Multi-hadron observables from lattice QCD

Maxwell T. Hansen

Confinement XIII

August 5th, 2018
Scattering and Spectroscopy

Experiments worldwide are exploring the exotic resonance spectrum

‘exotic’ = outside the quark model

- glueballs
- tetraquarks
- hybrids
Scattering and Spectroscopy

Understanding these requires the full machinery of...

Quantum Chromodynamics (QCD)

Experiments worldwide are exploring the exotic resonance spectrum

‘exotic’ = outside the quark model

| glueballs | tetraquarks | hybrids |

Quarks and gluons (interactions constrained by symmetries)

Simple underlying structure leads to a rich variety of phenomena

Difficult to extract predictions from the underlying theory
Many resonances, many questions

Belle

$X \rightarrow \pi^+\pi^- J/\psi$

$X(3872)$

PRL 91, 262001 (2003)

BaBar

$Y(4260)$

PRL95, 142001 (2005)

see very nice talk from Ryan Mitchell
Many resonances, many questions

- How does this rich structure emerge from such a simple underlying theory?
- How do resonances quantitatively modify scattering and production rates?
- Why are some states well described by the quark model and others not?
- How do resonance properties depend on QCD’s fundamental parameters?

see very nice talk from Ryan Mitchell
Definition of a resonance

- Roughly speaking, a bump in $|\mathcal{M}(E)|^2 \propto |e^{2i\delta(E)} - 1|^2 \propto \sin^2 \delta(E)$

Scattering rate

Unitarity relation

Protopopescu et al. (1972)

$\pi\pi \rightarrow \pi\pi$

Pions
Definition of a resonance

- Roughly speaking, a bump in

\[ |M(E)|^2 \propto |e^{2i\delta(E)} - 1|^2 \propto \sin^2 \delta(E) \]

scattering rate

unitarity relation

Consider this curve in the complex plane

Protopopescu et al. (1972)

ππ → ππ
Definition of a resonance

- Roughly speaking, a bump in $|\mathcal{M}(E)|^2 \propto |e^{2i\delta(E)} - 1|^2 \propto \sin^2 \delta(E)$

scattering rate

unitarity relation

Protopopescu et al. (1972)

Consider this curve in the complex plane

Expand about some arbitrary point
Definition of a resonance

- Roughly speaking, a bump in \( |\mathcal{M}(E)|^2 \propto |e^{2i\delta(E)} - 1|^2 \propto \sin^2 \delta(E) \)

scattering rate

\[
\rho
\]

\( I^G(J^{PC}) = 1^+(1--) \)

unitarity relation

\[ \pi \pi \rightarrow \pi \pi \]

Consider this curve in the complex plane

Expand about some arbitrary point

Protopopescu et al. (1972)
Definition of a resonance

- Roughly speaking, a bump in $|\mathcal{M}(E)|^2 \propto |e^{2i\delta(E)} - 1|^2 \propto \sin^2 \delta(E)$

**scattering rate**

**unitarity relation**

Roughly speaking, a bump in $|\mathcal{M}(E)|^2 \propto |e^{2i\delta(E)} - 1|^2 \propto \sin^2 \delta(E)$

Consider this curve in the complex plane

We see a feature that limits the convergence of the Taylor expansion
Definition of a resonance

- Roughly speaking, a bump in $|\mathcal{M}(E)|^2 \propto |e^{2i\delta(E)} - 1|^2 \propto \sin^2 \delta(E)$

  **scattering rate**

  **unitarity relation**

  Roughly speaking, a bump in $E (\text{GeV})$

  Protopopescu et al. (1972)

  Analytic continuation reveals that the bump corresponds to a pole in the complex plane

  This bridges the gap between bound states and resonances

  $E_R = M_R + i\Gamma_R/2$

  $E_B = M_B$
Multiple Riemann sheets

- It is more useful to analytically continue the scattering amplitude
  \[ M_\ell(s) = \frac{1}{K_\ell(s)^{-1} + \rho(s)} \]
  \[ \rho(s) \propto i \sqrt{s - (2M_\pi)^2} \]

- Each two-particle channel generates a square-root branchcut and doubles the number of Riemann sheets
Towards a detailed, first-principles understanding of resonances

Energies and decay channels

- Locate complex poles in scattering amplitudes
- The residues at the poles measure couplings to multi-particle decay products
Towards a detailed, first-principles understanding of resonances

