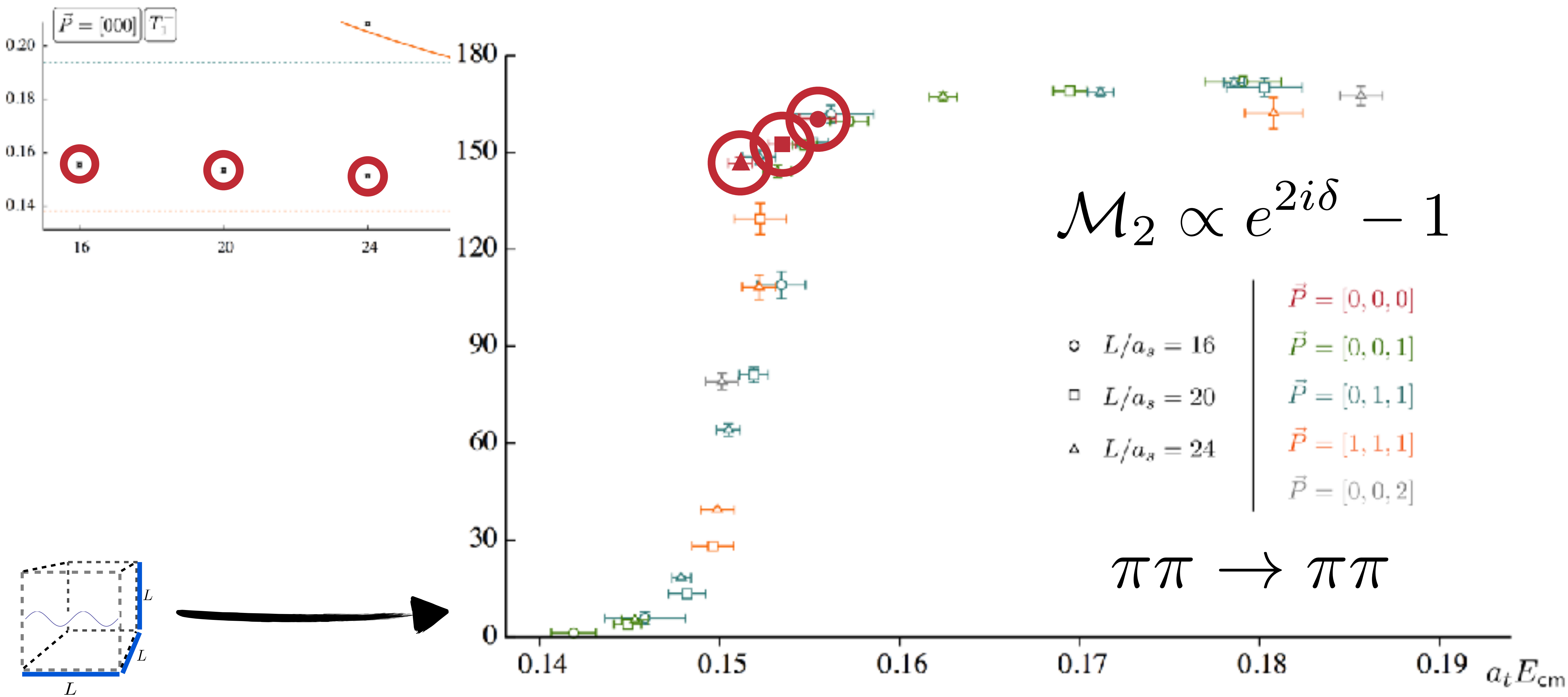

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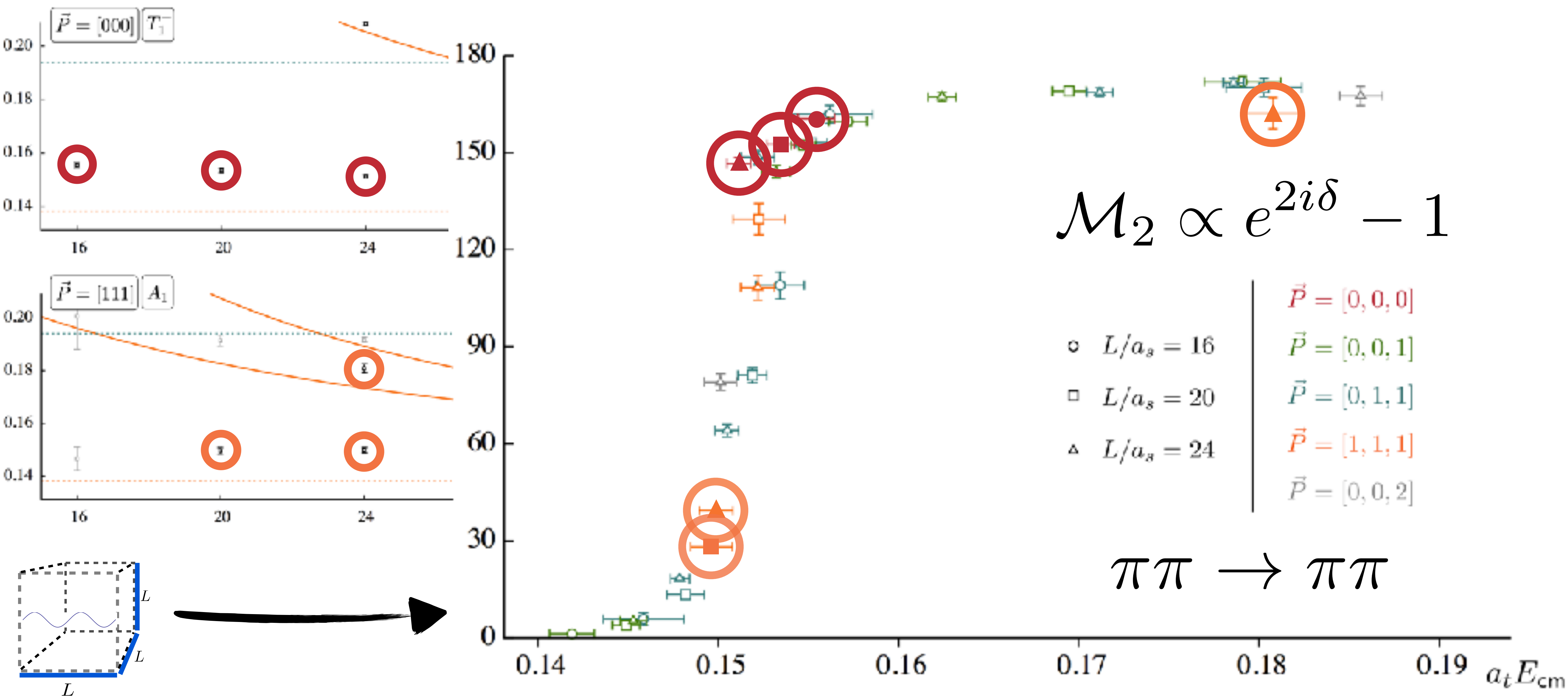

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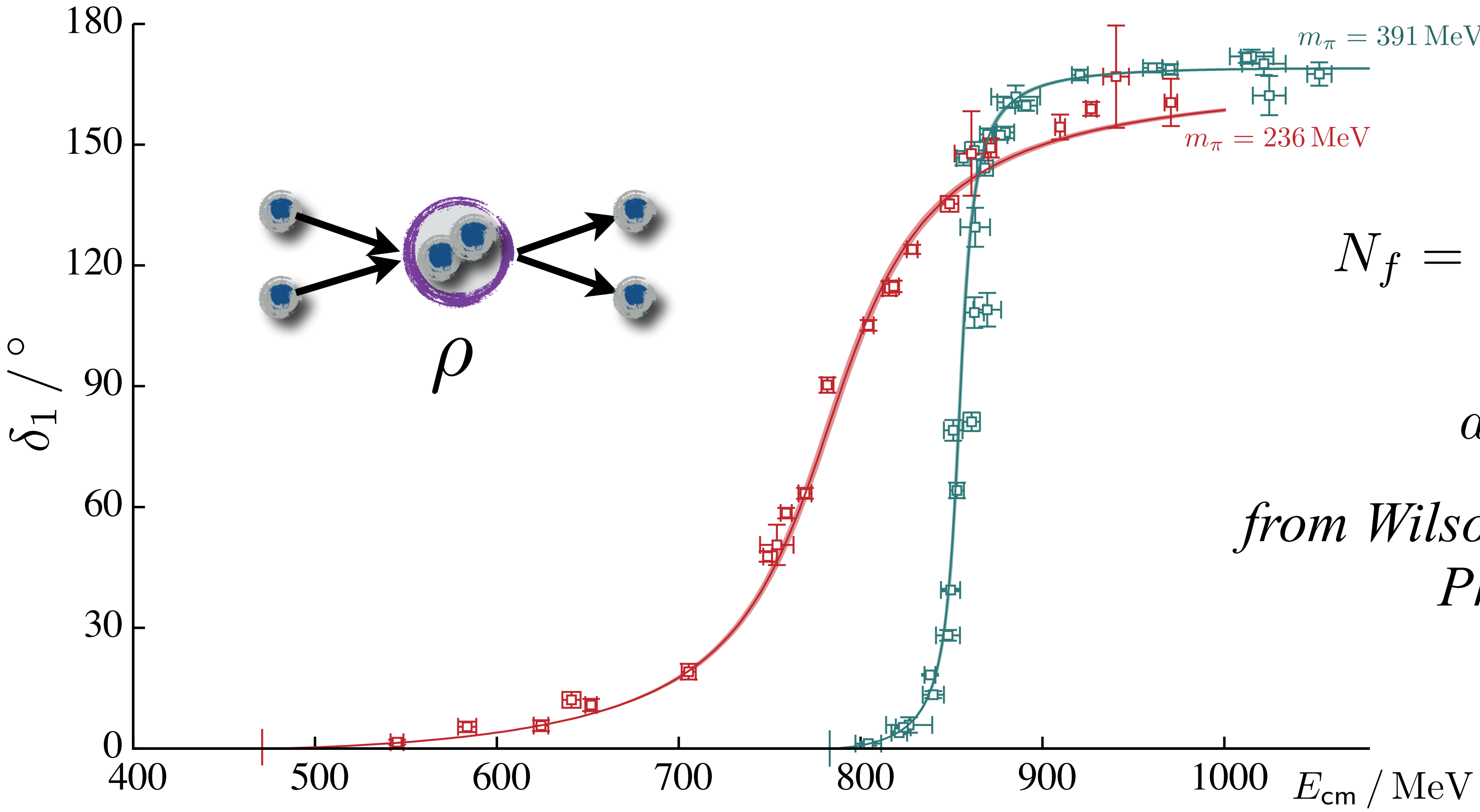
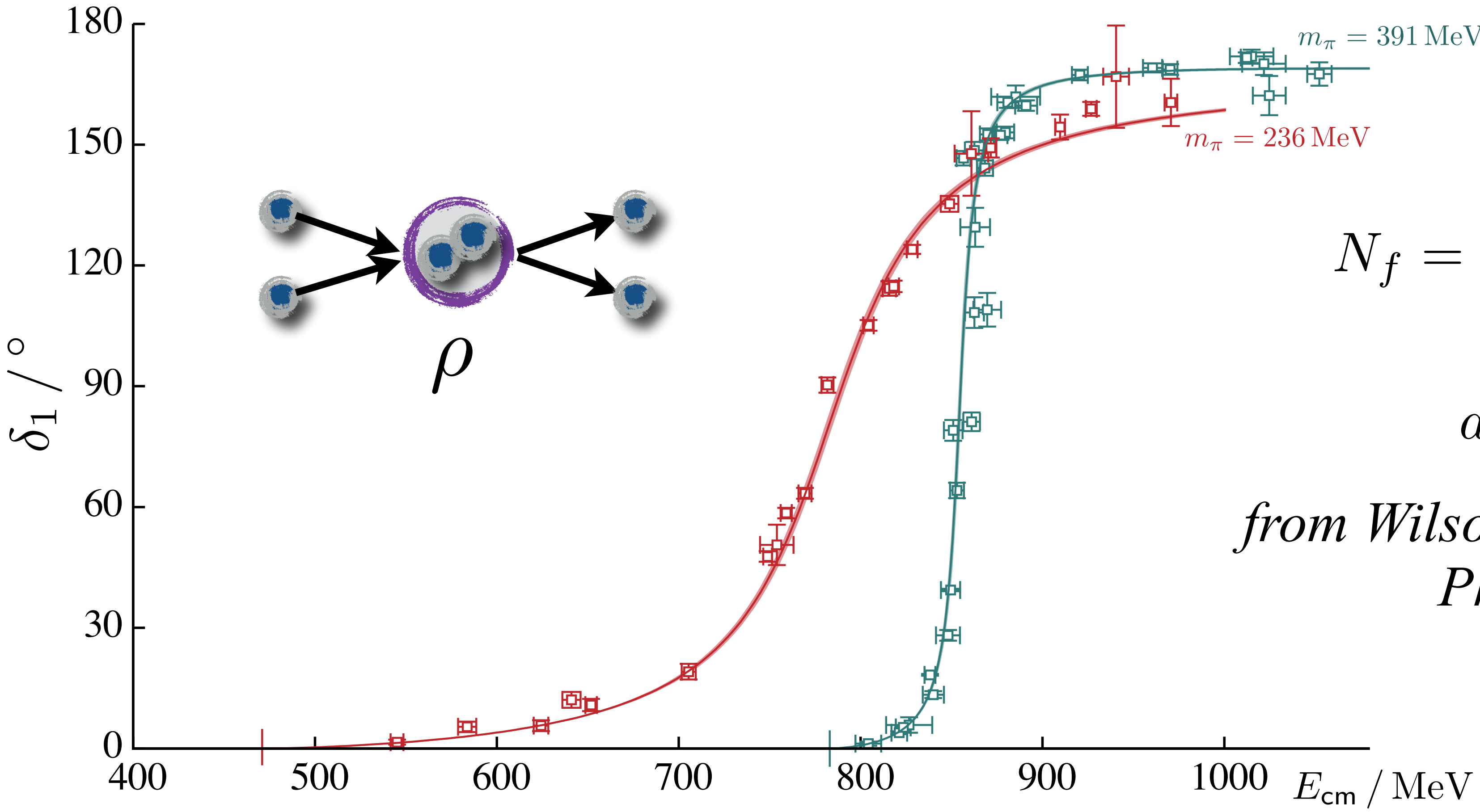


$$N_f = 2+1\,, \quad M_\pi \sim 240 \text{ and } 390\,\mathrm{MeV}\,,$$
$$M_\pi L \sim 4-6\,,$$
$$a_s \sim 0.12\,\mathrm{fm}\,, \quad a_t \sim 0.035\,\mathrm{fm}$$

from Wilson, Briceño, Dudek, Edwards, Thomas, Phys. Rev. D 92, 094502 (2015)



