Axion properties at zero and finite T from Lattice QCD

$SU(3)$ with light fermions at or close to the physical point

- Trunin, Burger, Ilgenfritz, Lombardo, Müller-Preussker
  J. Phys. Conf. Ser. **668**, 012123 (2016) [arXiv:1510.02265 [hep-lat]].
- Bonati, D'Elia, Mariti, Martinelli, Mesiti, Negro, Sanfilippo, Villadoro
  JHEP **1603**, 155 (2016) [arXiv:1512.06746 [hep-lat]].
- Petreczky, Schadler, Sharma
  Phys. Lett. B **762**, 498 (2016) [arXiv:1606.03145 [hep-lat]].
- Borsanyi, Fodor, Kampert, Katz, Kawanai, Kovacs, Mages, Pasztor, Pittler, Redondo, Ringwald, Szabo
  Nature **539**, 69 (2016) [arXiv:1606.07494 [hep-lat]].
- Burger, Ilgenfritz, Lombardo, Trunin
  arXiv:1805.06001 [hep-lat].

C. Bonati, M. D'Elia, G. Martinelli, F. Negro, F. Sanfilippo, A. Todaro **+ .....**
arXiv:1807.07954 [hep-lat]

At zero and non zero temperature T the Axion Potential can be derived from the dependence of the vacuum free energy F[ϑ,T] on ϑ

The general form of $F(\theta, T)$

$$F(\theta, T) = -\frac{1}{V_4} \log \int [\mathscr{D}A][\mathscr{D}\bar{\psi}][\mathscr{D}\psi] \exp\left(-\int_0^{1/T} \mathrm{d}t \int \mathrm{d}^3x\, \mathcal{L}_\theta^E\right)$$

Assuming analiticity at $\theta = 0$ the free energy density can be written as:

$$F(\theta, T) - F(0, T) = \frac{1}{2}\chi(T)\theta^2\left[1 + b_2(T)\theta^2 + b_4(T)\theta^4 + \cdots\right],$$

and it is easy to see that

$$\chi = \frac{1}{V_4}\langle Q^2\rangle_0 \qquad b_2 = -\frac{\langle Q^4\rangle_0 - 3\langle Q^2\rangle_0^2}{12\langle Q^2\rangle_0}$$

Thus for example the axion mass is related to the topological susceptibility

$$m_\phi^2 = \frac{1}{f_\phi^2}\chi_t = \frac{1}{f_\phi^2}\int d^4x\, \langle Q(x)Q(0)\rangle$$

AT ZERO TEMPERATURE WE DO NOT NEED THE LATTICE !

Using the Ward ids, expanding at small quark masses and saturating the T-products with Goldston boson intermediate states (or using χPT) we get

$$\chi_t = -\frac{3}{4} f_\pi^2 \left(m_\pi^2 + m_\eta^2\right) \frac{m_u m_d m_s}{(m_u + m_d + m_s)(m_u m_d + m_d m_s + m_u m_s)}$$

- The result depends on the number of flavours
- The topological susceptibility, and consequently the mass of the axion, vanishes whenever the mass of a quark is equal to zero

P. Di Vecchia and G. Veneziano, Giovanni Grilli di Cortona, Edward Hardy, Javier Pardo Vega, A. Shifman, A. I. Vainshtein and V. I. Zakharov ; W. A. Bardeen and S.-H.H. Tye, , Giovanni Villadoro et al.

Dilute Instanton Gas Approximation

At very high T ($T \gg \Lambda_{QCD}$) one can show that the θ dependence is dominated by weakly interacting objects of topological charge ± 1 and the free energy is given by (Gross, Pisarski, Yaffe 1981)

$$F(\theta, T) - F(0, T) \approx \chi(T)(1 - \cos\theta)$$

so that

$$b_2 = -\frac{1}{12} \quad b_4 = \frac{1}{360} \quad b_{2n} = (-1)^n \frac{2}{(2n+2)!}$$

and the susceptibility scales with the temperature as following

$$\chi(T) \propto m^{N_f} T^{4 - \frac{11}{3}N - \frac{1}{3}N_f}$$

when N_f light flavours are present.

THE AXION AT NON ZERO TEMPERATURE
That is WHEN LQCD ENTERS THE GAME

Chiral Lagrangians allow to study the temperature dependence of the axion potential and its mass to finite temperatures below the crossover region Tc ~150 MeV.

Around Tc there is no known reliable perturbative expansion under control and non-perturbative methods, such as lattice QCD are required.

S. Borsanyi et al., Phys. Lett. B 752, 175 (2016) [arXiv:1508.06917 [hep-lat]]
C. Bonati, M. D'Elia, H. Panagopoulos and E. Vicari, Phys. Rev. Lett. 110, 252003 (2013) [arXiv:1301.7640 [hep-lat]].
C. Bonati, JHEP 1503, 006 (2015) [arXiv:1501.01172 [hep-lat]]
G. Y. Xiong, J. B. Zhang, Y. Chen, C. Liu, Y. B. Liu and J. P. Ma, Phys. Lett. B 752, 34 (2016) [arXiv:1508.07704
A. Trunin, F. Burger, E.-M. Ilgenfritz, M. P. Lombardo and M. Muller-Preussker, arXiv:1510.02265 [hep-lat].
M. I. Buchoff et al., Phys. Rev. D 89, 054514 (2014) [arXiv:1309.4149 [hep-lat]].

HOWEVER

i) $\chi(T)$ is usually determined from the probability distribution $P(Q)$ and in particular from its variance, $\chi(T) = \langle Q^2 \rangle / V$

The suppression of $\chi(T)$ at large T implies that, on volumes reasonably achievable in lattice simulations, one has $\langle Q^2 \rangle = V \chi(T) \ll 1$.

Since Q is integer valued, that means that configurations with $Q \neq 0$ become very rare, leading to the need of unaffordably large statistics in order to achieve a correct sampling of the topological charge distribution.