Transition amplitudes
- Measure how photons and other currents mediate exotic resonance production

Energies and decay channels
- Locate complex poles in scattering amplitudes
- The residues at the poles measure couplings to multi-particle decay products

Resonant form factors
- Predict how currents couple to the resonance

\[ |\text{Res}\rangle = a|\omega\rangle + b|\chi\rangle + c|\delta\rangle \]
Lattice QCD is a powerful tool for extracting QCD predictions

\[ \text{LQCD} = \text{evaluating a difficult integral numerically} \]

\[ \text{observable} = \int D\phi \ e^{iS} \left[ \text{interpolator for observable} \right] \]
Lattice QCD is a powerful tool for extracting QCD predictions

\[ \text{LQCD} = \text{evaluating a difficult integral numerically} \]

\[ \left( \begin{array}{c} \text{observable?} \\ \text{discretized, finite volume,} \\ \text{Euclidean, heavy pions} \end{array} \right) = \int \prod_{i}^{N} d\phi_i \ e^{-S} \left[ \text{interpolator} \right] \]

To do so we have to make three compromises

1. **nonzero lattice spacing**
2. **finite volume, \( L \)**
3. **Euclidean signature**

Also... **Unphysical pion masses** \( M_{\pi, \text{lattice}} > M_{\pi, \text{our universe}} \)

But calculations at the physical pion are becoming common
Difficulties for scattering

- The most important modification for scattering is the **finite volume**...
  - Discretizes the spectrum
  - Eliminates the branch cuts
  - Removes the second Riemann sheet
  - Hides the resonance poles
Difficulties for scattering

The most important modification for scattering is the **finite volume**...

- Discretizes the spectrum
- Eliminates the branch cuts
- Removes the second Riemann sheet
- Hides the resonance poles

Finite-volume analytic structure

Infinite-volume analytic structure
Observables available in LQCD

‘On the lattice’ one can calculate finite-volume energies and matrix elements

\[
\langle \mathcal{O}_j(\tau)\mathcal{O}_{i}^\dagger(0) \rangle = \sum_{n} \langle 0|\mathcal{O}_j(\tau)|E_n\rangle \langle E_n|\mathcal{O}_{i}^\dagger(0)|0 \rangle = \sum_{n} e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^* \]
Observables available in LQCD

‘On the lattice’ one can calculate finite-volume energies and matrix elements

\[
\langle \mathcal{O}_j(\tau) \mathcal{O}_i^\dagger(0) \rangle = \sum_n \langle 0 | \mathcal{O}_j(\tau) | E_n \rangle \langle E_n | \mathcal{O}_i^\dagger(0) | 0 \rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*
\]

A key technical breakthrough: **Distillation**

M. Peardon et al. 2009

the extraction of the essential meaning or most important aspects of something

Construct smeared quark propagators via Laplacian eigenvectors
Observables available in LQCD

‘On the lattice’ one can calculate finite-volume energies and matrix elements

\[
\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0) \rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle \langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^* 
\]

A key technical breakthrough: **Distillation**

M. Peardon et al. 2009

the extraction of the essential meaning or most important aspects of something

Construct smeared quark propagators via Laplacian eigenvectors

Can determine **optimized operators** by ‘diagonalizing’

the **correlator matrix** (GEVP)

This gives a method to determine **energies** and **matrix elements**

\[
\langle \Omega_m(\tau)\Omega_m^\dagger(0) \rangle \sim e^{-E_m(L)\tau} + \cdots \\
\langle \Omega_m'(\tau) \mathcal{J}(0) \Omega_m^\dagger(-\tau) \rangle \sim e^{-E_{m'}\tau} e^{-E_m\tau} \langle E_{m'}|\mathcal{J}(0)|E_m\rangle + \cdots 
\]
Scattering in LQCD relies on relations between finite- and infinite-volume quantities.
Scattering in LQCD relies on relations between finite- and infinite-volume quantities.

Energies and decay channels

Finite-volume energies related to scattering

Transition amplitudes

Energies and finite-volume matrix elements are related to transitions

Resonant form factors

See talk by Raúl Briceño
Scattering in LQCD relies on relations between finite- and infinite-volume quantities.

Transition amplitudes

Energies and finite-volume matrix elements are related to transitions.