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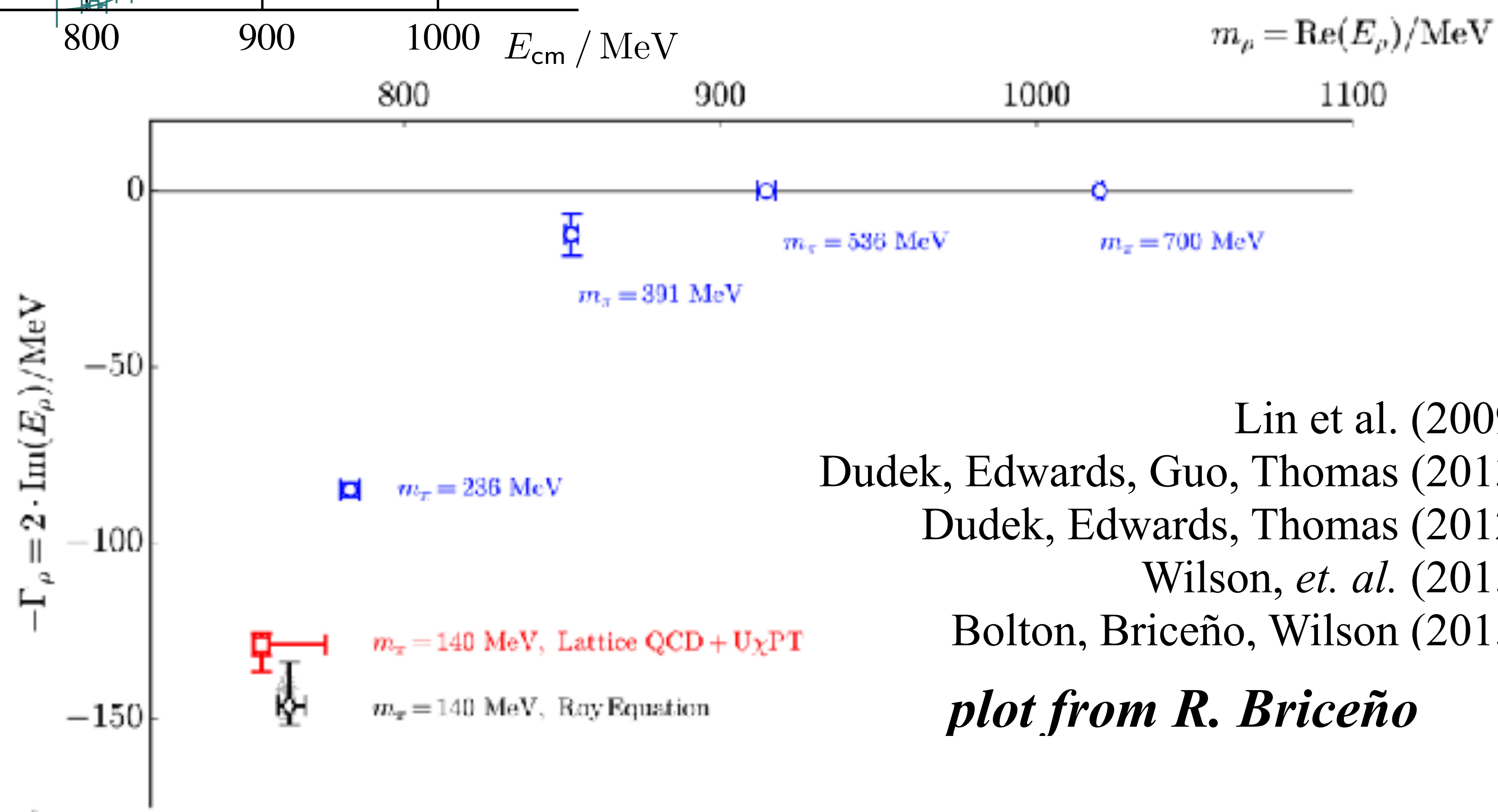
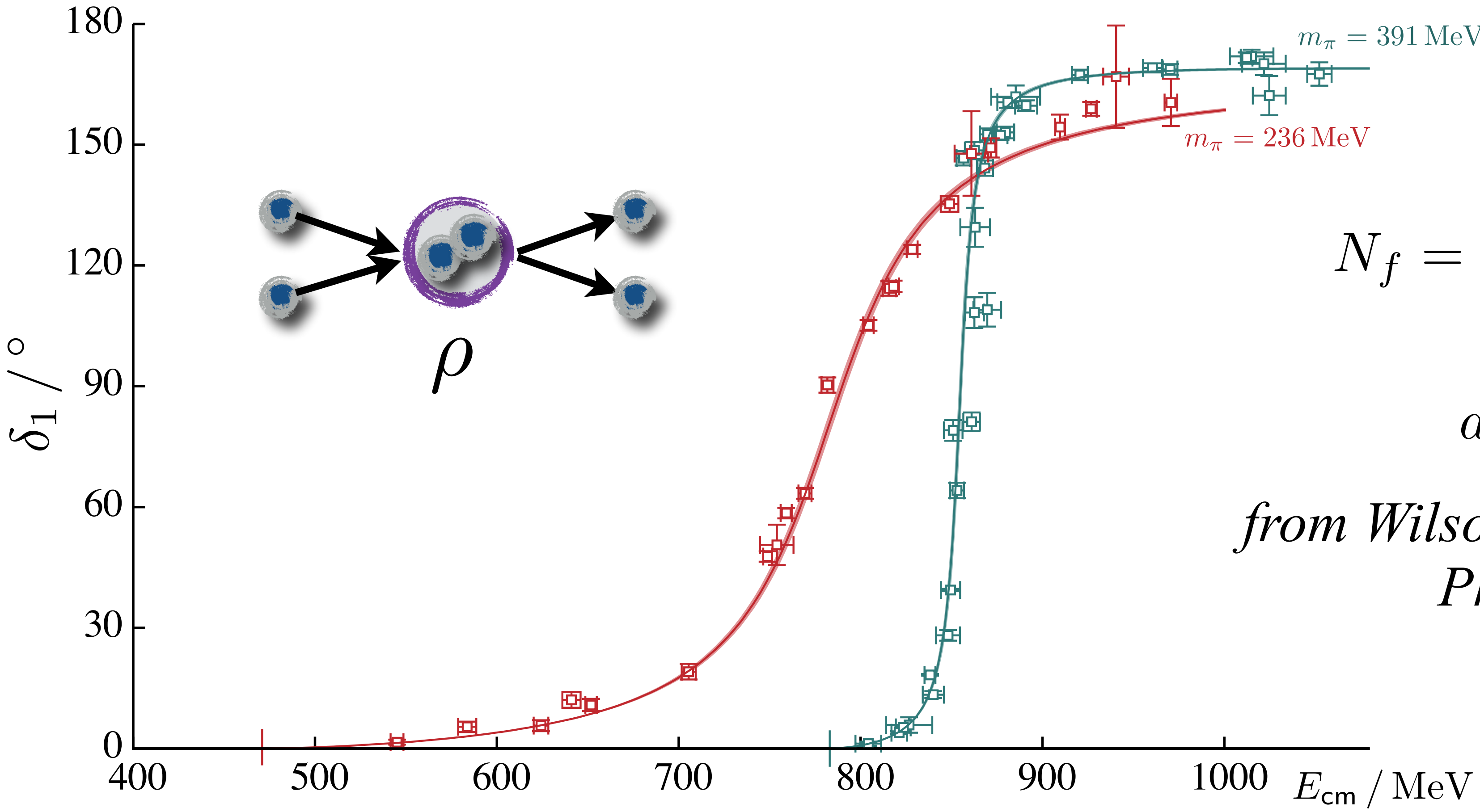


Lin et al. (2009)
Dudek, Edwards, Guo, Thomas (2013)
Dudek, Edwards, Thomas (2012)
Wilson, *et. al.* (2015)
Bolton, Briceño, Wilson (2015)

plot from R. Briceño



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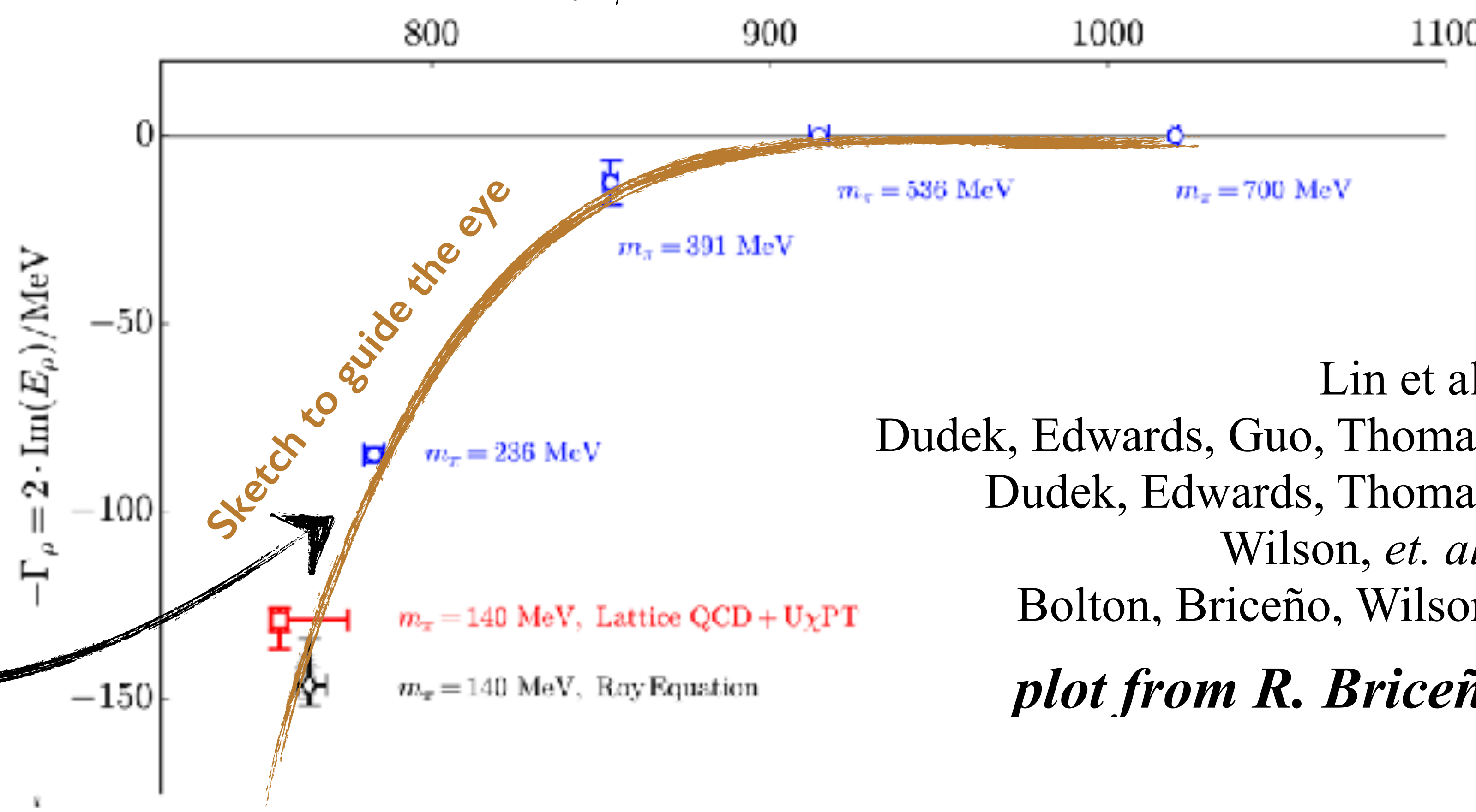
from Wilson, Briceño, Dudek, Edwards, Thomas, Phys. Rev. D 92, 094502 (2015)

Quantitatively trace the pole position in the complex plane



Lin et al. (2009)
Dudek, Edwards, Guo, Thomas (2013)
Dudek, Edwards, Thomas (2012)
Wilson, *et. al.* (2015)
Bolton, Briceño, Wilson (2015)

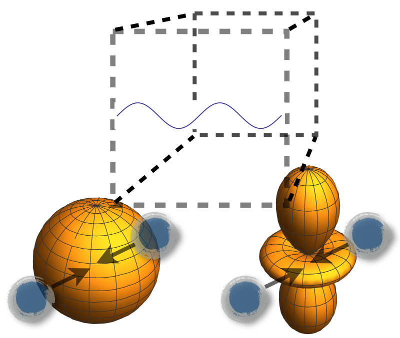
plot from R. Briceño

Coupled channels

- The cubic volume mixes different partial waves…

e.g. $K\pi \to K\pi$
$\vec{P} \neq 0$

$$\longrightarrow \det\left[\begin{pmatrix} \mathcal{M}_s^{-1} & 0 \\ 0 & \mathcal{M}_p^{-1} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix}\right] = 0$$



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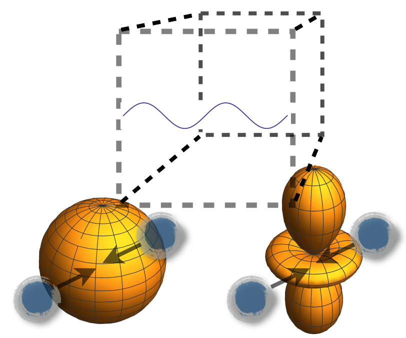

…as well as different flavor channels…

e.g. $a = \pi\pi$, $b = K\overline{K}$

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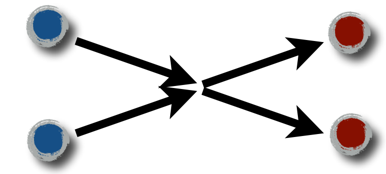

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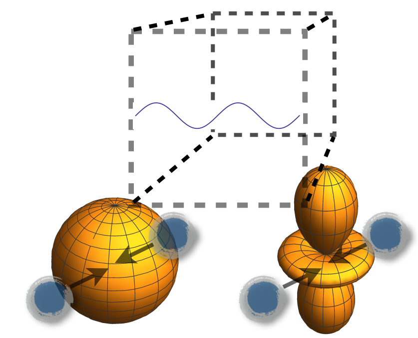

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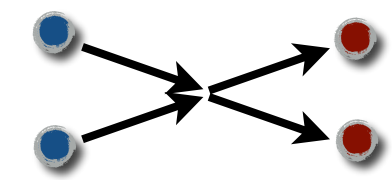


- The road to physics…

Calculate a matrix of correlators with a large & varied operator basis

$$\langle \mathcal{O}_a(\tau)\mathcal{O}_b^\dagger(0)\rangle$$

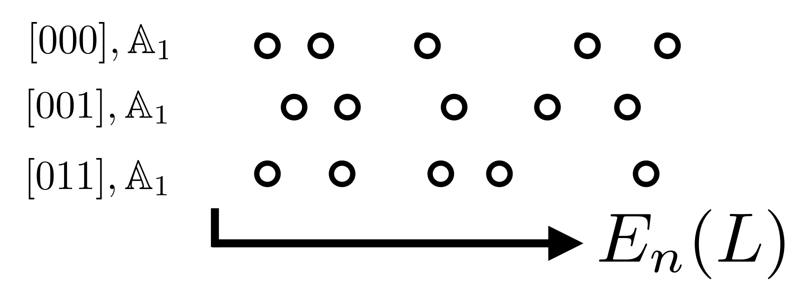
→

Diagonalize (GEVP) to reliably extract finite-volume energies

$$\langle \Omega_m(\tau)\Omega_m^\dagger(0)\rangle \sim e^{-E_m(L)\tau}$$

→

Vary *L* and *P* to recover a dense set of energies



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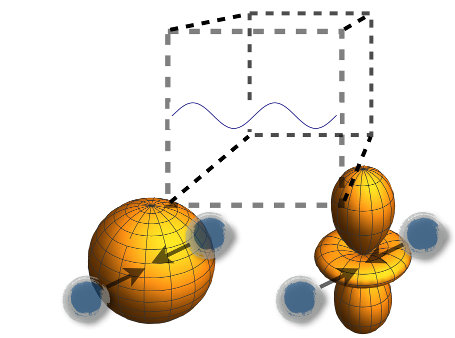

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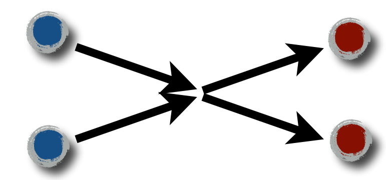

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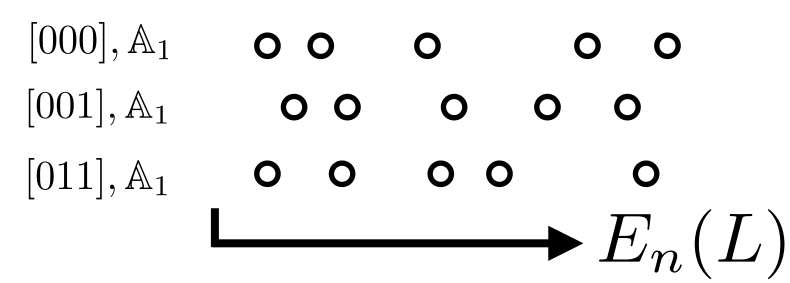