HOWEVER

ii) Topology is strongly correlated with the chiral properties of the theory (the θ -dependence would be strictly zero in the presence of massless quarks)

The explicit chiral symmetry breaking that is present in most fermion discretizations can lead to significant lattice artifacts, thus requiring very small lattice spacings to achieve a reliable continuum extrapolation

HOWEVER

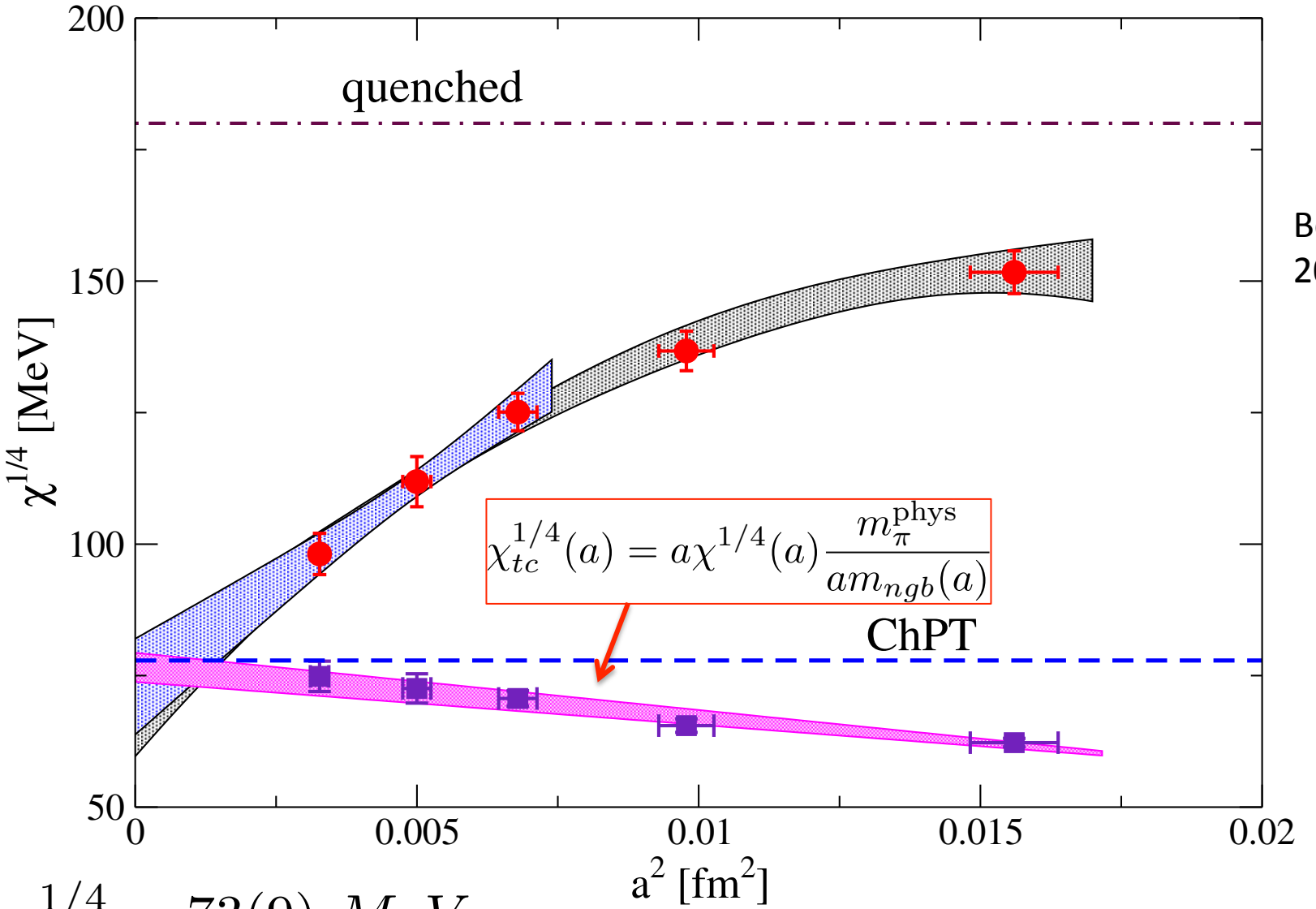
iii) Closer to the continuum limit, because of the topological nature of the problem, standard updating algorithms fail to correctly sample the distribution of Q and get trapped in path integral sectors with fixed topology.

The freezing of topological charge leads to a severe critical slowing down of numerical simulations

HOWEVER

iv) Finally, since finite temperature numerical simulations are usually performed with $L_s^3 \times N_t$ at fixed L_s/N_t , as $T = 1/(N_t a)$ becomes large also the physical spatial volume $a^3 L_s^3$ becomes small, so that the possible presence of finite volume effects should be checked.