Energies and decay channels

Finite-volume energies related to scattering

Resonant form factors

See talk by Raúl Briceño
Basic idea

- **Finite-volume set-up**

- **cubic**, spatial volume (extent \( L \))

- **periodic** boundary conditions

\[
\tilde{\mathbf{p}} = \frac{2\pi}{L} \tilde{\mathbf{n}}, \quad \tilde{\mathbf{n}} \in \mathbb{Z}^3
\]

- \( L \) is large enough to neglect

\[
e^{-M \pi L}
\]
Basic idea

- Finite-volume set-up

- **cubic**, spatial volume (extent $L$)

- **periodic** boundary conditions

\[ \vec{p} = \frac{2\pi}{L} \vec{n}, \quad \vec{n} \in \mathbb{Z}^3 \]

- $L$ is large enough to neglect $e^{-M\pi L}$

- Scattering observables leave an imprint on finite-volume quantities
Basic idea

- **Finite-volume set-up**
  - **cubic**, spatial volume (extent $L$)
  - **periodic boundary conditions**
    \[ \vec{p} = \frac{2\pi}{L} \vec{n}, \quad \vec{n} \in \mathbb{Z}^3 \]
  - $L$ is large enough to neglect $e^{-M_\pi L}$

- **Scattering observables leave an imprint on finite-volume quantities**
  
  Consider a **weakly-interacting, two-body system** with no bound states

\[ E_0 = 2M_\pi \]

Infinite-volume ground state

\[ \mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a \]

Information is in the scattering amplitude
Basic idea

- Finite-volume set-up
  - cubic, spatial volume (extent $L$)
  - periodic boundary conditions
    \[ \tilde{p} = \frac{2\pi}{L} \tilde{n}, \quad \tilde{n} \in \mathbb{Z}^3 \]
  - $L$ is large enough to neglect $e^{-M_\pi L}$

Scattering observables leave an imprint on finite-volume quantities

Consider a weakly-interacting, two-body system with no bound states

\[ E_0 = 2M_\pi \]

Infinite-volume ground state

\[ \mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a \]

Information is in the scattering amplitude

\[ E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4) \]

Huang, Yang (1958)

Finite-volume ground state
Scattering observables leave an imprint on finite-volume quantities

**Basic idea**

- **Finite-volume set-up**
  - cubic, spatial volume (extent $L$)
  - periodic boundary conditions
  \[ \mathbf{p} = \frac{2\pi}{L} \mathbf{n}, \quad \mathbf{n} \in \mathbb{Z}^3 \]
  - $L$ is large enough to neglect $e^{-M_\pi L}$

- **Scattering observables leave an imprint on finite-volume quantities**

Consider a **weakly-interacting, two-body system** with no bound states

\[ E_0 = 2M_\pi \]

Infinite-volume ground state

\[ M_{\ell=0}(2M_\pi) = -32\pi M_\pi a \]

Information is in the scattering amplitude

\[ E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + O(1/L^4) \]

Huang, Yang (1958)
Hint of the derivation

In the infinite-volume world…

\[ M_{\ell=0}(E_{\text{cm}}) = \circ \cdot \circ \cdot \cdot = -\lambda + O(\lambda^2) \]

Lines represent low-energy degrees of freedom (e.g. pions in QCD)
Hint of the derivation

- In the infinite-volume world...

\[ \mathcal{M}_{\ell=0}(E_{\text{cm}}) = \ldots = -\lambda + \mathcal{O}(\lambda^2) \rightarrow a = \frac{\lambda}{32\pi M_{\pi}} + \mathcal{O}(\lambda^2) \]

scattering length
Hint of the derivation

In the infinite-volume world...

\[ M_{\ell=0}(E_{\text{cm}}) = 0 + \cdots = -\lambda + \mathcal{O}(\lambda^2) \]

\[ a = \frac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2) \]

In the finite-volume world...

\[ M_L(E_{\text{cm}}) = 0 + \cdots \]

At leading order, the finite-volume amplitude has no poles
Hint of the derivation

In the infinite-volume world...
\[ M_{\ell=0}(E_{\text{cm}}) = -\lambda + O(\lambda^2) \rightarrow a = \frac{\lambda}{32\pi M_\pi} + O(\lambda^2) \]

In the finite-volume world...
\[ M_L(E_{\text{cm}}) = -\lambda - \frac{1}{2L^3} \sum_k \frac{1}{(2\omega_k)^2(E_{\text{cm}} - 2\omega_k)}\lambda + \cdots \]