Identify a broad list of K-matrix parametrizations

polynomials and poles

EFT based

dispersion theory based

hadspec

Coupled channels

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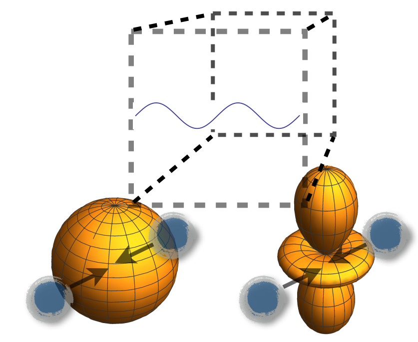

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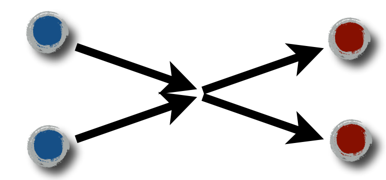

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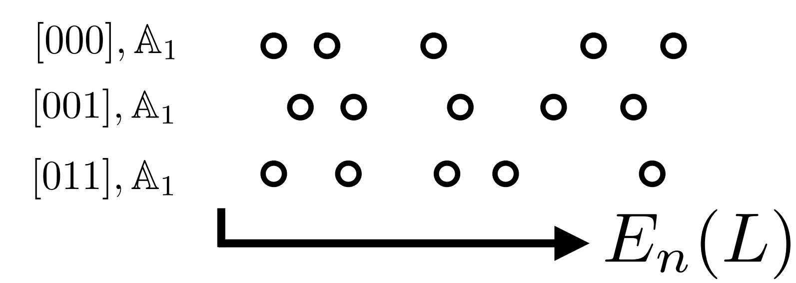

Identify a broad list of K-matrix parametrizations

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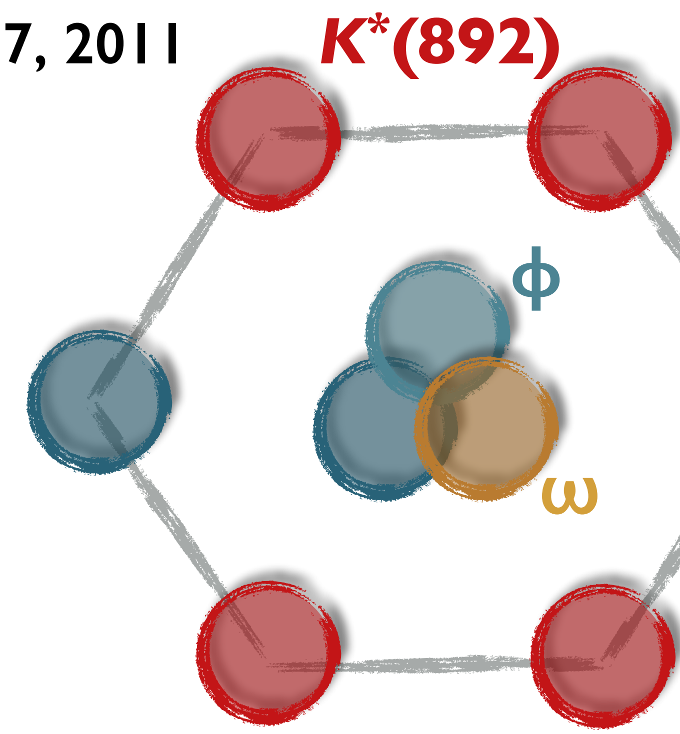
Perform global fits to the finite-volume spectrum

Lots of activity!

$\rho \to \pi\pi$

- CP-PACS/PACS-CS 2007, 2011
- ETMC 2010
- Lang et al. 2011
- HadSpec 2012, 2016
- Pellisier 2012
- RQCD 2015
- Guo et al. 2016
- Fu et al. 2016
- Bulava et al. 2016
- Alexandrou et al. 2017



$J^P = 1^-$

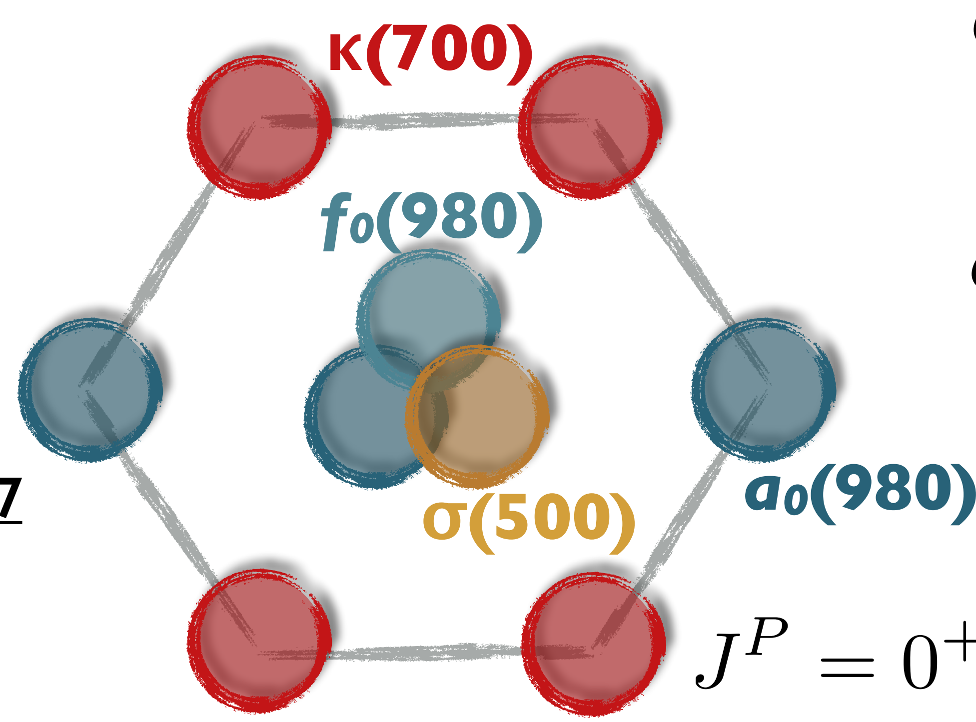
$\kappa \to K\pi$

$K^* \to K\pi$

- Lang et al. 2012
- Prelovsek et al. 2013
- Wilson et al. 2015
- RQCD 2015
- Brett et al. 2018

$\sigma \to \pi\pi$

- Prelovsek et al. 2010
- Fu 2013
- Wakayama 2015
- Howarth and Giedt 2017
- Briceño et al. 2017



$J^P = 0^+$

$a_0(980) \to \pi\eta, K\overline{K}$

- Dudek et al. 2016

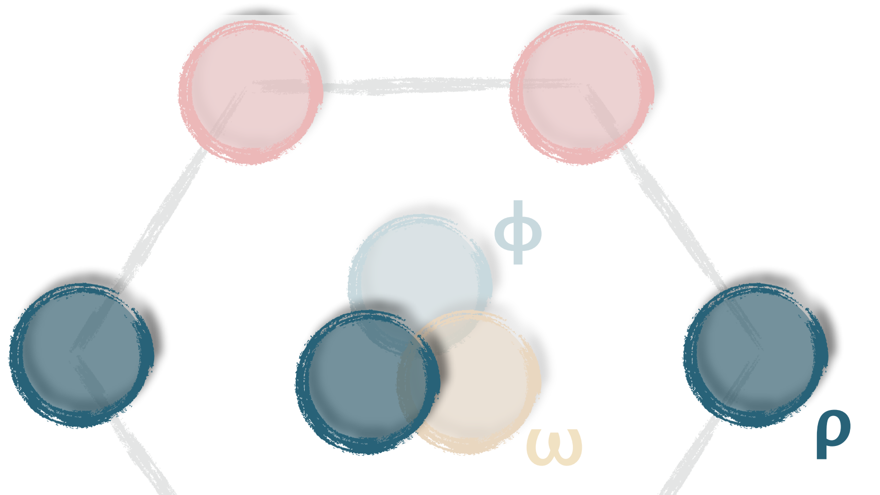
$\sigma, f_0, f_2 \to \pi\pi, K\overline{K}, \eta\eta$

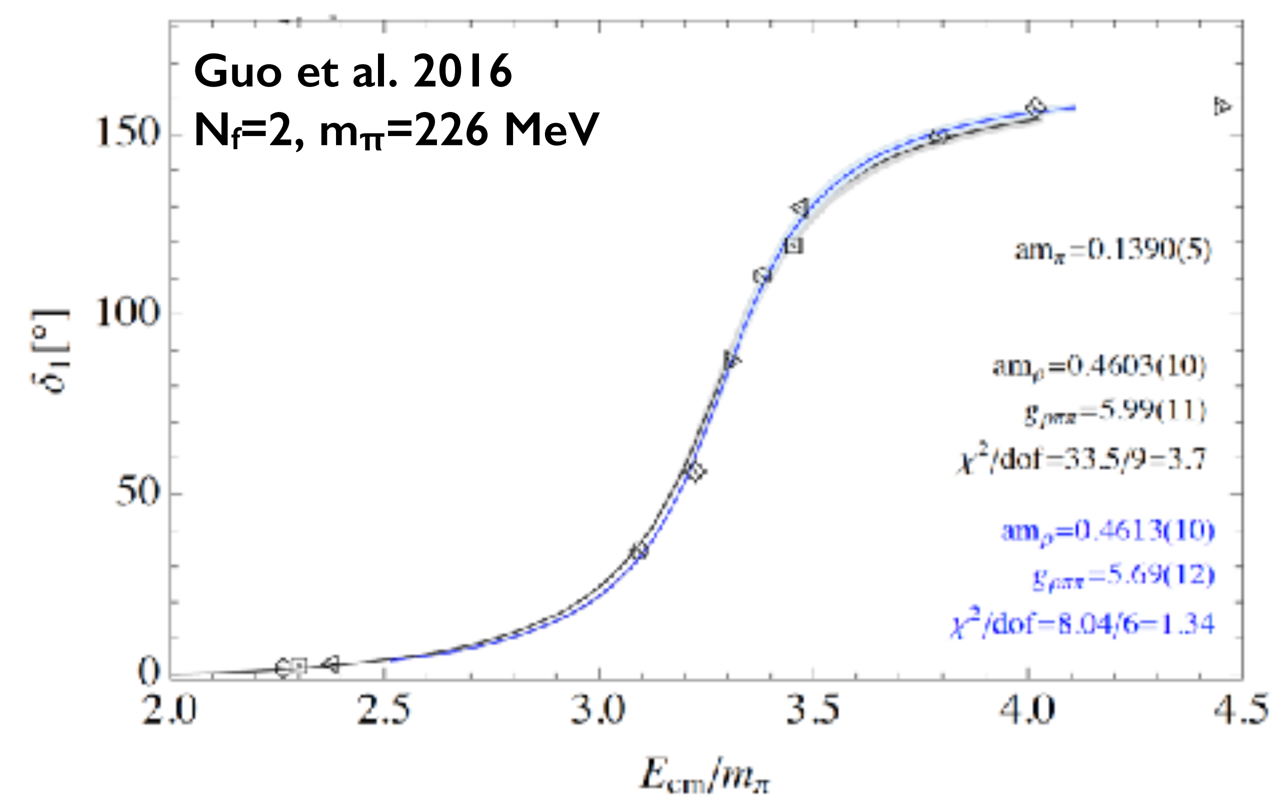
- Briceño et al. 2017

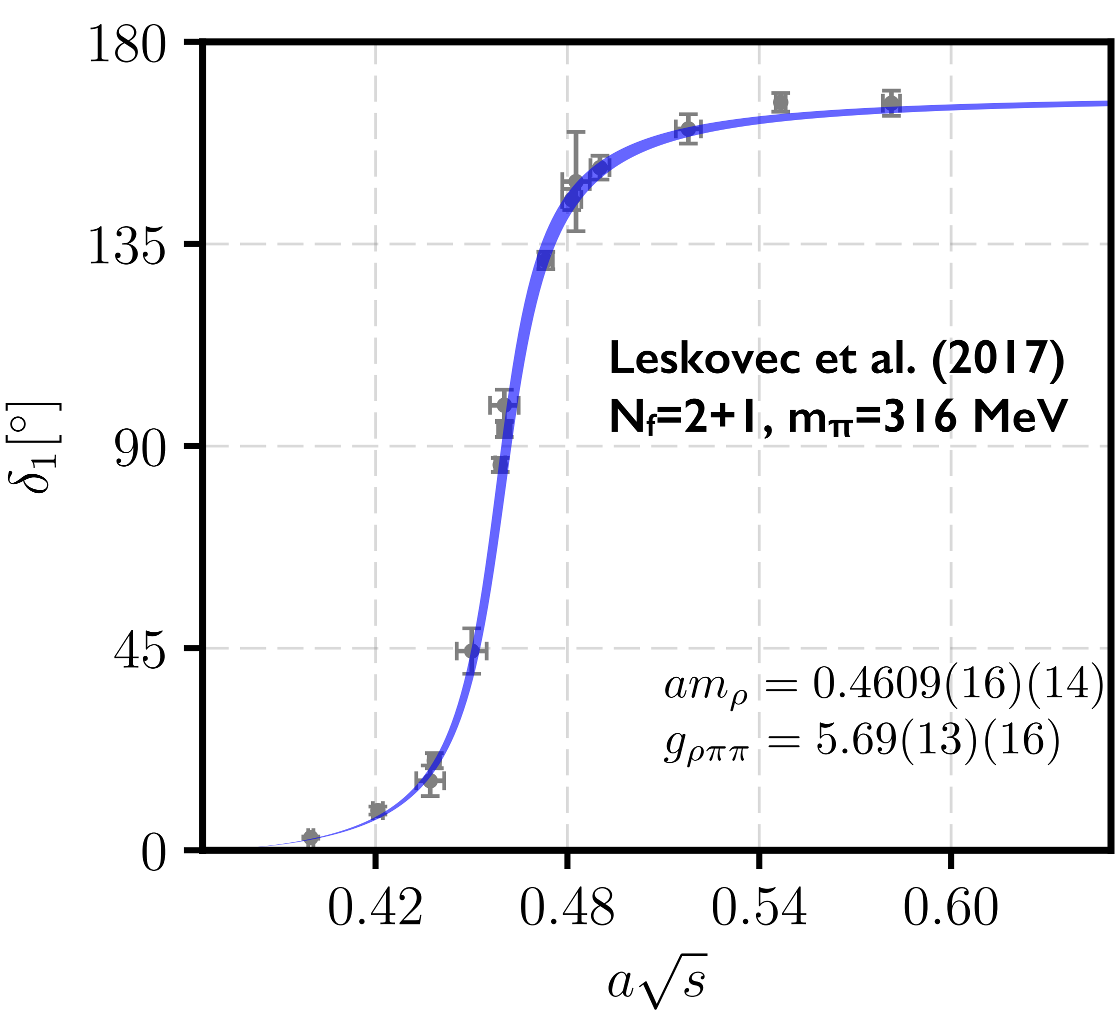
See the recent review by Briceño, Dudek and Young