ii) Large Cutoff (Discretization) Effects

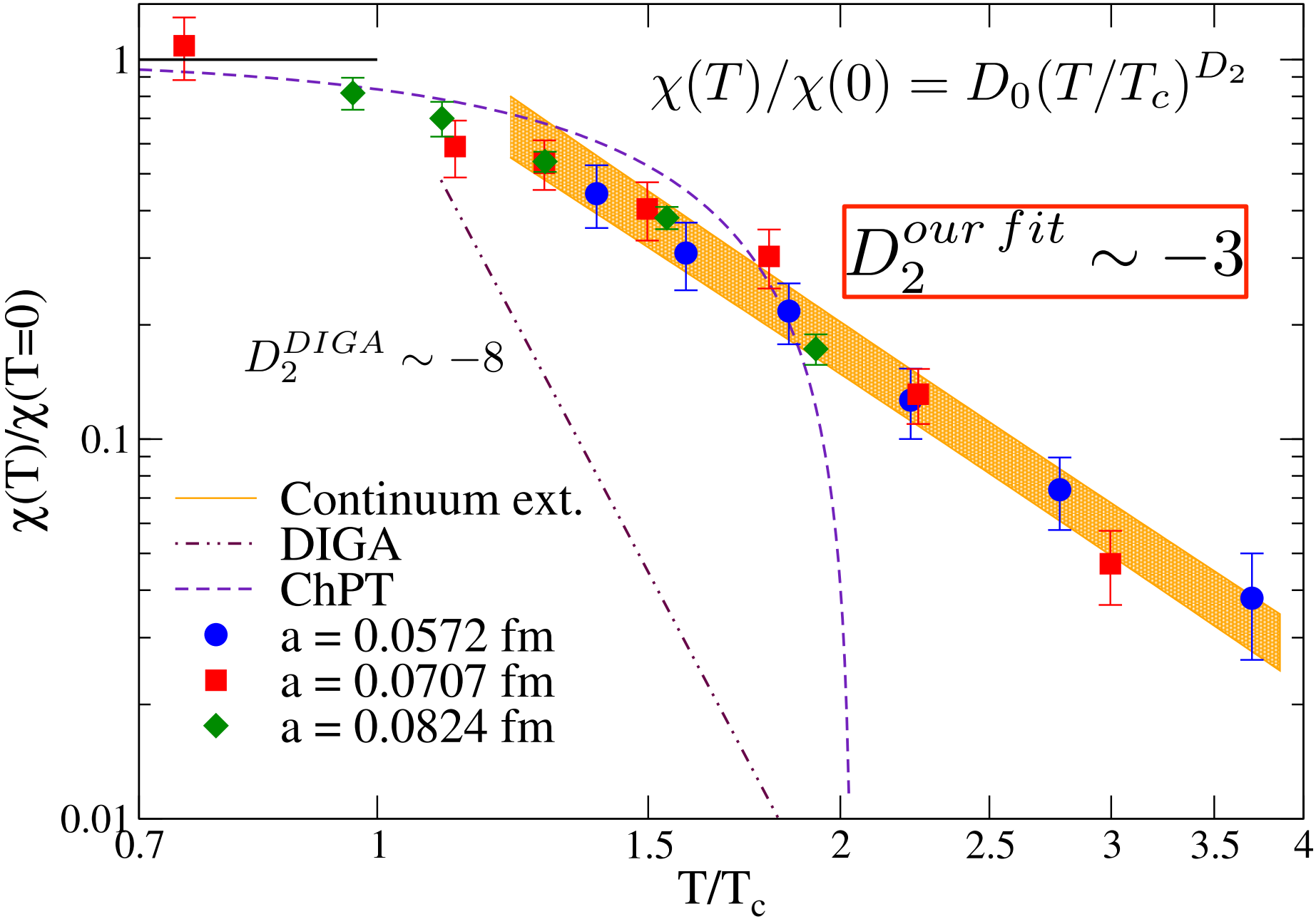


Bonati et al. 2015

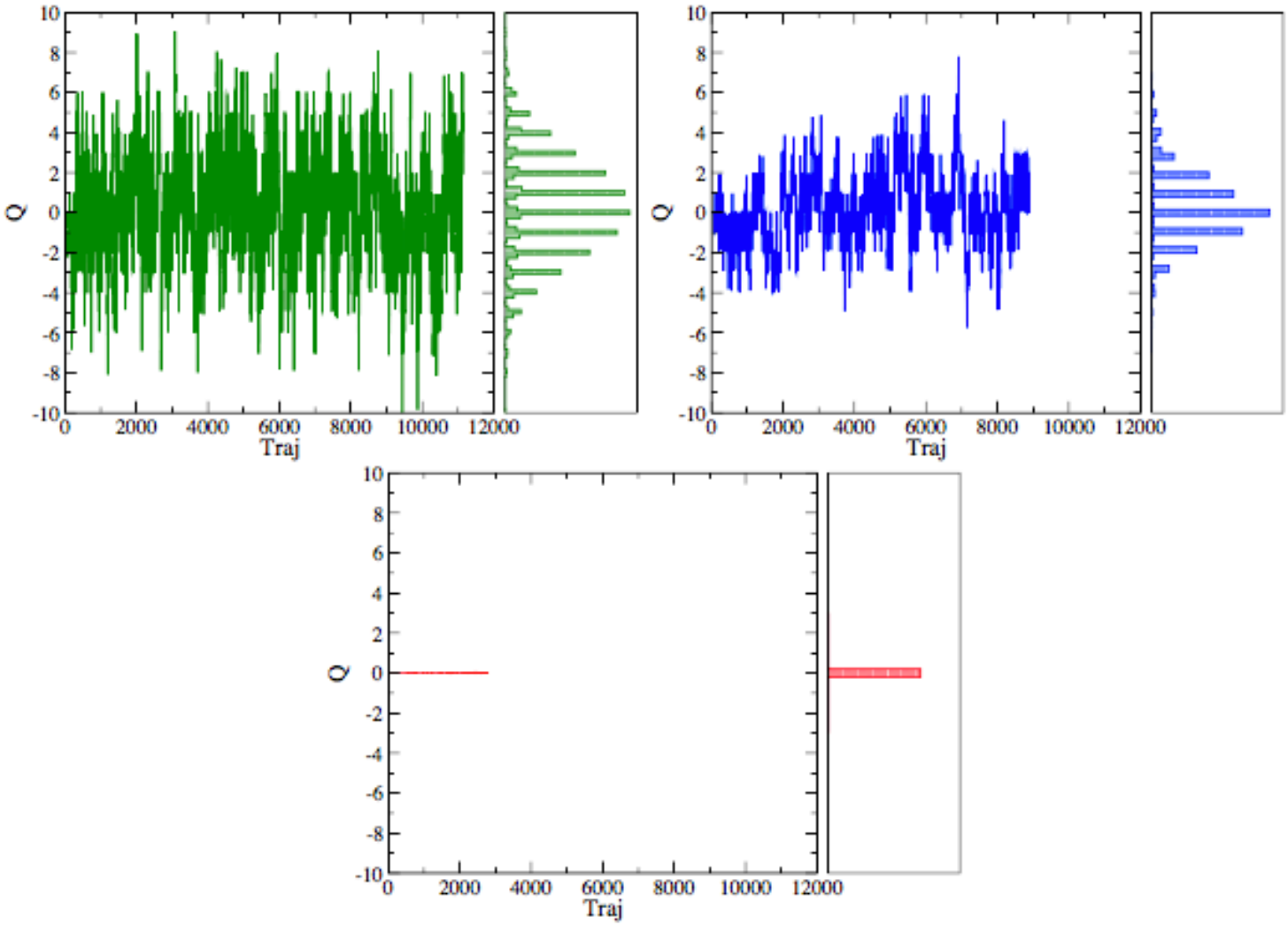
$$\chi^{1/4} = 73(9)\ MeV$$

in reasonable agreement with $\chi^{1/4}_{\rm ChPT} = 77.8(4)\ MeV$

ii) Reducing cutoff effects



iii) Freezing of the topological charge at small lattice spacings



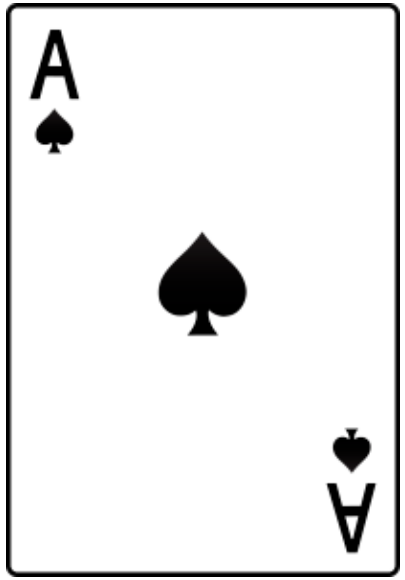