At next-to-leading order, we see poles of two non-interacting particles
\[ \omega_k = \sqrt{k^2 + M_\pi^2} \quad \text{where} \quad k = \frac{2\pi}{L}n, \quad n \in \mathbb{Z}^3 \]
Hint of the derivation

- In the infinite-volume world...

\[ \mathcal{M}_{\ell=0}(E_{\text{cm}}) = -\lambda + \mathcal{O}(\lambda^2) \xrightarrow{\text{scattering length}} a = \frac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2) \]

- In the finite-volume world...

\[ \mathcal{M}_L(E_{\text{cm}}) = -\lambda - \lambda \frac{1}{2} \frac{1}{L^3} \sum_k \frac{1}{(2\omega_k)^2(E_{\text{cm}} - 2\omega_k)^2} \lambda + \cdots \]

\[ = -\lambda - \lambda \frac{1}{2} \frac{1}{L^3} \frac{1}{(2M_\pi)^2(E_{\text{cm}} - 2M_\pi)^2} \lambda + \cdots \]

The truncated series is failing because we are interested in \( E_{\text{cm}} - 2M_\pi = \mathcal{O}(\lambda) \)
Hint of the derivation

☐ In the infinite-volume world…

\[ \mathcal{M}_{\ell=0}(E_{\text{cm}}) = -\lambda + \mathcal{O}(\lambda^2) \rightarrow a = \frac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2) \]

☐ In the finite-volume world…

\[ \mathcal{M}_L(E_{\text{cm}}) = -\lambda - \lambda \frac{1}{2L^3} \sum_k \frac{1}{(2\omega_k)^2(E_{\text{cm}} - 2\omega_k)} + \cdots \]

\[ = -\lambda - \lambda \frac{1}{2L^3} \left( \frac{1}{(2M_\pi)^2(E_{\text{cm}} - 2M_\pi)} \right) + \cdots \]

\[ = -\lambda \sum_{n=0}^{\infty} [f(E_{\text{cm}}, L) \lambda^n] = \frac{1}{-1/\lambda + f(E_{\text{cm}}, L)} \]
Hint of the derivation

- In the infinite-volume world...
  \[ M_{\ell=0}(E_{\text{cm}}) = -\lambda + \mathcal{O}(\lambda^2) \quad \Rightarrow \quad a = \frac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2) \]

- In the finite-volume world...
  \[ M_L(E_{\text{cm}}) = -\lambda - \frac{1}{2} \frac{1}{L^3} \sum_k \frac{1}{(2\omega_k)^2(E_{\text{cm}} - 2\omega_k)} \lambda + \cdots \]
  \[ = -\lambda - \frac{1}{2} \frac{1}{L^3} \frac{1}{(2M_\pi)^2(E_{\text{cm}} - 2M_\pi)} \lambda + \cdots \]
  \[ = -\lambda \sum_{n=0}^{\infty} \lambda^n = \frac{1}{-1/\lambda + f(E_{\text{cm}}, L)} \]

Summing this **singular contribution** to all orders gives the final expression...

\[ -1/\lambda + f(E_{\text{cm}}, L) = 0 \quad \Rightarrow \quad E_{\text{cm}} = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4) \]

- This result can be generalized dramatically...
Two-to-two scattering

- Lüscher’s formalism + extensions give a general mapping

\[
\det \left[ M_2^{-1}(E^*_n) + F(E_n, \vec{P}, L) \right] = 0
\]

scattering amplitude known geometric function

Matrices in angular momentum, spin and channel space

- All results are contained in a generalized quantization condition
Two-to-two scattering

- Lüscher’s formalism + extensions give a general mapping

\[
\det \left[ M_2^{-1}(E^*_n) + F(E_n, \vec{P}, L) \right] = 0
\]

scattering amplitude known geometric function

Matrices in angular momentum, spin and channel space

- All results are contained in a generalized quantization condition

- Varying \( E, \vec{P} \) gives more constraints on functions of \( E^* = \sqrt{E^2 - \vec{P}^2} \)

- Derivation ignores (drops) suppressed volume effects (\( e^{-M_\pi L} \))

Using the result

- Simplest case is a single channel
  (e.g. for pions in a p-wave the relation reduces to)
Using the result