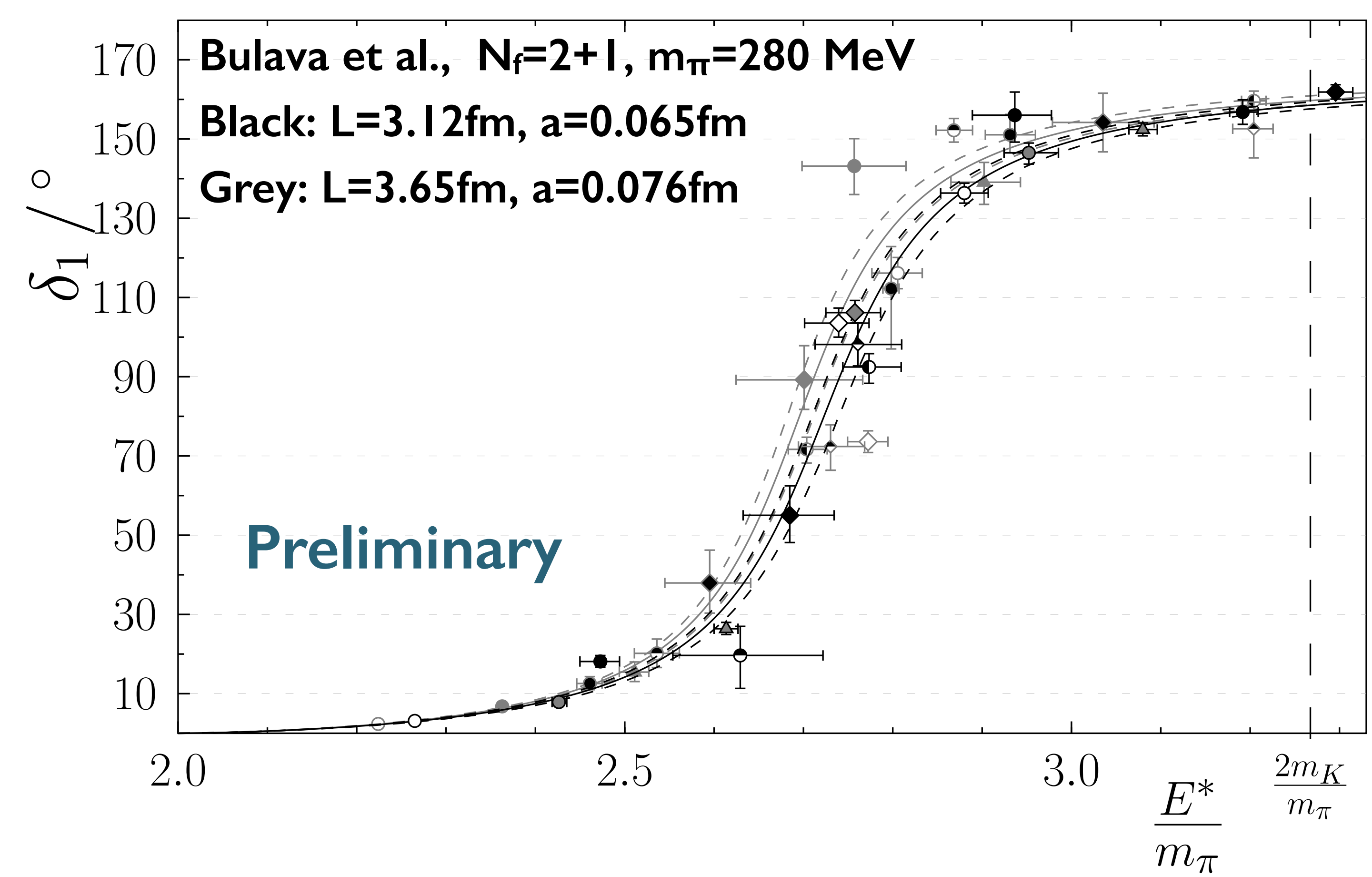
$$\rho \to \pi\pi$$

$$I^G(J^{PC}) = 1^+(1^{--})$$

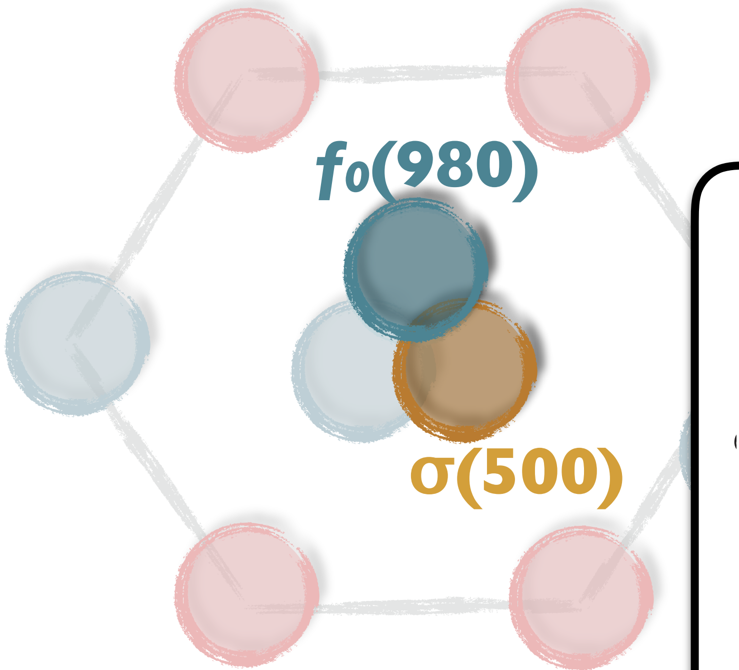




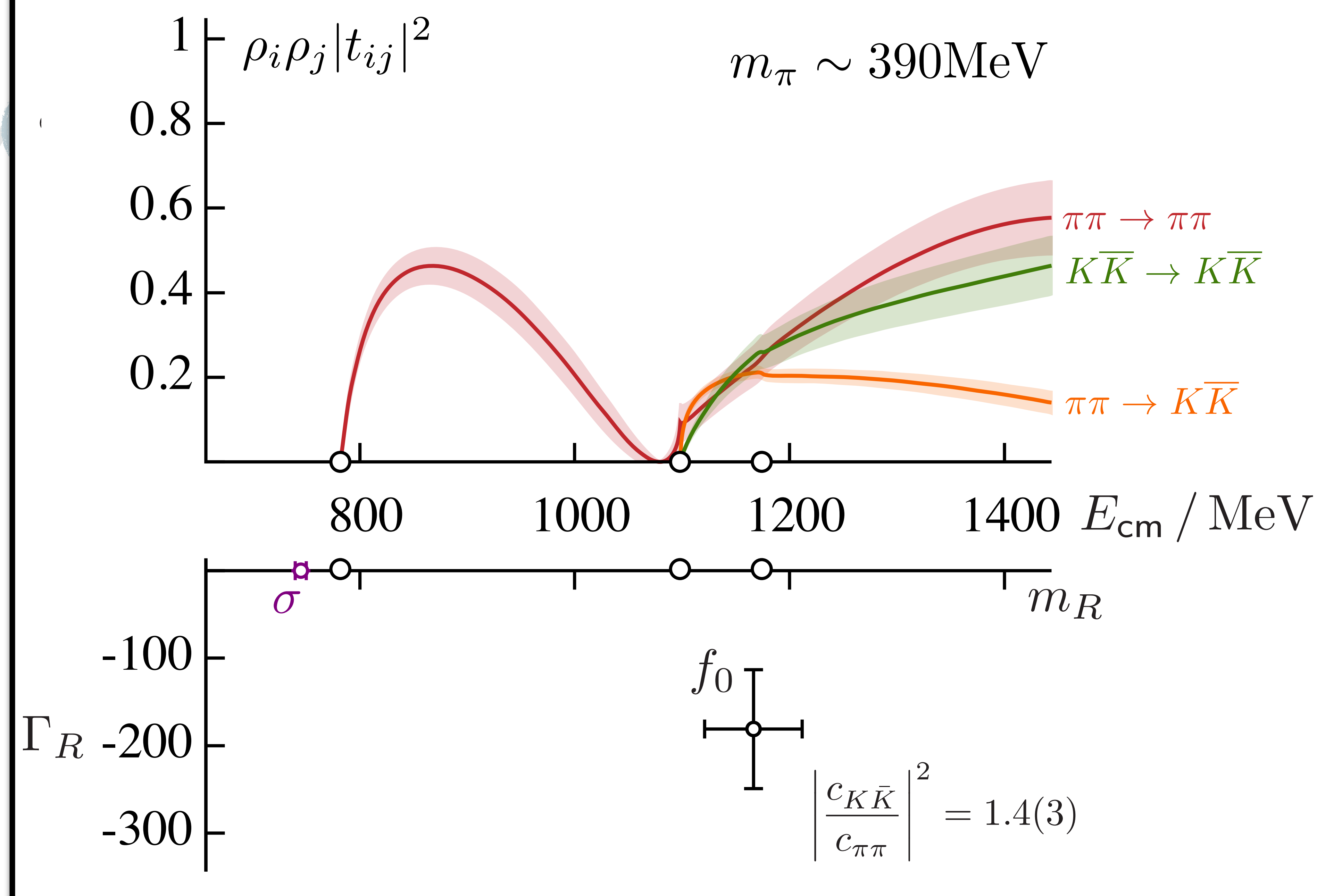




$$I^G(J^{PC}) = 0^+(0^{++})$$



Coupled-channel scattering!



Briceño, Dudek, Edwards & Wilson (2017)

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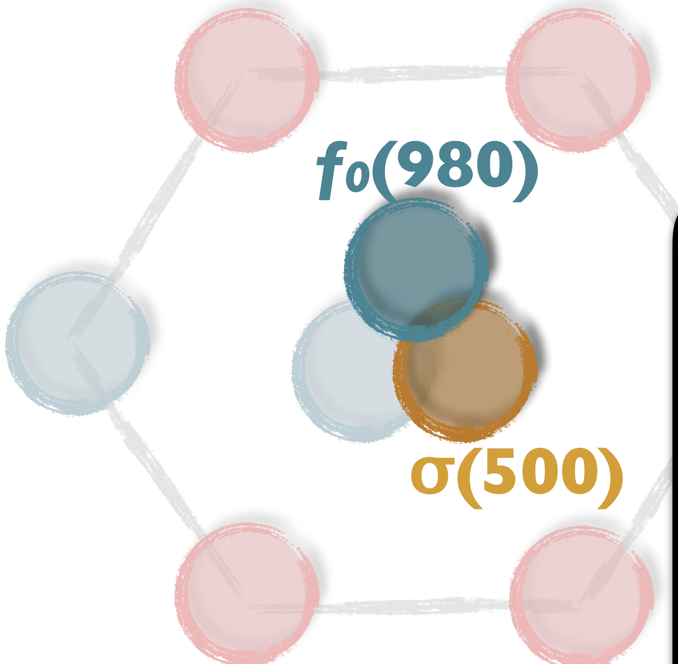

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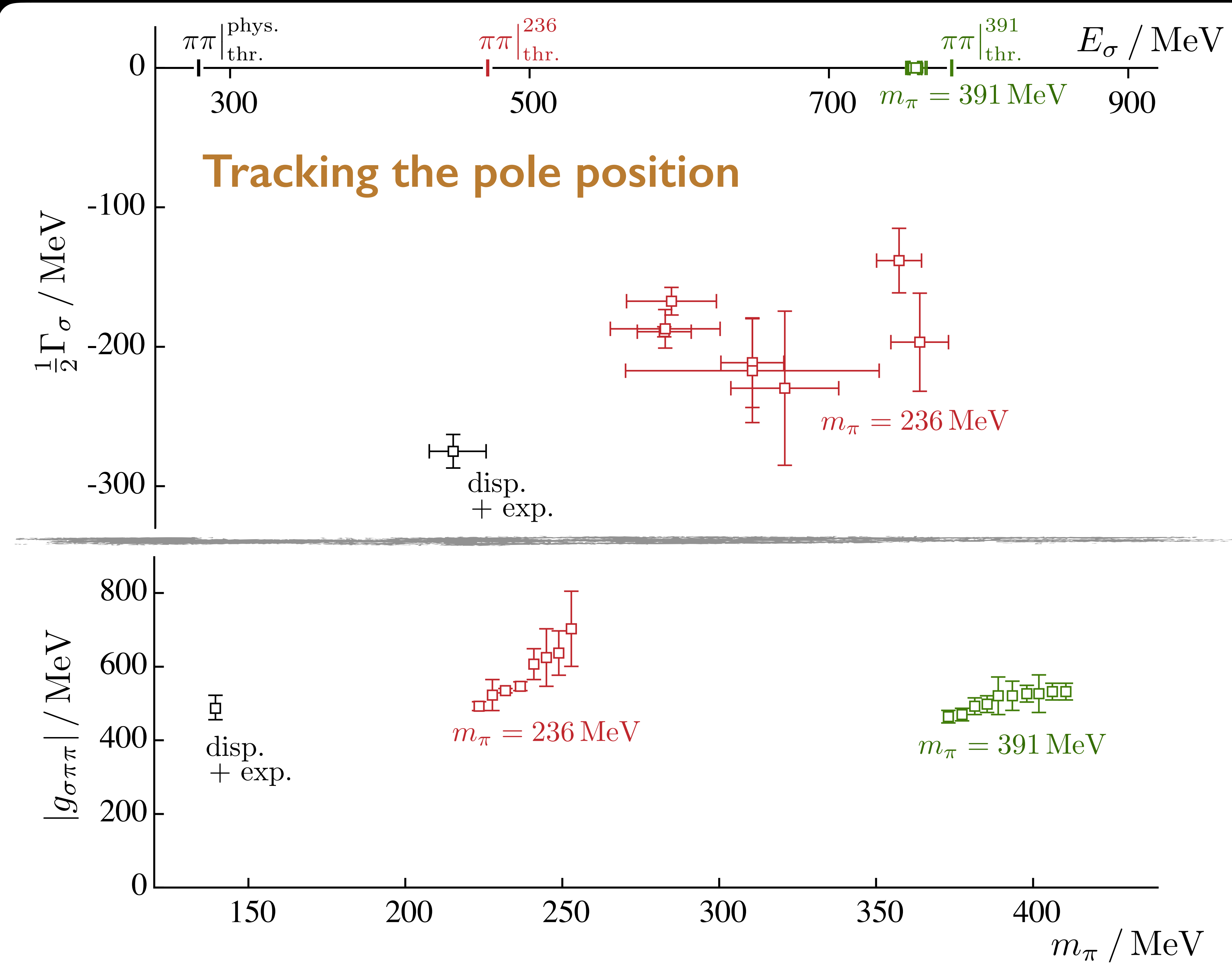

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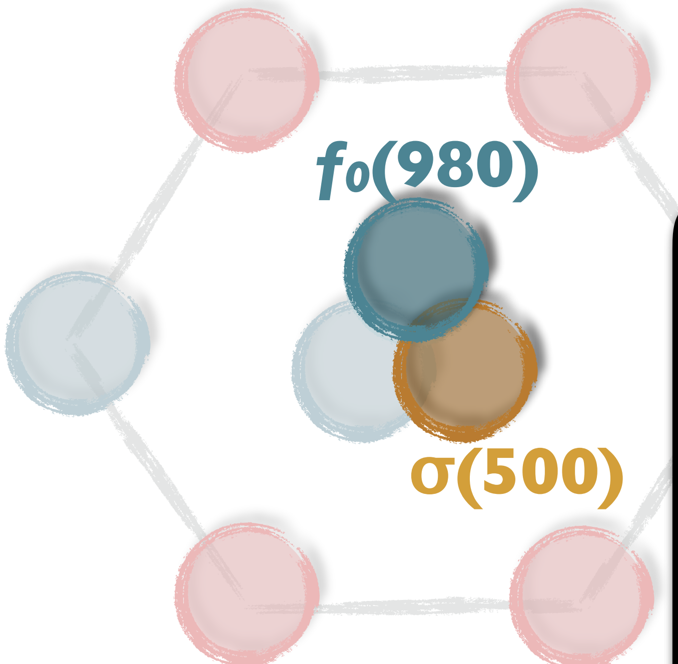