Topological charge time history and histogram for a = 0.0824 fm on a 32^4 lattice, for a = 0.0572 fm on a 48^4 lattice, for a = 0.0397 fm on a 40^4 lattice

THE FODOR TRUMP CARDS

or how these smart people obtained results at large values of T

Borsanyi et al.

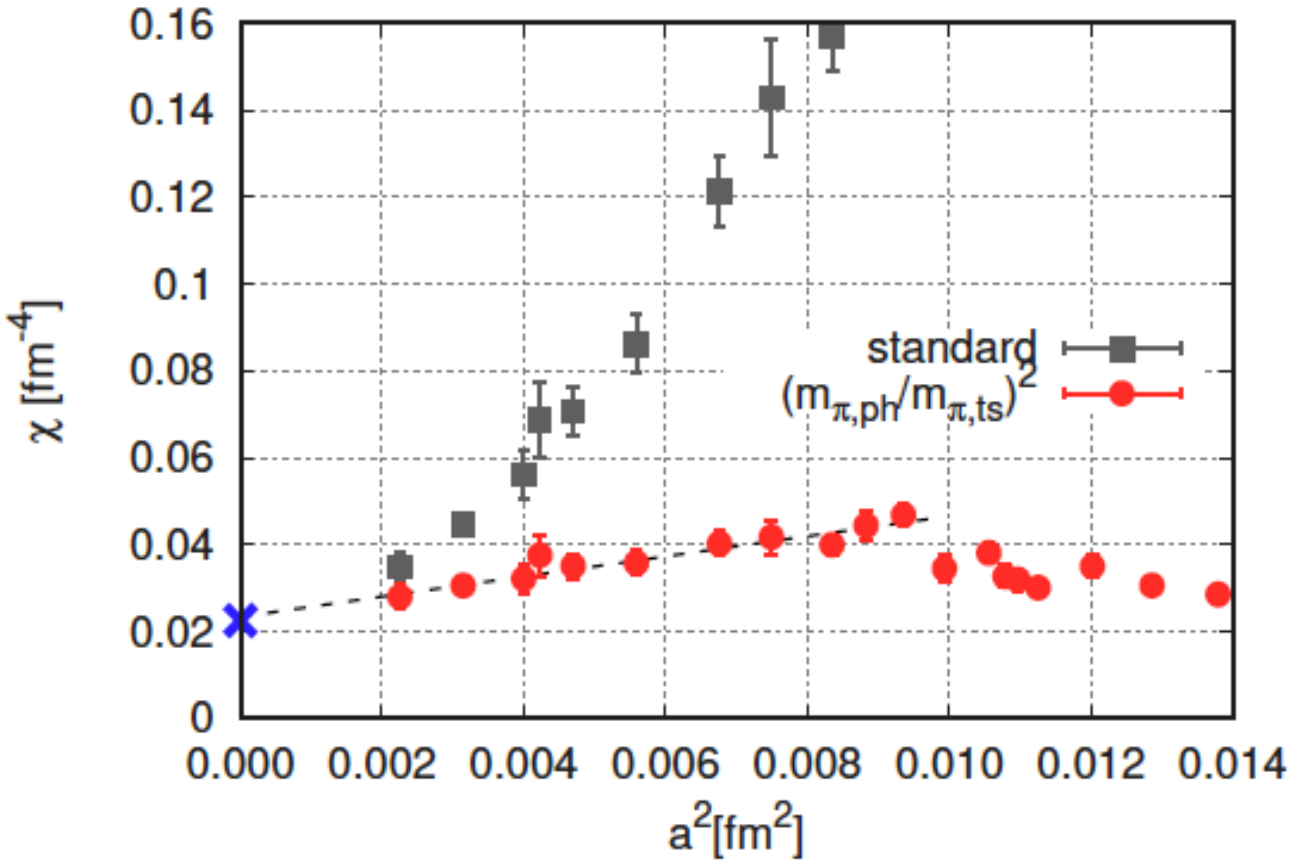
Some approximations have been used to compute the Equation of State (EOS)
But they will not be discussed here