- Simplest case is a single channel
  (e.g. for pions in a p-wave the relation reduces to)

\[ \mathcal{M}_2(E_n^*) = -1/F(E_n, \vec{P}, L) \]

\[ \mathcal{M}_2 \propto e^{2i\delta} - 1 \]

\[ \pi \pi \rightarrow \pi \pi \]

Using the result

- Simplest case is a single channel
  (e.g. for pions in a p-wave the relation reduces to)

\[ \mathcal{M}_2(E_n^*) = -\frac{1}{F(E_n, \vec{P}, L)} \]

\[ \mathcal{M}_2 \propto e^{2i\delta} - 1 \]

Using the result

- Simplest case is a single channel (e.g. for pions in a p-wave the relation reduces to)

\[ \mathcal{M}_2(E_n^*) = -\frac{1}{F(E_n, \vec{P}, L)} \]

\[ \mathcal{M}_2 \propto e^{2i\delta} - 1 \]

Using the result

- Simplest case is a single channel (e.g. for pions in a p-wave the relation reduces to)

\[ M_2(E_n^*) = -1/F(E_n, \vec{P}, L) \]

Using the result

☐ Simplest case is a single channel (e.g. for pions in a p-wave the relation reduces to)

\[ \mathcal{M}_2(E_n^*) = -1/F(E_n, \mathbf{P}, L) \]

\[ \mathcal{M}_2 \propto e^{2i\delta} - 1 \]

\[ \pi \pi \rightarrow \pi \pi \]

\[ N_f = 2 + 1, \quad M_\pi \sim 240 \text{ and } 390 \text{ MeV}, \]
\[ M_\pi L \sim 4 - 6, \]
\[ a_s \sim 0.12 \text{ fm, } \quad a_t \sim 0.035 \text{ fm} \]

*from Wilson, Briceño, Dudek, Edwards, Thomas, Phys. Rev. D 92, 094502 (2015)*
\begin{align*}
N_f &= 2 + 1, \quad M_\pi \sim 240 \text{ and } 390 \text{ MeV}, \\
M_\pi L &\sim 4 - 6, \\
a_s &\sim 0.12 \text{ fm}, \quad a_t \sim 0.035 \text{ fm}
\end{align*}

\textit{from Wilson, Briceño, Dudek, Edwards, Thomas, Phys. Rev. D 92, 094502 (2015)}

Lin et al. (2009)
Dudek, Edwards, Guo, Thomas (2013)
Dudek, Edwards, Thomas (2012)
Wilson, et. al. (2015)

\textit{plot from R. Briceño}
Quantitatively trace the pole position in the complex plane

\[ N_f = 2 + 1, \quad M_\pi \sim 240 \text{ and } 390 \text{ MeV}, \]
\[ M_\pi L \sim 4 - 6, \]
\[ a_s \sim 0.12 \text{ fm}, \quad a_t \sim 0.035 \text{ fm} \]

*from Wilson, Briceño, Dudek, Edwards, Thomas, Phys. Rev. D 92, 094502 (2015)*
Coupled channels

- The cubic volume mixes different partial waves...

\[
\begin{align*}
K \pi & \rightarrow K \pi, \quad \vec{P} \neq 0 \\
\det & \left[ \begin{pmatrix} \mathcal{M}_s^{-1} & 0 \\ 0 & \mathcal{M}_p^{-1} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix} \right] = 0
\end{align*}
\]

*e.g.*
Coupled channels

The cubic volume mixes different partial waves...

\[ \begin{align*}
&\text{e.g. } \ K\pi \rightarrow K\pi \quad \vec{P} \neq 0 \quad \rightarrow \quad \text{det} \left( \begin{pmatrix} M_s^{-1} & 0 \\ 0 & M_p^{-1} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix} \right) = 0
\end{align*} \]

...as well as different flavor channels...

\[ \begin{align*}
&\text{e.g. } \ a = \pi\pi \quad b = K\bar{K} \quad \rightarrow \quad \text{det} \left( \begin{pmatrix} M_{a\rightarrow a} & M_{a\rightarrow b} \\ M_{b\rightarrow a} & M_{b\rightarrow b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix} \right) = 0
\end{align*} \]
Coupled channels

- The cubic volume mixes different partial waves...
  
  e.g. \( K\pi \to K\pi \)
  \( \vec{P} \neq 0 \)
  \[
  \det \left[ \begin{pmatrix} \mathcal{M}_s^{-1} & 0 \\ 0 & \mathcal{M}_p^{-1} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix} \right] = 0
  \]

  ...as well as different flavor channels...
  
  e.g. \( a = \pi\pi \)
  \( b = K\overline{K} \)
  \[
  \det \left[ \begin{pmatrix} \mathcal{M}_{a\to a} & \mathcal{M}_{a\to b} \\ \mathcal{M}_{b\to a} & \mathcal{M}_{b\to b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix} \right] = 0
  \]

- The road to physics...