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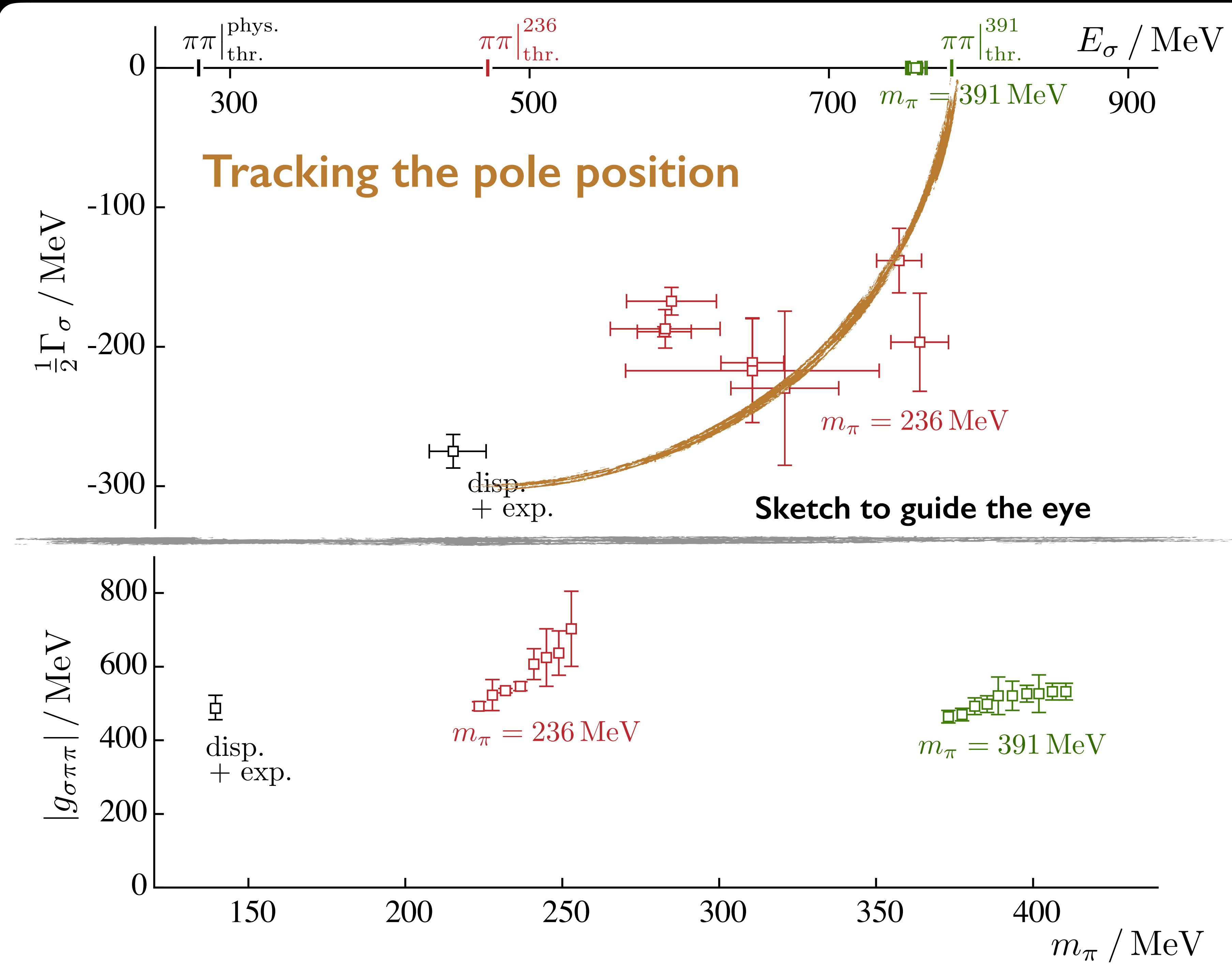
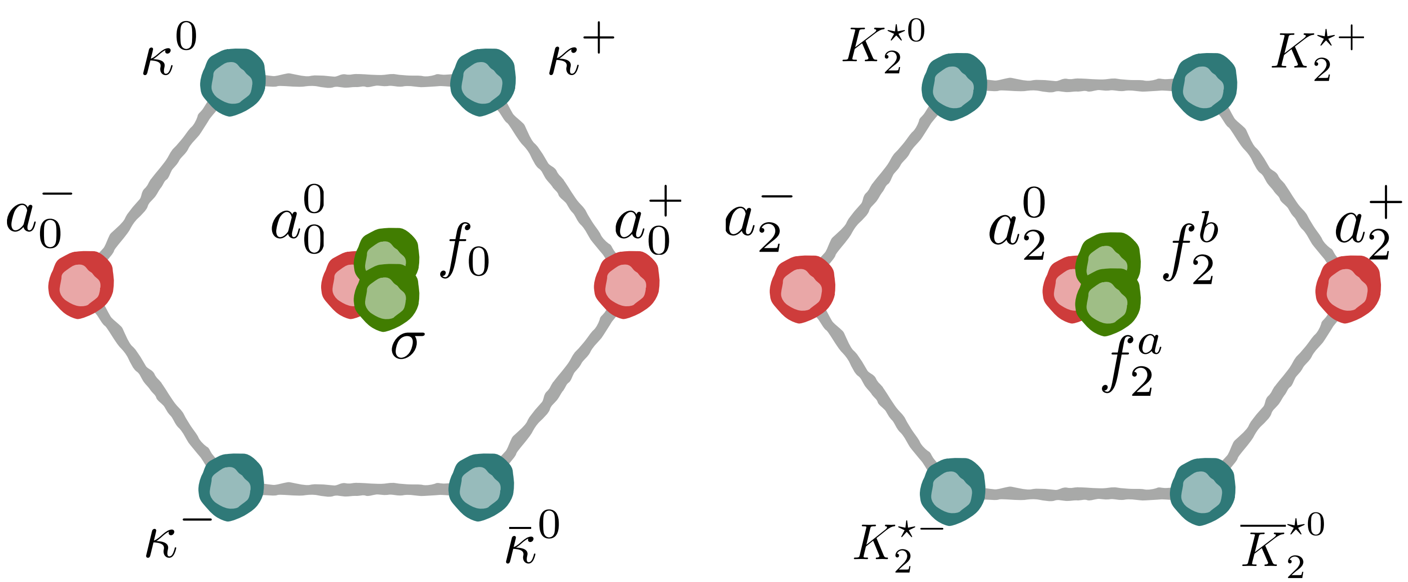


Briceño, Dudek, Edwards & Wilson (2016)

There's so much more!…

- Much more activity in the light-quark sector
  - e.g. first complete determination of the scalar and tensor nonets



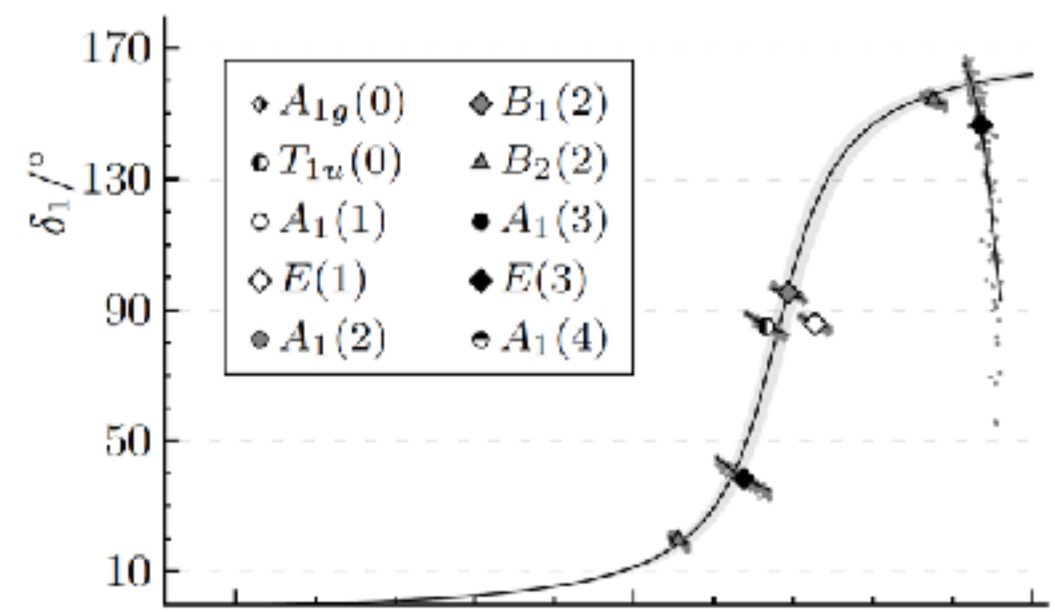
hadspec

ππ, KK, ηη: Briceño, Dudek, Edwards - PRL (2017)
Briceño, Dudek, Edwards - arXiv (2017)

Kπ, Kη: Dudek, Edwards, Thomas, Wilson - PRL (2015)
Wilson, Dudek, Edwards, Thomas - PRD (2015)

πη, KK: Dudek, Edwards, Wilson - PRD (2016)

- recent result for K*(892) from Brett et al. 2018



There's so much more!…

- **Scattering calculations are also being performed in the charm sector!**
  - e.g. $I = 0,\ DK \to DK$ scattering examining the $D^*_{s0}(2317)$
    - Lang et al. (2014)
    - Bali et al. (2017)
  - e.g. $I = 1/2,\ D\pi,\ D\eta,\ D_s\overline{K}$ scattering examining the $D^*_0(2400)$
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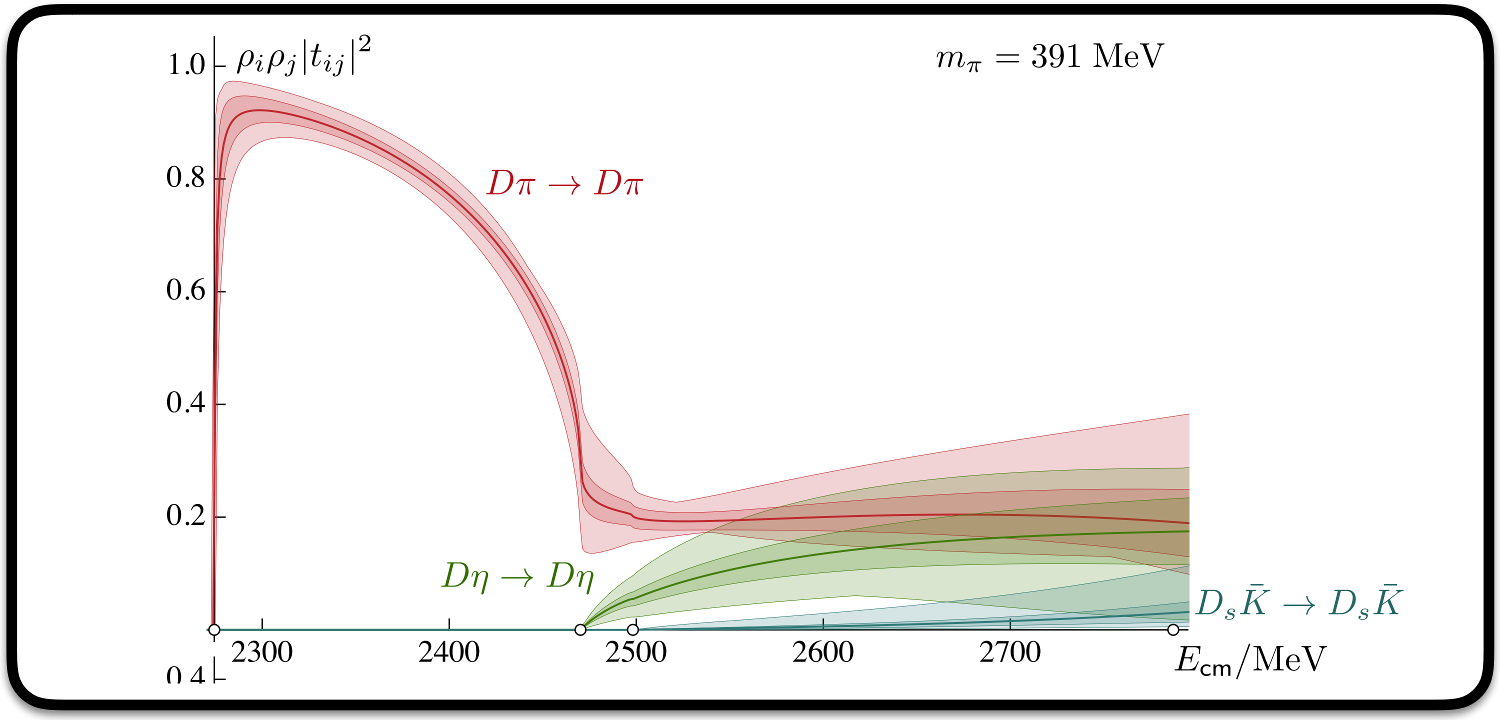

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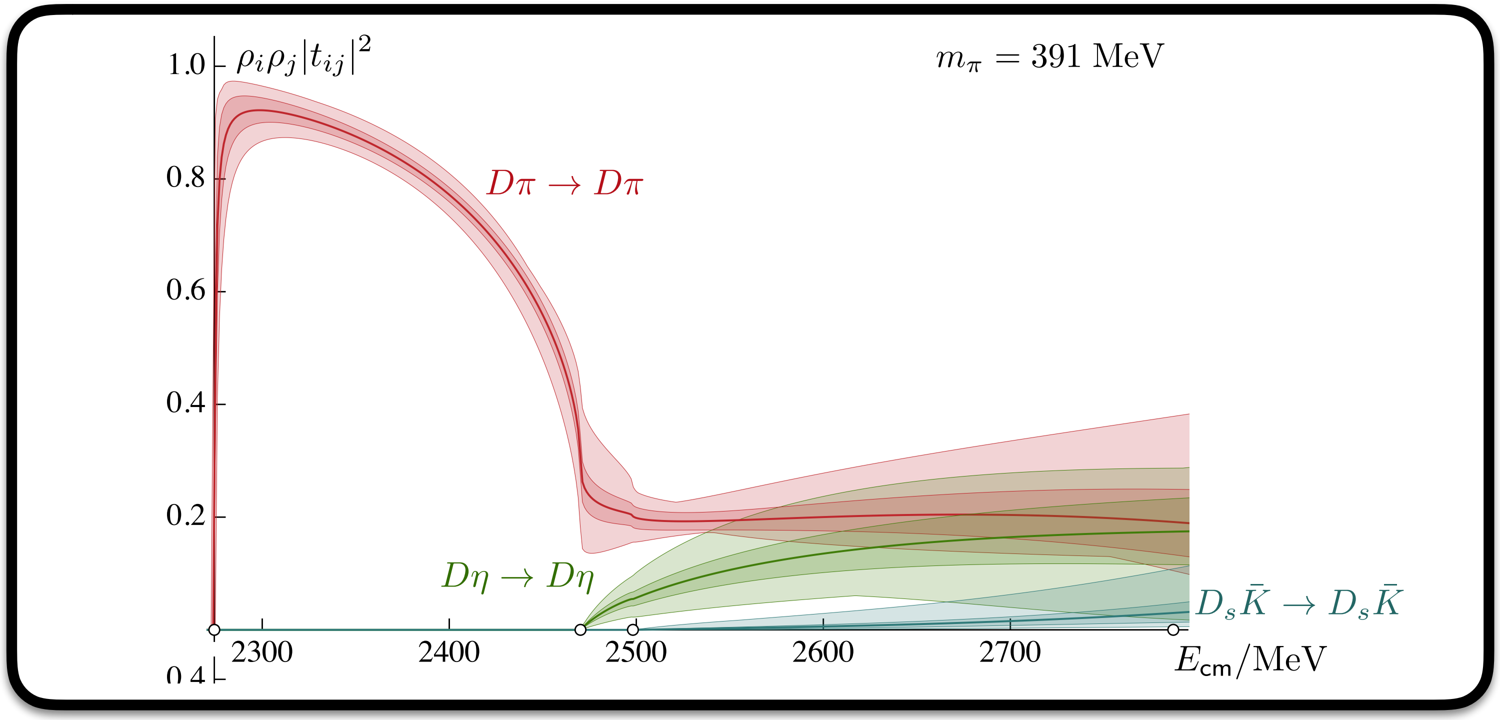