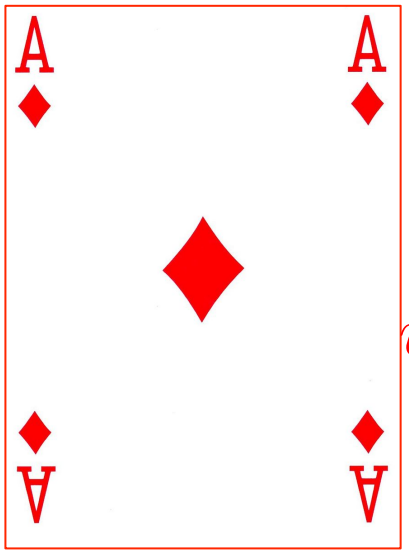


Trying to reduce discretization errors which for staggered fermions are rather large.
At zero temperature

$$\tilde{\chi} = \left(\frac{m_\pi}{m_{\pi I}}\right)^2 \chi$$

A serious study of the dependence on the quark masses is still missing however

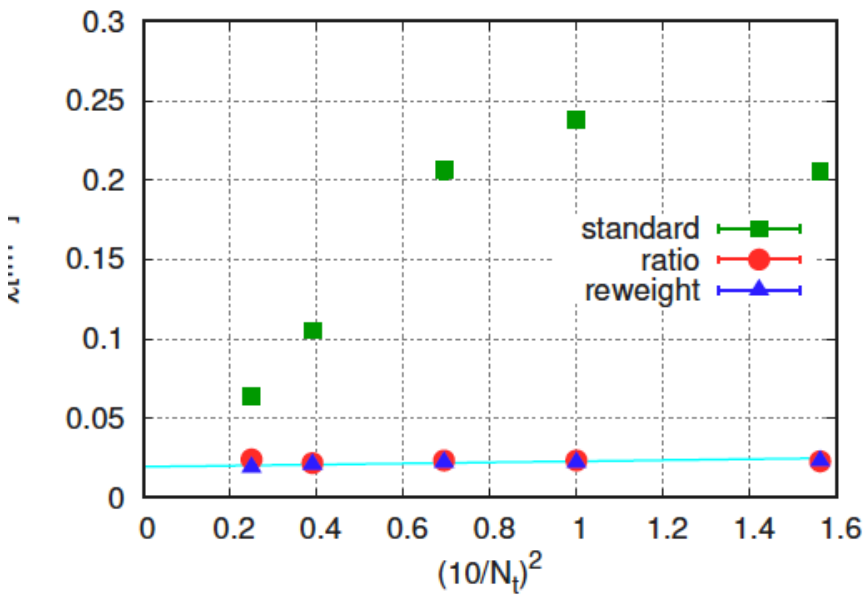


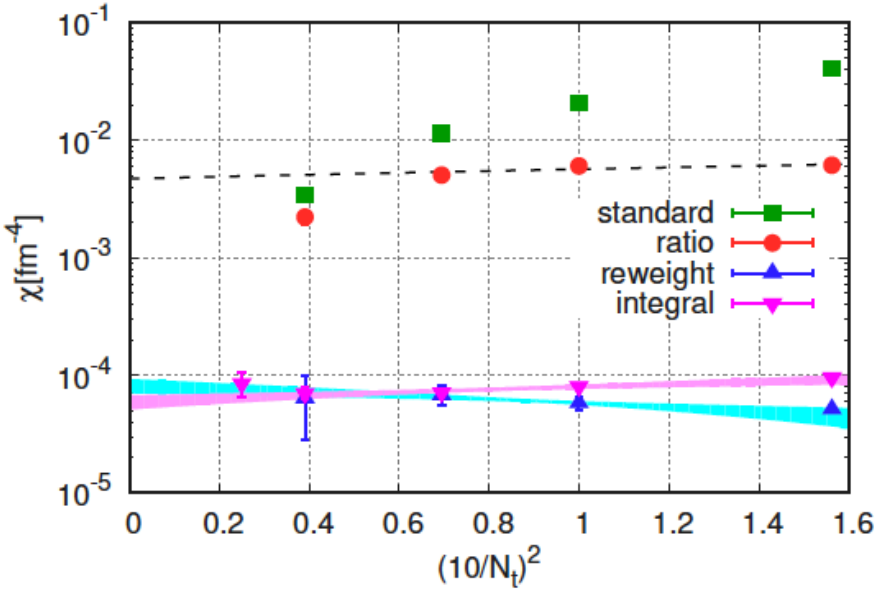


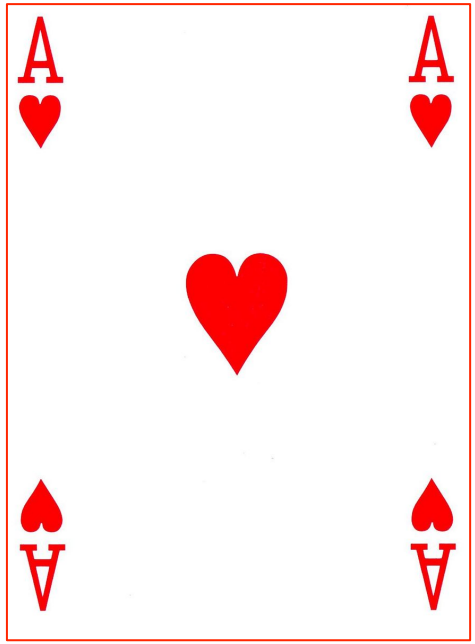
Reweighting of the chiral condensate to reduce discretization errors

$$w[U] = \prod_f \prod_{n=1}^{2Q[U]} \prod_{\sigma=\pm 1} \left(\frac{2\,m_f}{i\sigma\lambda_n[u] + 2\,m_f} \right)^{n_f/4}$$

The choice of Q[U] looks rather arbitrary and has a huge effect (1-2 order of magnitude)