Calculate a matrix of correlators with a large & varied operator basis

\[ \langle O_a(\tau) O_b^\dagger(0) \rangle \]

Diagonalize (GEVP) to reliably extract
finite-volume energies

\[ \langle \Omega_m(\tau) \Omega_m^\dagger(0) \rangle \sim e^{-E_m(L)\tau} \]

Vary \( L \) and \( P \) to recover a dense set of energies

\[ [000], A_1 \]
\[ [001], A_1 \]
\[ [011], A_1 \]

\[ E_n(L) \]
Coupled channels

- The cubic volume mixes different partial waves...
  
  \[
  \det \left[ \begin{pmatrix} M_s^{-1} & 0 \\ 0 & M_p^{-1} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix} \right] = 0
  \]
  
  ...as well as different flavor channels...

  \[
  \det \left[ \begin{pmatrix} M_{a \rightarrow a} & M_{a \rightarrow b} \\ M_{b \rightarrow a} & M_{b \rightarrow b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix} \right] = 0
  \]

- The road to physics...
  
  Calculate a matrix of correlators with a large & varied operator basis

  \[
  \langle O_a(\tau) O_b^\dagger(0) \rangle
  \]

  Diagonalize (GEVP) to reliably extract finite-volume energies

  \[
  \langle \Omega_m(\tau) \Omega^\dagger_m(0) \rangle \sim e^{-E_m(L)\tau}
  \]

  Vary \( L \) and \( P \) to recover a dense set of energies

  \[
  E_n(L)
  \]

Identify a broad list of K-matrix parametrizations

- Polynomials
- EFT based
- Dispersion theory based

\[\text{had\textsuperscript{spec}}\]
Coupled channels

- The cubic volume mixes different partial waves...
  - e.g. $K \pi \rightarrow K \pi$
  - $P \neq 0$ 
  - $\det \left[ \begin{pmatrix} M_s^{-1} & 0 & 0 \\ 0 & M_p^{-1} & 0 \\ 0 & 0 & M_{ps} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix} \right] = 0$

- ...as well as different flavor channels...
  - e.g. $a = \pi \pi$
  - $b = K \overline{K}$
  - $\det \left[ \begin{pmatrix} (M_{a\rightarrow a} & M_{a\rightarrow b} \\ M_{b\rightarrow a} & M_{b\rightarrow b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix} \right] = 0$

The road to physics...

- Calculate a matrix of correlators with a large & varied operator basis
  - $\langle O_a(\tau)O_b^\dagger(0) \rangle$

- Diagonalize (GEVP) to reliably extract finite-volume energies
  - $\langle \Omega_m(\tau)\Omega_m^\dagger(0) \rangle \sim e^{-E_m(L)\tau}$

- Vary $L$ and $P$ to recover a dense set of energies
  - [000], $A_1$
  - [001], $A_1$
  - [011], $A_1$
  - $E_n(L)$

- Identify a broad list of K-matrix parametrizations
  - polynomials
  - EFT based
  - dispersion theory based

- Perform global fits to the finite-volume spectrum
Lots of activity!

\[ \rho \rightarrow \pi \pi \]

- CP-PACS/PACS-CS 2007, 2011
- ETMC 2010
- Lang et al. 2011
- Pellisier 2012
- RQCD 2015
- Guo et al. 2016
- Fu et al. 2016
- Bulava et al. 2016
- Alexandrou et al. 2017

\[ K^*(892) \]

\[ \kappa \rightarrow K \pi \]

\[ K^* \rightarrow K \pi \]

- Lang et al. 2012
- Prelovsek et al. 2013
- Wilson et al. 2015
- RQCD 2015
- Brett et al. 2018

\[ \kappa \rightarrow K \pi \]

\[ J^P = 1^- \]