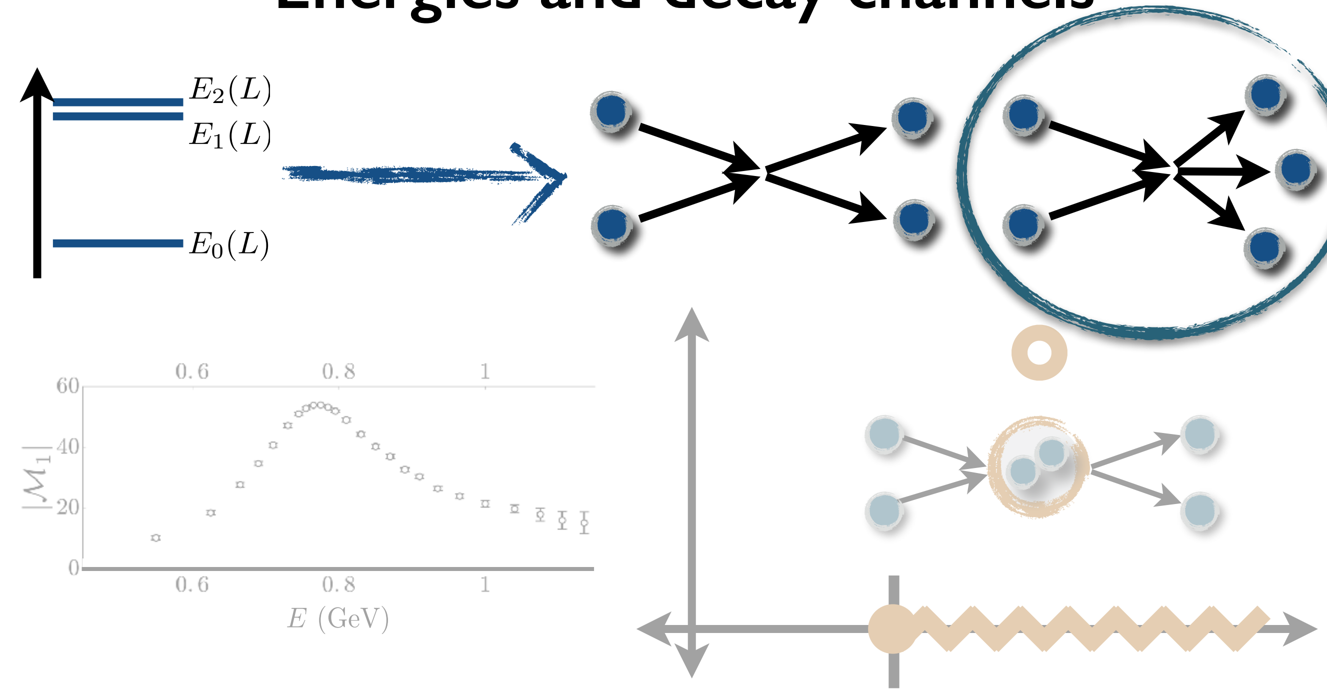
- Also significant progress in nucleon-meson and nucleon-nucleon scattering

Talks at this workshop from…

Evan Berkowitz, Raúl Briceño, Will Detmold, Jozef Dudek, Ciaran Hughes, Nilmani Mathur, Daniel Mohler, Amy Nicholson, Phiala Shanahan, Christopher Thomas

Scattering in LQCD relies on relations between finite- and infinite-volume quantities

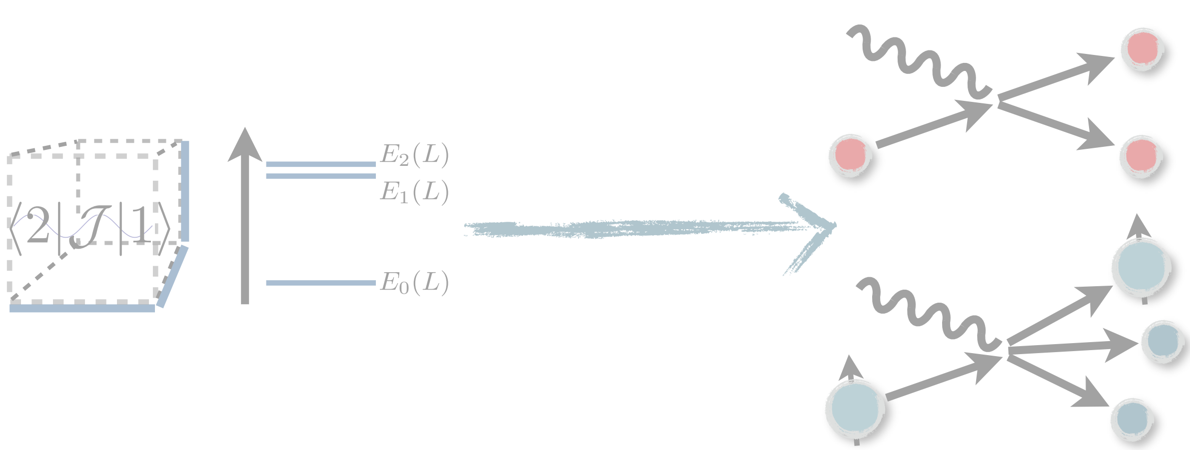
Energies and decay channels



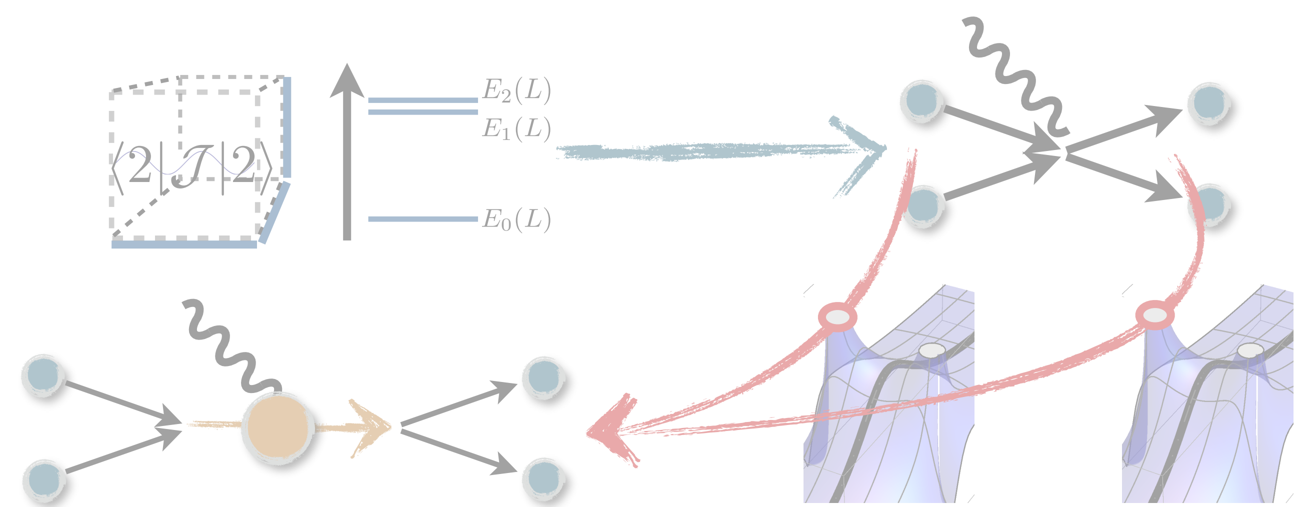
Finite-volume energies related to scattering

Transition amplitudes

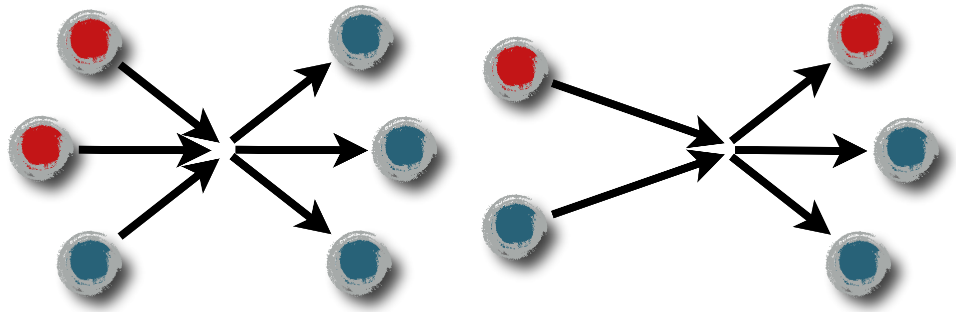
Energies and finite-volume matrix elements are related to transitions



Resonant form factors



The aim is to derive a formalism for studying relativistic two- and three-particle systems from lattice QCD



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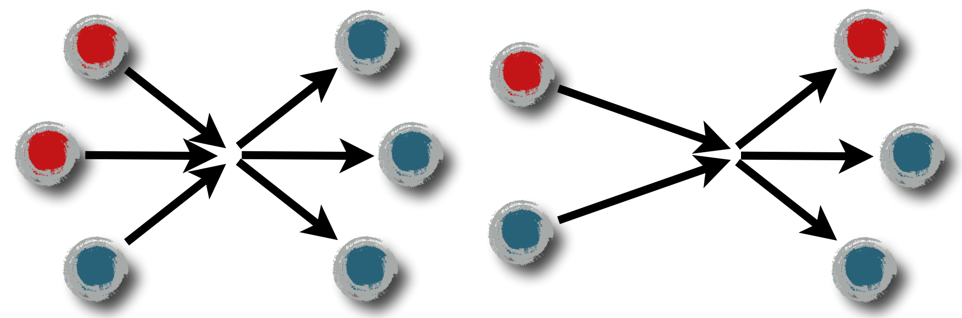


Potential applications…

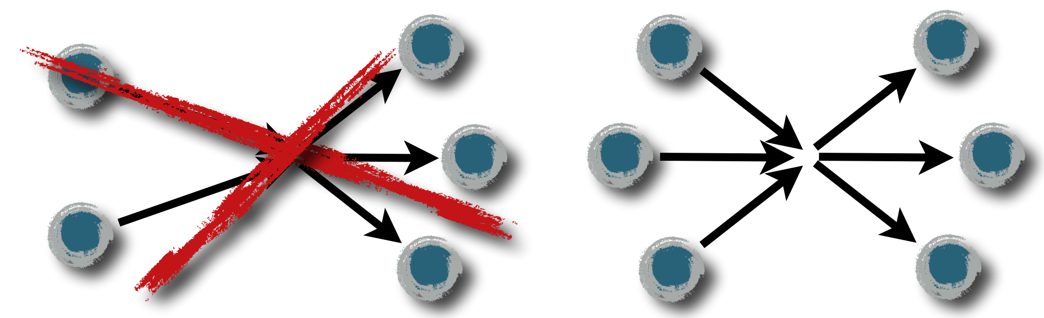
- **Studying three-particle resonances**

$$\omega(782),\ a_1(1420) \to \pi\pi\pi$$
$$\pi_1(1400) \to ?$$
$$N(1440) \to N\pi, N\pi\pi$$
$$\eta(1405) \to a_0(980)\pi$$
$$\eta(1475) \to K^*(892)\overline{K}$$

- **Calculating weak decays, form factors and transitions**

$$K \to \pi\pi\pi \qquad N\gamma^* \to N\pi\pi$$

First consider identical scalar particles with a Z_2 symmetry



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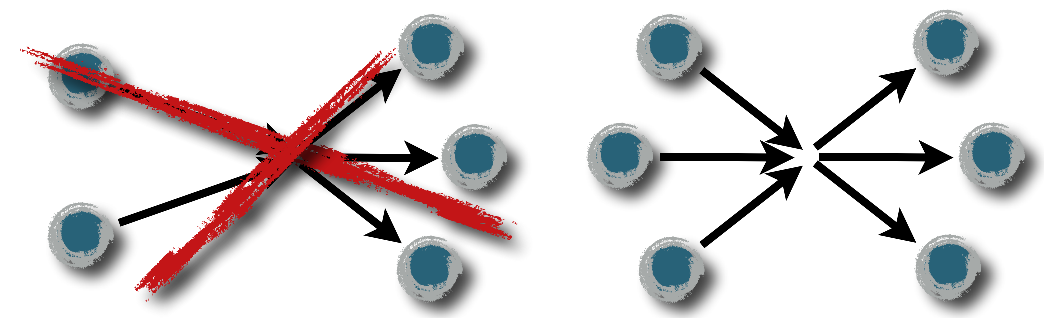

The three-to-three scattering amplitude has ***kinematic singularities***

$$i\mathcal{M}_{3\to3} \equiv$$ fully connected correlator with six external legs amputated and projected on shell

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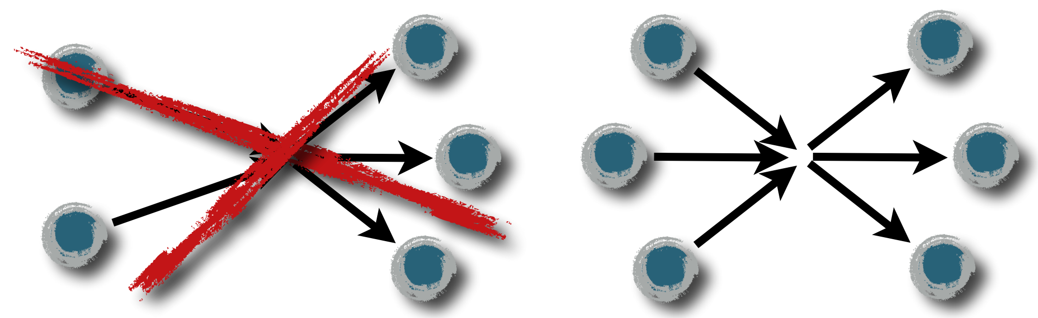

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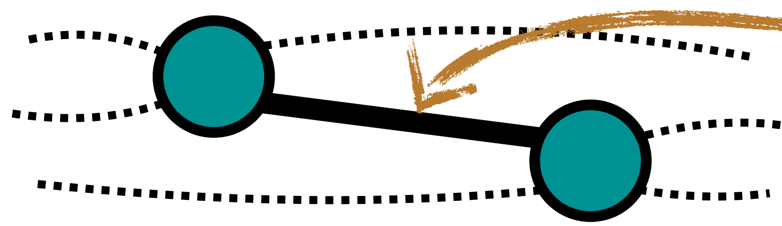
$$=$$  $$+ \cdots$$

Certain external momenta put this on-shell!