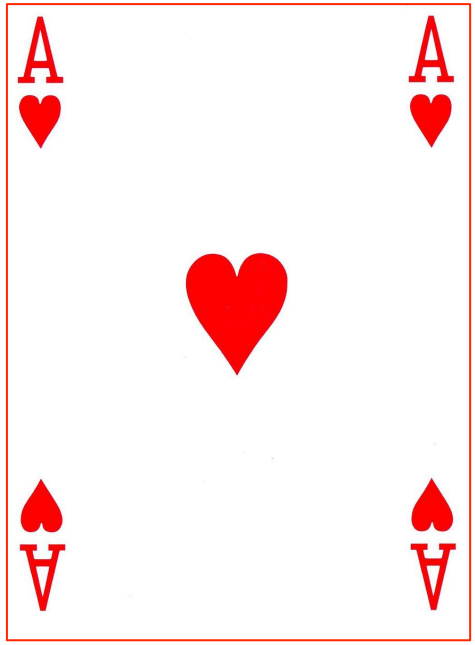
BUT THE REAL TRUMP CARD IS:

1) At high T calculation at fixed topological sector Q=0,1 only;
2) $Z_Q/Z_0(T)$ computed via average action and condensate

Compute $$b_Q = \frac{d\log Z_Q/Z_0}{d\log T} =$$

$$\frac{d\beta}{d\log a}\langle S_g\rangle_{Q-0} + \sum_f \frac{d\log\, m_f}{d\log a}\langle\bar{\psi}\psi_f\rangle_{Q-0}$$

$$\boxed{\frac{Z_Q}{Z_0}(T) = e^{\int_{T_0}^{T} d\log T' b_Q(T')}\,\frac{Z_Q}{Z_0}(T_0)}$$



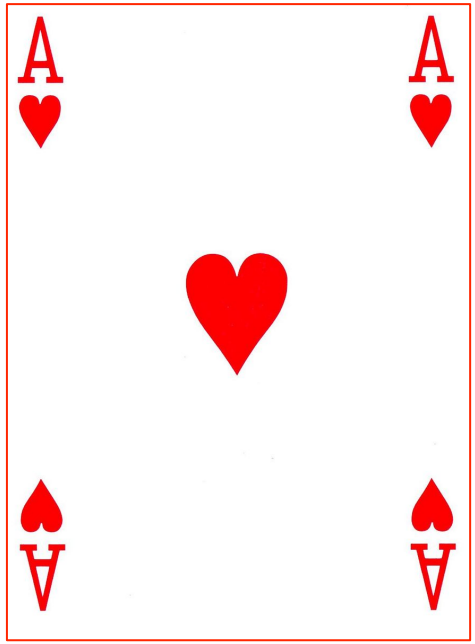
Somehow a circular argument: Freezing at high temperatures and small lattice spacing ➔ small Q ➔ only Q=0,1 -> diluite instanton gas

Freezing and or physical small volumes can mimic large discretization errors

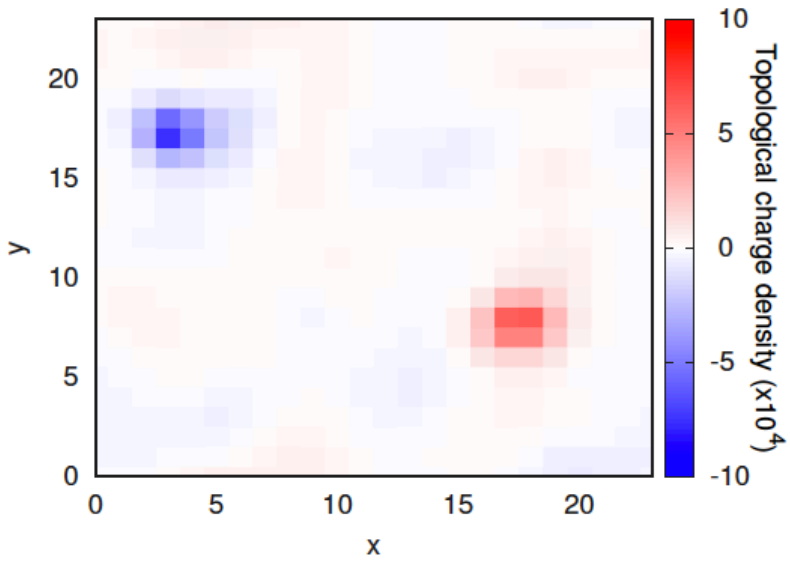
But it is not enough yet:

$$b_Q^{\rm rw} = \frac{d\log Z_Q^{\rm rw}/Z_0^{\rm rw}}{d\log T} = \frac{d\beta}{d\log a}\langle S_g\rangle_{Q-0}^{\rm rw} + \sum_f \frac{d\log m_f}{d\log a} m_f \langle\overline{\psi}\psi_f\rangle_{Q-0}^{\rm rw+zm},$$

$$\langle\overline{\psi}\psi_f\rangle_{Q-0}^{\rm rw+zm} = \langle\overline{\psi}\psi_f\rangle_{Q-0}^{\rm rw} + \frac{|Q|}{m_f} - \left\langle \frac{1}{2m_f}\sum_{n=1}^{2|Q|}\frac{4m_f^2}{\lambda_n^2[U]+4m_f^2}\right\rangle_Q^{\rm rw}.$$

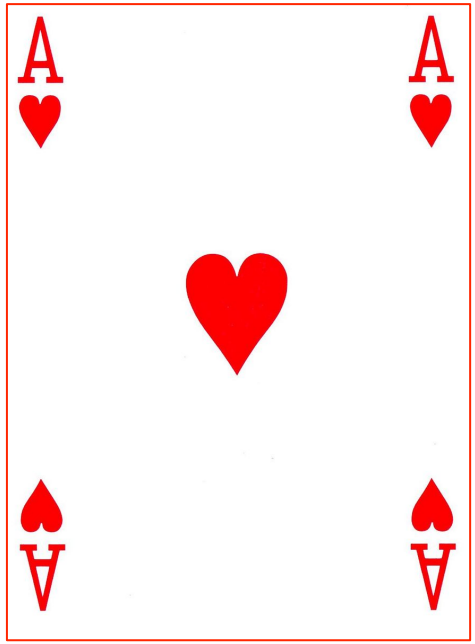


But it is not enough yet: by hand treatment of Instanton-Antistanton configurations