- Prelovsek et al. 2010
- Fu 2013
- Wakayama 2015
- Howarth and Giedt 2017
- Briceño et al. 2017

\[ a_0(980) \rightarrow \pi \eta, K \bar{K} \]

- Dudek et al. 2016

\[ \sigma, f_0, f_2 \rightarrow \pi \pi, K \bar{K}, \eta \eta \]

- Briceño et al. 2017

See the recent review by Briceño, Dudek and Young.
\[ \rho \rightarrow \pi \pi \]

\[ I^G(J^{PC}) = 1^+(1^{--}) \]

Leskovec et al. (2017)
\[ N_f=2+1, m_\pi=316 \text{ MeV} \]

Guo et al. 2016
\[ N_f=2, m_\pi=226 \text{ MeV} \]

Black: \( L=3.12 \text{ fm}, a=0.065 \text{ fm} \)
Grey: \( L=3.65 \text{ fm}, a=0.076 \text{ fm} \)

Black: \( L=3.12 \text{ fm}, a=0.065 \text{ fm} \)
Grey: \( L=3.65 \text{ fm}, a=0.076 \text{ fm} \)

Preliminary
$I^G(J^{PC}) = 0^+(0^{++})$

Coupled-channel scattering!

$I^G(J^{PC}) = 0^+(0^{++})$

Coupled-channel scattering!

Tracking the pole position

$I^G(J^{PC}) = 0^+(0^{++})$

Coupled-channel scattering!

Tracking the pole position

There’s so much more!...

- Much more activity in the light-quark sector
  - e.g. first complete determination of the scalar and tensor nonets

\[ \pi\pi, K\eta, \eta: \text{Briceño, Dudek, Edwards - PRL (2017)} \]
\[ \eta\eta: \text{Briceño, Dudek, Edwards - arXiv (2017)} \]
\[ K\pi, K\eta: \text{Dudek, Edwards, Thomas, Wilson - PRL (2015)} \]
\[ \pi\eta, KK: \text{Dudek, Edwards, Wilson - PRD (2016)} \]

- Recent result for $K^*(892)$ from Brett et al. 2018
There’s so much more!...

- Scattering calculations are also being performed in the charm sector!
  - e.g. $I = 0$, $DK \rightarrow DK$ scattering examining the $D_{s0}^*(2317)$
    - Lang et al. (2014)
    - Bali et al. (2017)
  - e.g. $I = 1/2$, $D\pi$, $D\eta$, $D_s\bar{K}$ scattering examining the $D_0^*(2400)$
    - Moir, Peardon, Ryan, Thomas & Wilson (2016)
There's so much more!...

- Scattering calculations are also being performed in the charm sector!
  - e.g. $I = 0, DK \to DK$ scattering examining the $D_{s0}^*(2317)$
    - Lang et al. (2014)
    - Bali et al. (2017)
  - e.g. $I = 1/2, D\pi, D\eta, D_s\bar{K}$ scattering examining the $D_0^*(2400)$
    - Moir, Peardon, Ryan, Thomas & Wilson (2016)

- Also significant progress in nucleon-meson and nucleon-nucleon scattering

Talks at this workshop from...

Evan Berkowitz, Raúl Briceño, Will Detmold, Jozef Dudek, Ciaran Hughes, Nilmani Mathur, Daniel Mohler, Amy Nicholson, Phiala Shanahan, Christopher Thomas
Scattering in LQCD relies on relations between finite- and infinite-volume quantities.

Transition amplitudes
Energies and finite-volume matrix elements are related to transitions.

Energies and decay channels
Finite-volume energies related to scattering.

Resonant form factors
The aim is to derive a formalism for studying relativistic two- and three-particle systems from lattice QCD.
The aim is to derive a formalism for studying relativistic two- and three-particle systems from lattice QCD

Potential applications...