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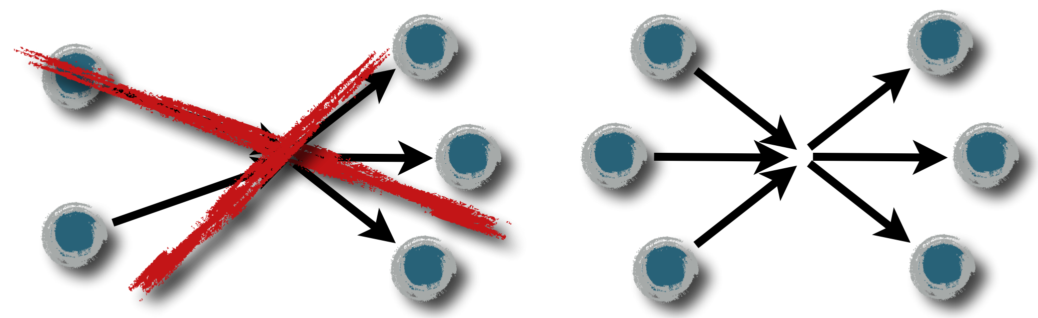

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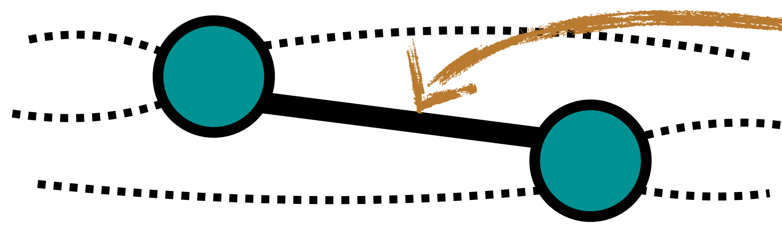
$=$  $+\cdots$

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The three-to-three scattering amplitude has *more degrees of freedom*

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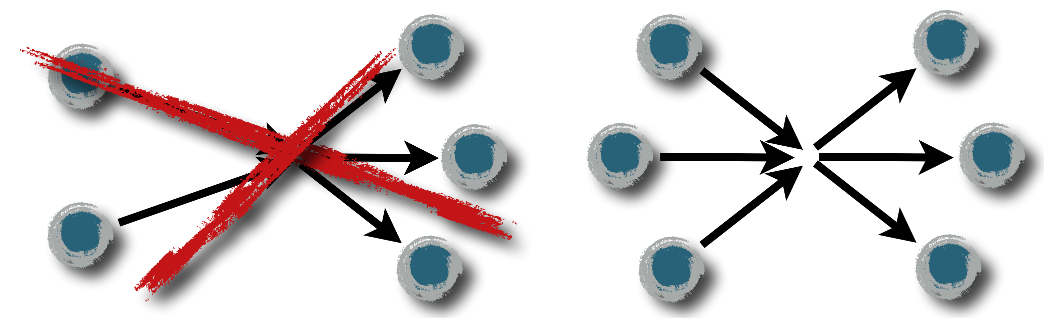

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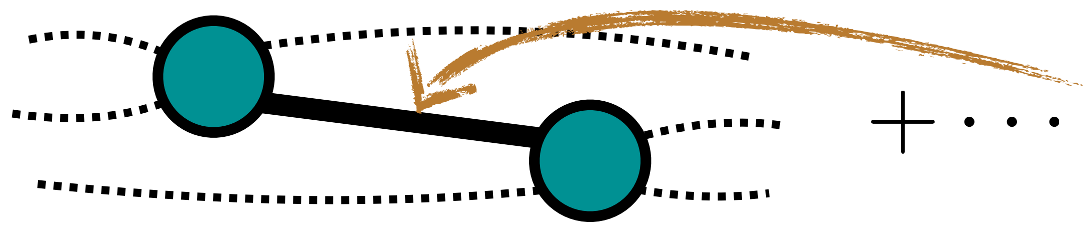
$$=$$  $$+\cdots$$

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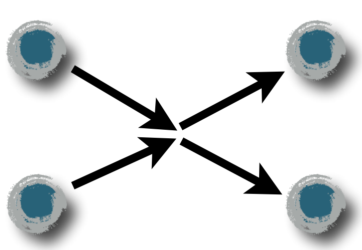

12 momentum components

-10 Poincaré generators

2 degrees of freedom

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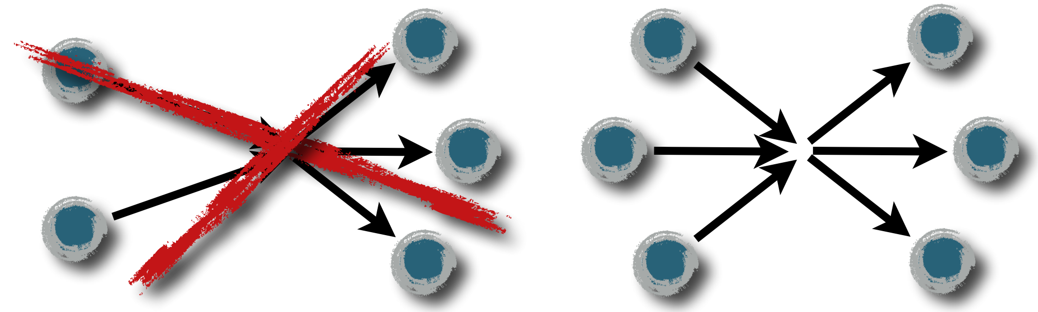

The three-to-three scattering amplitude has *kinematic singularities*

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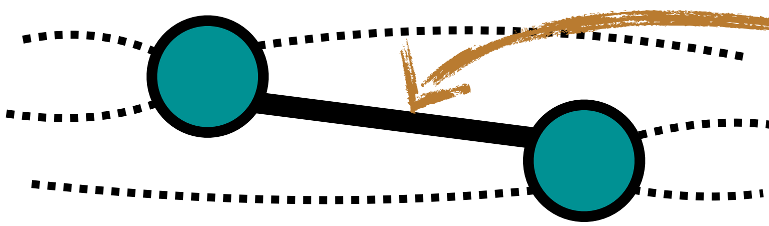
$=$  $+ \cdots$

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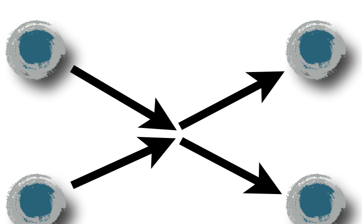

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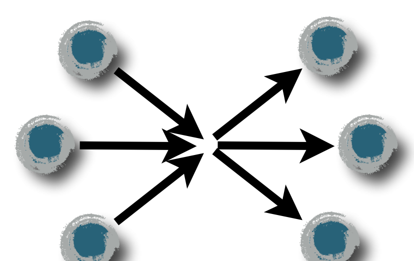
-10 Poincaré generators

2 degrees of freedom



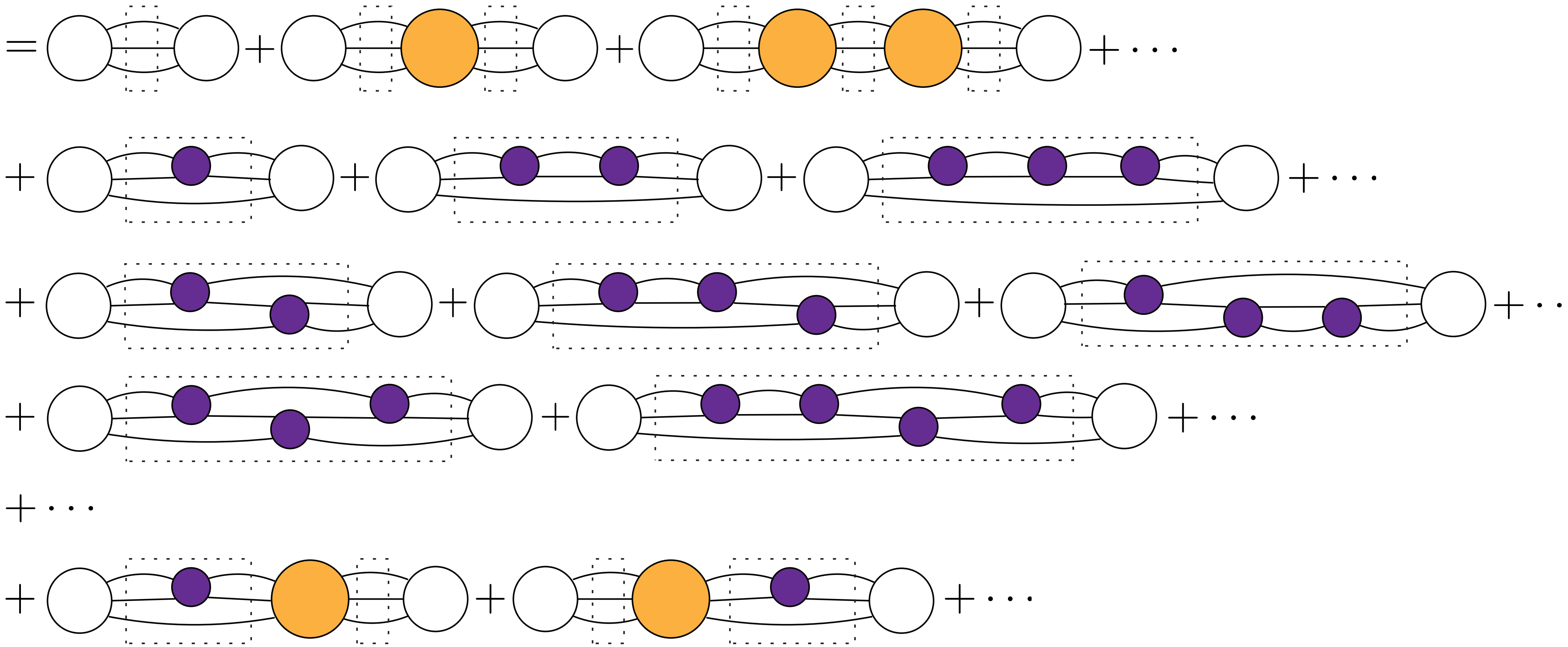
18 momentum components

-10 Poincaré generators

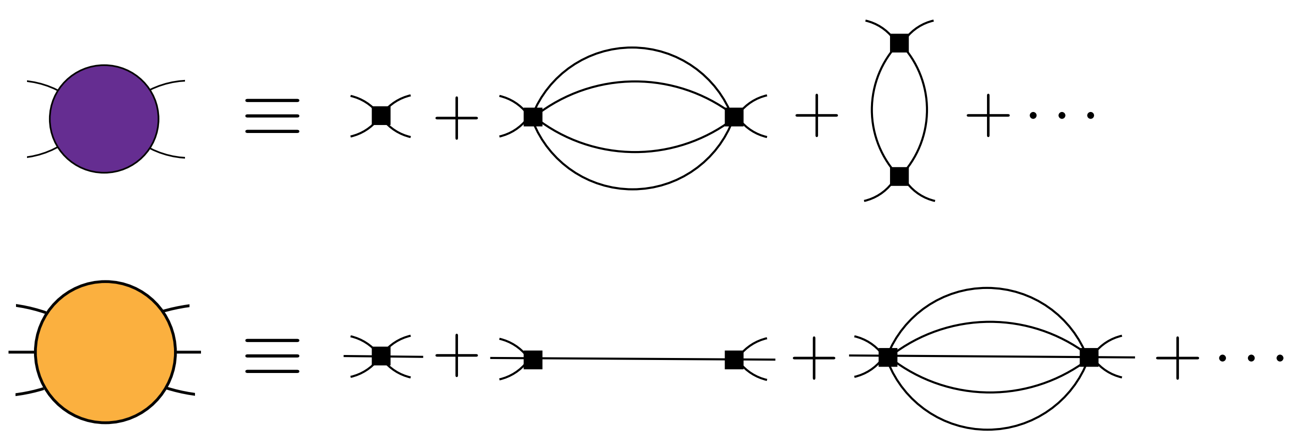
8 degrees of freedom

Skeleton expansion

$C_L(E, \vec{P}) =$



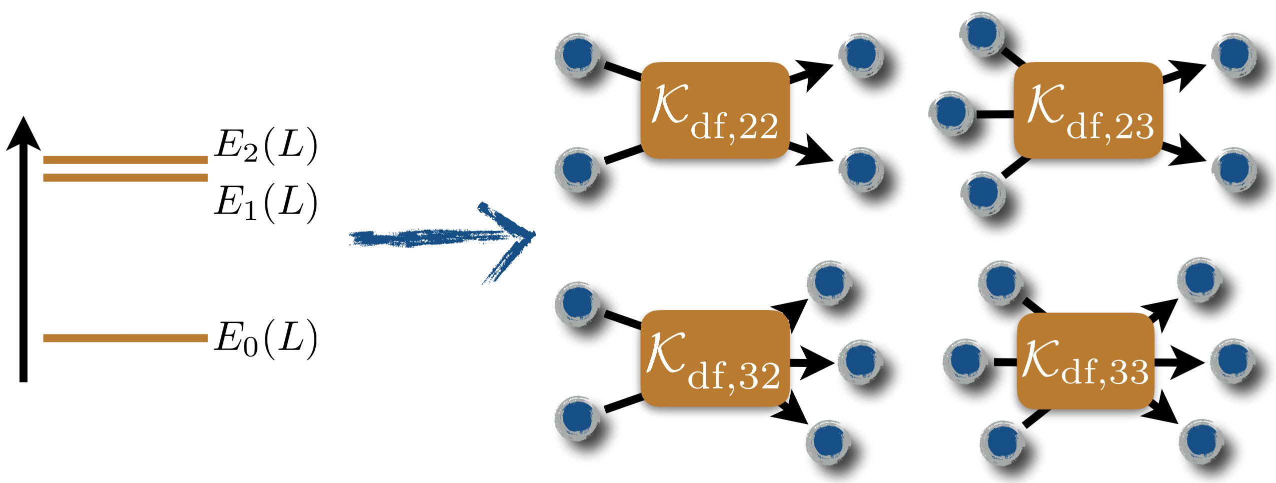
Kernel definitions:



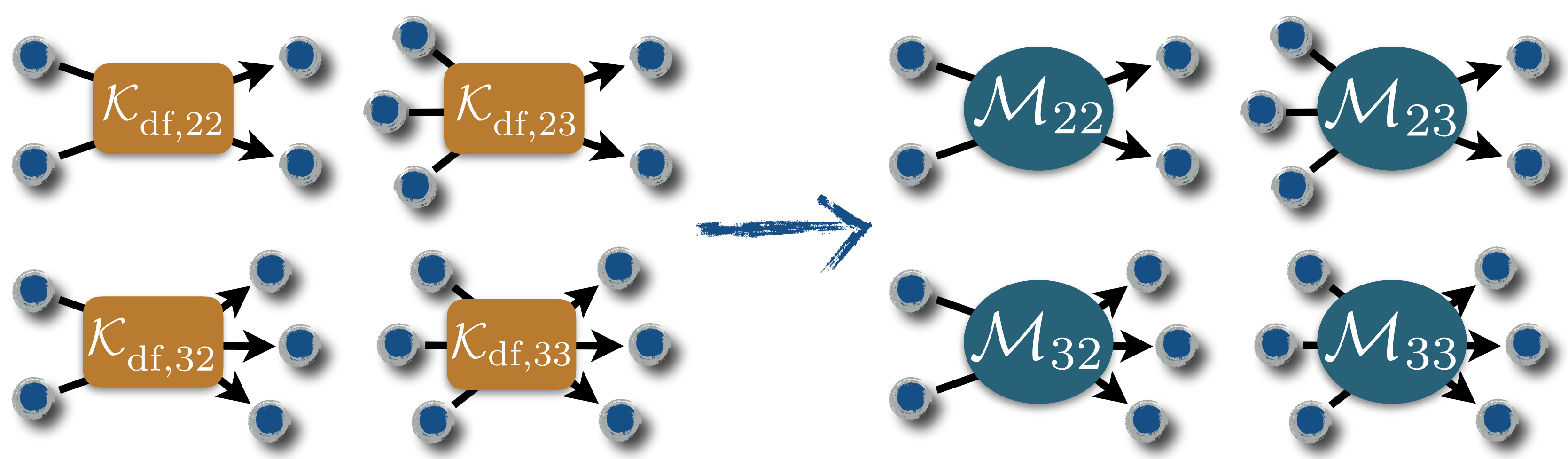
- All lines represent pions
- Boxes represent sums over finite-volume momenta

Current status:

Complete (model- & EFT-independent relation) between finite-volume energies and two-and-three particle scattering for identical scalars



$$\det\left[\begin{pmatrix}\mathcal{K}_{\mathrm{df},22} & \mathcal{K}_{\mathrm{df},23}\\ \mathcal{K}_{\mathrm{df},32} & \mathcal{K}_{\mathrm{df},33}\end{pmatrix}^{-1}+\begin{pmatrix}F_2 & 0\\ 0 & F_3\end{pmatrix}\right]=0$$