But it is not enough yet: 3+1 versus 2+1+1 T high temperatures (indeed they approximate with R^2)

$$\left.\frac{Z_1}{Z_0}\right|_{2+1+1} = \exp\left(\int_{m_{ud}^{phys}}^{m_s^{phys}} d\log m_{ud}\; m_{ud}\langle\overline{\psi}\psi_{ud}\rangle\right)\cdot\left.\frac{Z_1}{Z_0}\right|_{3+1}$$



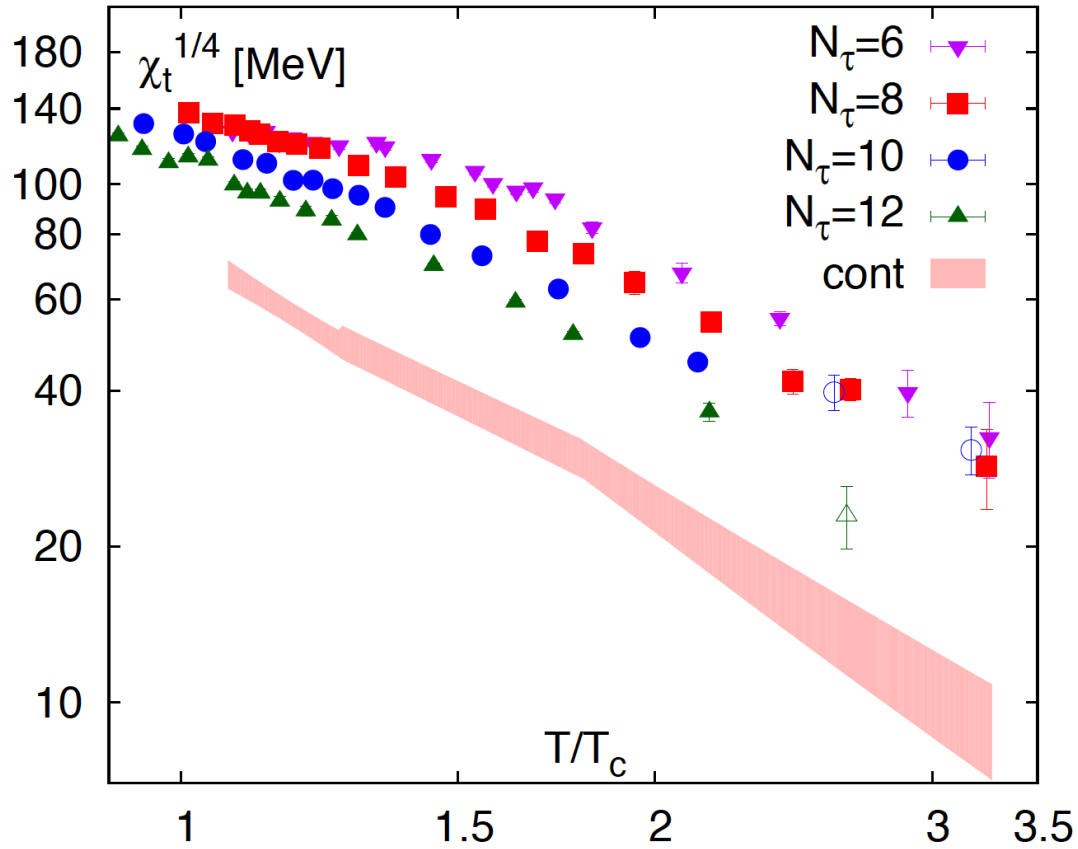
But it is not enough yet: since it is difficult to compute $<S_g>_{1-0}$

they propose to compute $<S_g>_{Q-0}$ with Q=2,3, etc

$$<S_g>_{1-0} = <S_g>_{Q-0}/|Q| \quad !!$$

Also a riscaling of the topological susceptibility of a factor
4 mu md /(mu+md)^2 = 0.88
by hand

The results indicate that above Tc, cutoff effects depend dominantly on $aT = 1/N_t$ rather than ~ a Λ_{QCD} . Moreover it is impossible to obtain an acceptable fit to the data with a single exponent in the entire temperature range. Rather one could fit the data using two different values for temperatures T > 1. 5Tc and T < 1. 5Tc .



P Petreczky, H Schadler S Sharma 1606.03145v2

The results extrapolated to the continuum confirm the results of Borsanyi et al. Although the constant does not agree with the perturbative calculation the exponent is in good agreement with DIGA