- Studying three-particle resonances
  
  $\omega(782), \ a_1(1420) \rightarrow \pi\pi\pi$
  
  $\eta(1405) \rightarrow a_0(980)\pi$
  
  $\pi_1(1400) \rightarrow ?$
  
  $\eta(1475) \rightarrow K^*(892)\bar{K}$
  
  $N(1440) \rightarrow N\pi, N\pi\pi$

- Calculating weak decays, form factors and transitions

  $K \rightarrow \pi\pi\pi$

  $N\gamma^* \rightarrow N\pi\pi$
First consider identical scalar particles with a $\mathbb{Z}_2$ symmetry
First consider identical scalar particles with a $Z_2$ symmetry

The three-to-three scattering amplitude has \textit{kinematic singularities}

$$iM_{3\rightarrow 3} \equiv \text{fully connected correlator with}$$
$$\text{six external legs amputated and projected on shell}$$
First consider identical scalar particles with a $Z_2$ symmetry

The three-to-three scattering amplitude has \textit{kinematic singularities}

$$i\mathcal{M}_{3\rightarrow3} \equiv$$

fully connected correlator with six external legs amputated and projected on shell

$$= \quad \text{Certain external momenta put this on-shell!}$$
First consider identical scalar particles with a $Z_2$ symmetry

The three-to-three scattering amplitude has *kinematic singularities*

\[ i \mathcal{M}_{3 \rightarrow 3} \equiv \text{fully connected correlator with} \]
\[ \text{six external legs amputated and projected on shell} \]
\[ = \]
\[ \text{Certain external momenta put this on-shell!} \]

The three-to-three scattering amplitude has *more degrees of freedom*
First consider identical scalar particles with a $Z_2$ symmetry.

The three-to-three scattering amplitude has **kinematic singularities**

$$iM_{\lambda\gamma} \equiv \text{fully connected correlator with six external legs amputated and projected on shell}$$

$$= \text{Certain external momenta put this on-shell!}$$

The three-to-three scattering amplitude has **more degrees of freedom**

- 12 momentum components
- 10 Poincaré generators

2 degrees of freedom
First consider identical scalar particles with a $Z_2$ symmetry.

The three-to-three scattering amplitude has *kinematic singularities*

$$i M_{3 \to 3} \equiv \text{fully connected correlator with six external legs amputated and projected on shell}$$

$$= \text{Certain external momenta put this on-shell!}$$

The three-to-three scattering amplitude has *more degrees of freedom*

12 momentum components 
-10 Poincaré generators 
2 degrees of freedom 

18 momentum components 
-10 Poincaré generators 
8 degrees of freedom
Skeleton expansion

$$C_L(E, \vec{P}) = \cdots + \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{kernel1} \\ + \ \begin{array}{c} \includegraphics[width=0.1\textwidth]{kernel2} \\ + \ \begin{array}{c} \includegraphics[width=0.1\textwidth]{kernel3} \\ + \ \begin{array}{c} \includegraphics[width=0.1\textwidth]{kernel4} \\ + \ \begin{array}{c} \includegraphics[width=0.1\textwidth]{kernel5} \end{array} \end{array} \end{array} \end{array} \end{array} + \cdots \right.$$
Current status:

Complete (model- & EFT-independent relation) between finite-volume energies and two-and-three particle scattering for identical scalars

\[
\det \left[ \begin{pmatrix} \mathcal{K}_{df,22} & \mathcal{K}_{df,23} \\ \mathcal{K}_{df,32} & \mathcal{K}_{df,33} \end{pmatrix}^{-1} + \begin{pmatrix} F_2 & 0 \\ 0 & F_3 \end{pmatrix} \right] = 0
\]

see also Hammer, Pang, Rusetsky (2017)  Döring, Mai (2017)
\[ \mathcal{K}_{df,3}^{\text{iso}}(E) = 0 \] solutions

Straightforward to vary \( a \) and to study large volumes

Threshold expansion requires very large \( L \)

Getting better and better

Repulsive works as well
Towards three-particle resonance extraction

Need to understand the finite-volume signature

\[ a = -10 \quad \mathcal{K}_{\text{iso,3}}^{\text{df}}(E) = -\frac{c \times 10^3}{E^2 - M_R^2} \]

Further investigation is needed to see if this gives a physical resonance description
Scattering in LQCD relies on relations between finite- and infinite-volume quantities.

Transition amplitudes

Energies and finite-volume matrix elements are related to transitions.

Energies and decay channels

Finite-volume energies related to scattering

Resonant form factors

See talk by Raúl Briceño
Scattering in LQCD relies on relations between finite- and infinite-volume quantities.

Using the finite-volume as a tool has proven to be a powerful approach.

The two-particle sector is increasingly under control.

(Many coupled channel scattering calculations already available)

Stay tuned for three-particle scattering observables from LQCD.

Thanks for your attention!