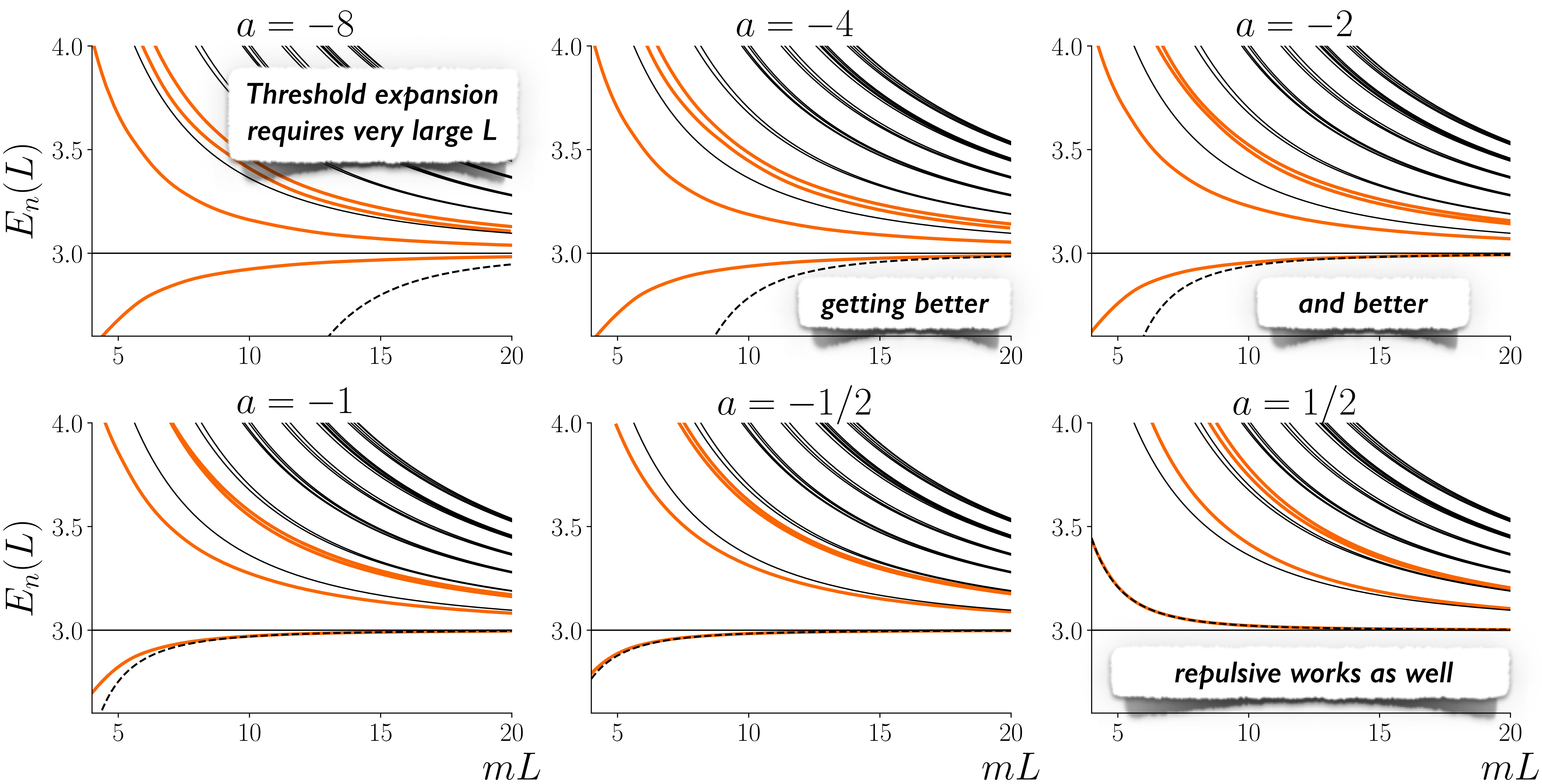


MTH, Sharpe (2014-2016) ○ **Briceño, MTH, Sharpe** (2017)

see also Hammer, Pang, Rusetsky (2017) ○ **Döring, Mai** (2017)

$\mathcal{K}^{\rm iso}_{\rm df,3}(E) = 0$ **solutions**

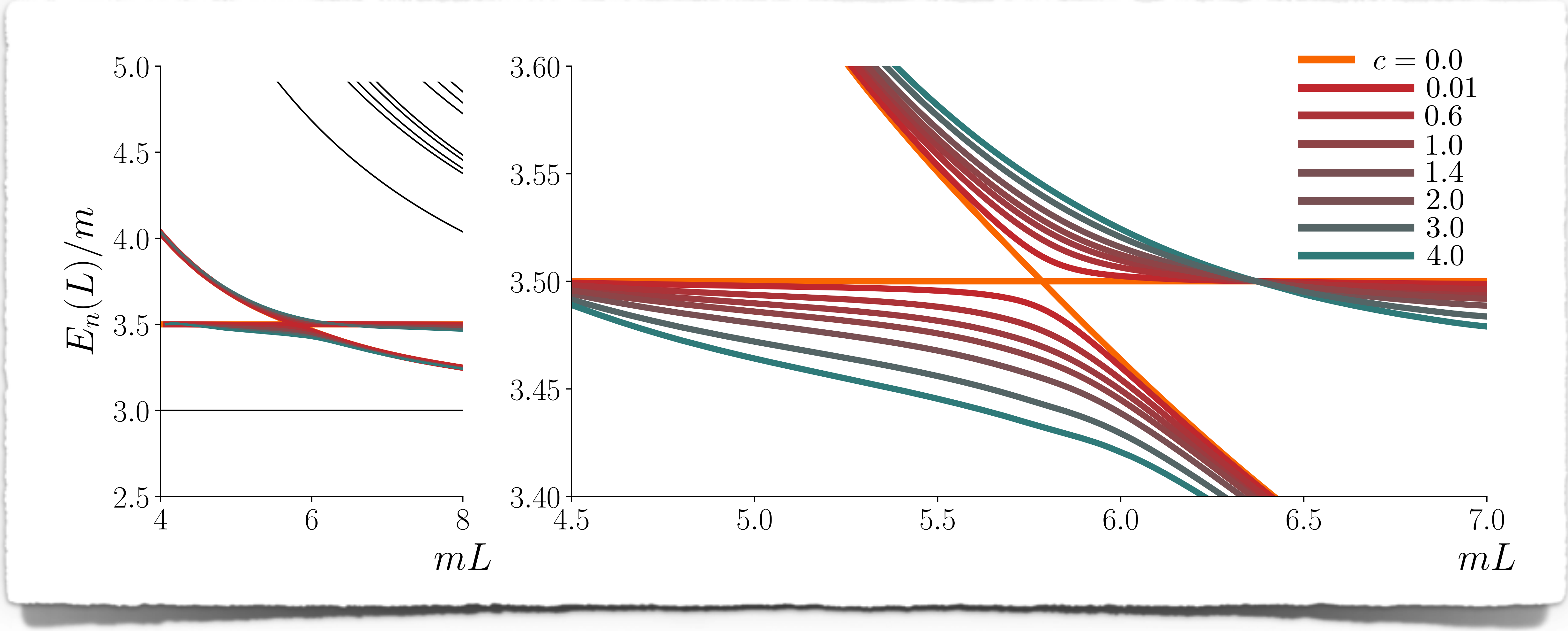
Straightforward to vary *a* and to study large volumes



Towards three-particle resonance extraction

- ❑ **Need to understand the finite-volume signature**

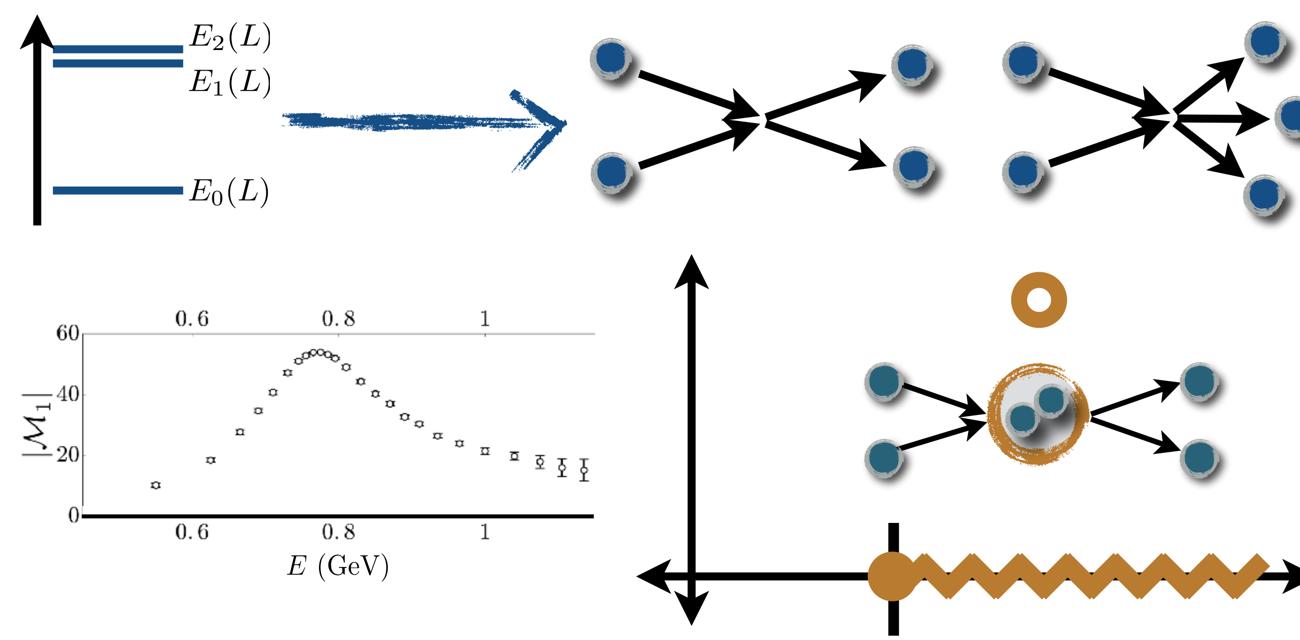
$$a = -10 \qquad \mathcal{K}_{\text{df},3}^{\text{iso}}(E) = -\frac{c \times 10^3}{E^2 - M_R^2}$$



Further investigation is needed to see if this gives a physical resonance description

Scattering in LQCD relies on relations between finite- and infinite-volume quantities

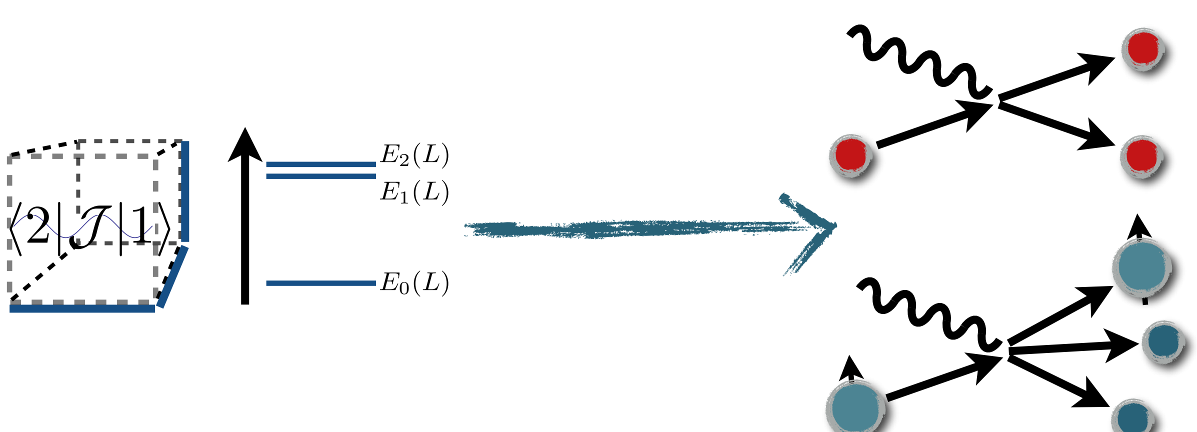
Energies and decay channels



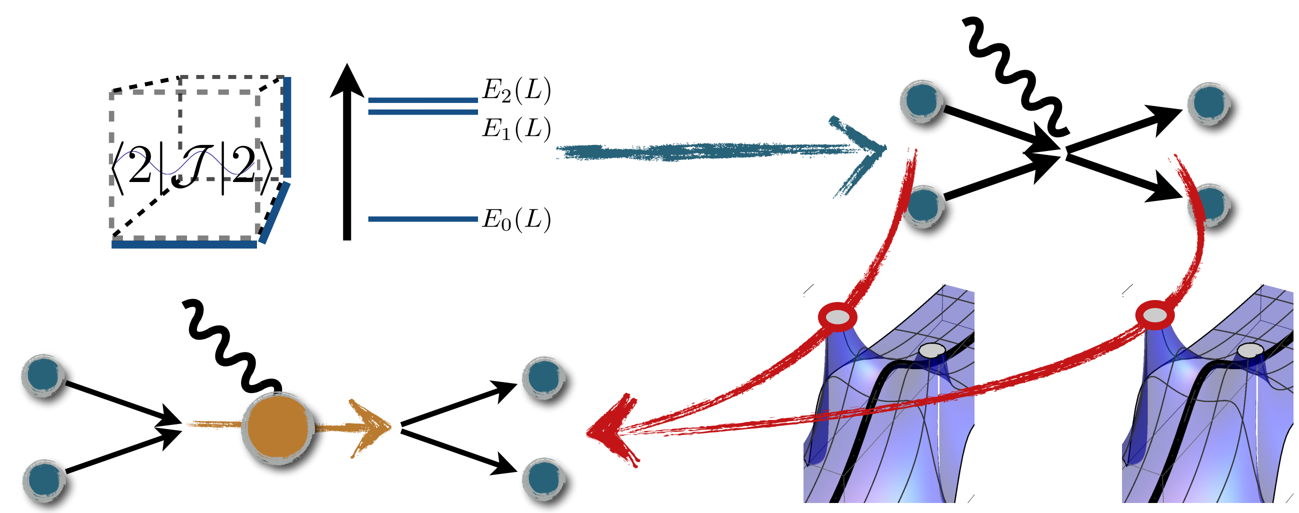
Finite-volume energies related to scattering

Transition amplitudes

Energies and finite-volume matrix elements are related to transitions



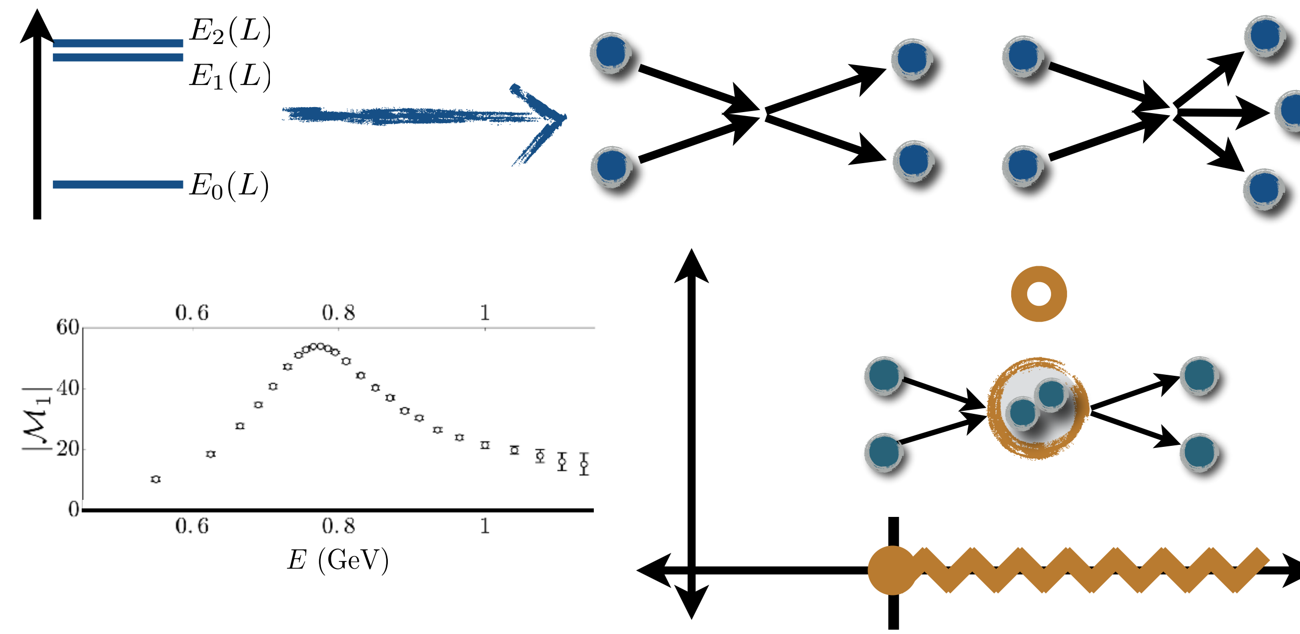
Resonant form factors



See talk by Raúl Briceño

Scattering in LQCD relies on relations between finite- and infinite-volume quantities

Energies and decay channels



Finite-volume energies related to scattering

- Using the finite-volume as a tool has proven to be a powerful approach
- The two-particle sector is increasingly under control
  (Many coupled channel scattering calculations already available)

- Stay tuned for three-particle scattering observables from LQCD

Thanks for your attention!