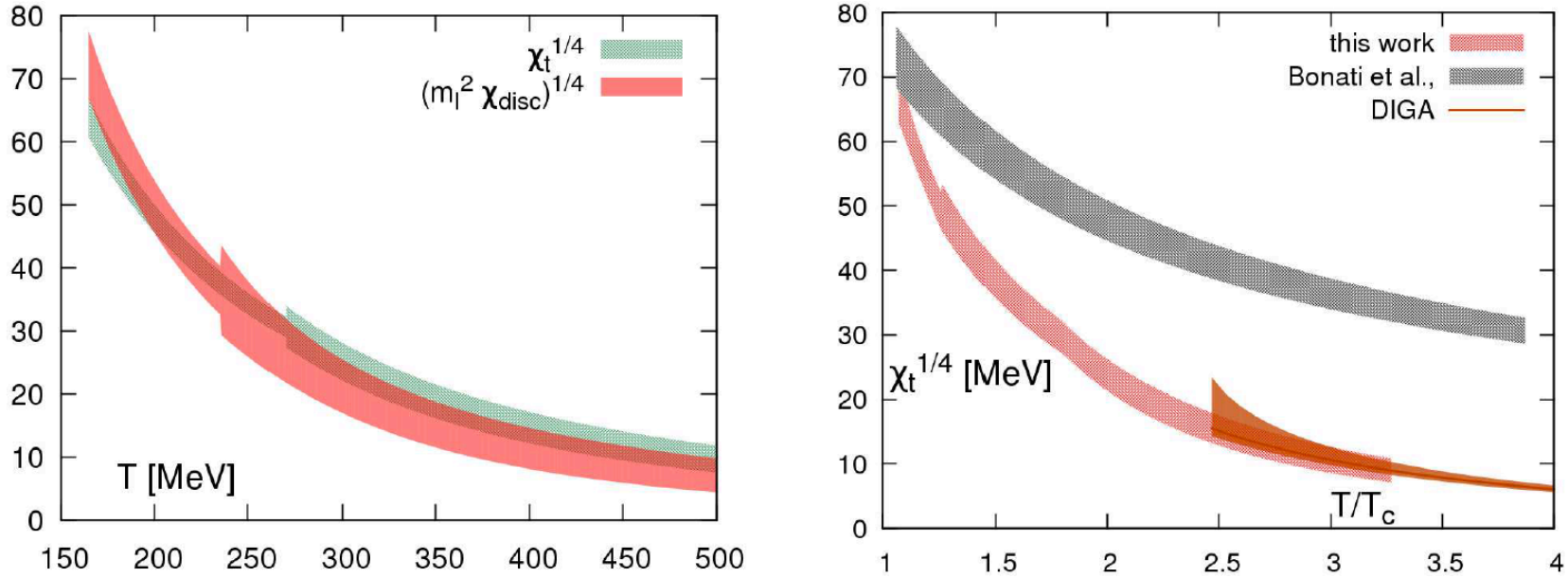


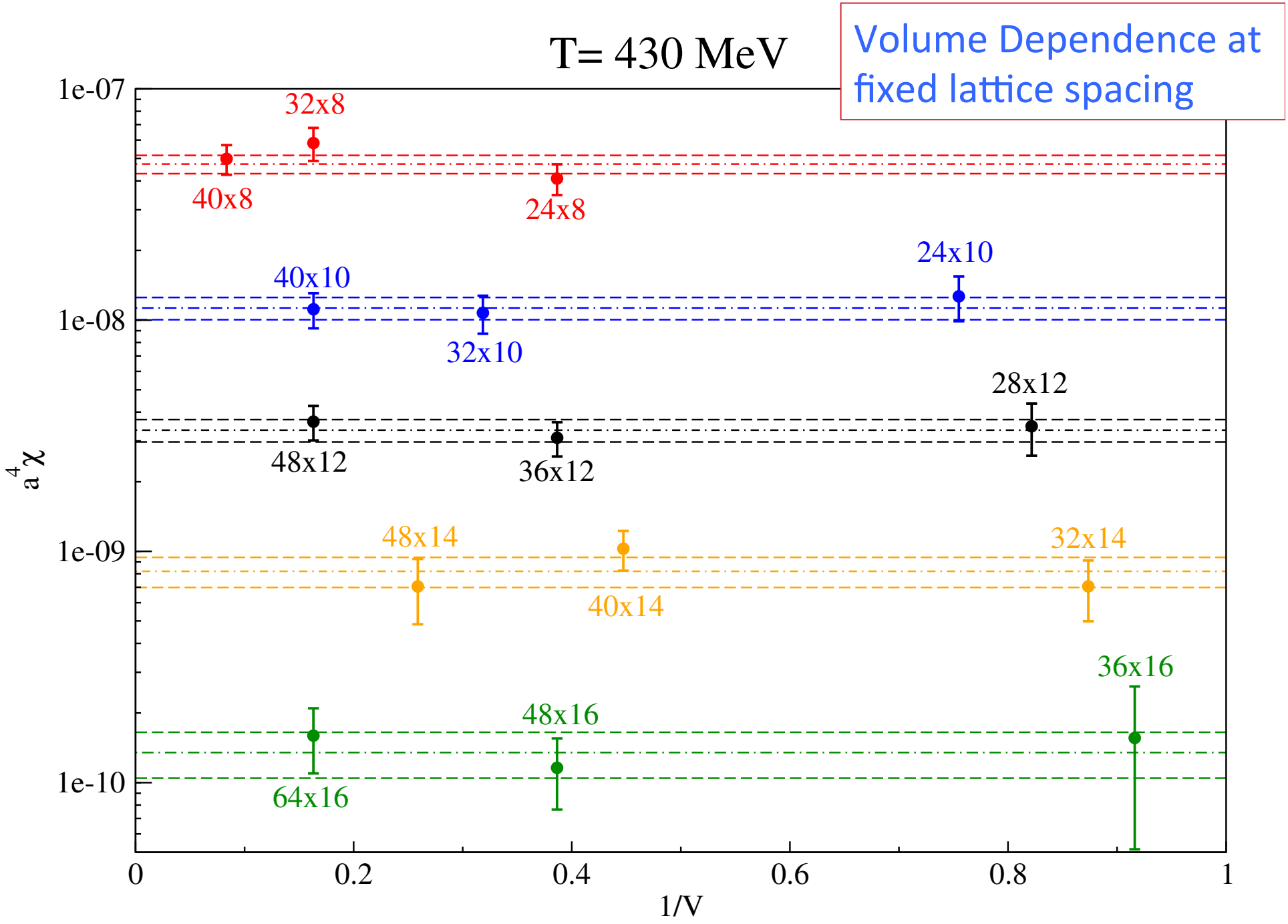
Figure 5: Continuum extrapolated results for $\chi_t^{1/4}$ measured from gluonic definition of topological charge and $(m_l^2\chi_{disc})^{1/4}$ fitted separately (left) and the joint continuum extrapolation of $\chi_t^{1/4}$ and $(m_l^2\chi_{disc})^{1/4}$ (right). In the right panel, our results are compared with the continuum extrapolated results obtained in Ref. [36]. The solid orange line corresponds to a partial two-loop DIGA calculation with $\mu = \pi T$, $K = 1.9$ and $\alpha_s(\mu = 1.5\ \text{GeV}) = 0.336$, while the band is obtained from the variation of α_s by 1σ around this central value as well as variation of the scale μ by a factor two (see text).

C. Bonati, A. Todaro et al. arXiv:1807.07954 [hep-lat]

1) Multicanonical

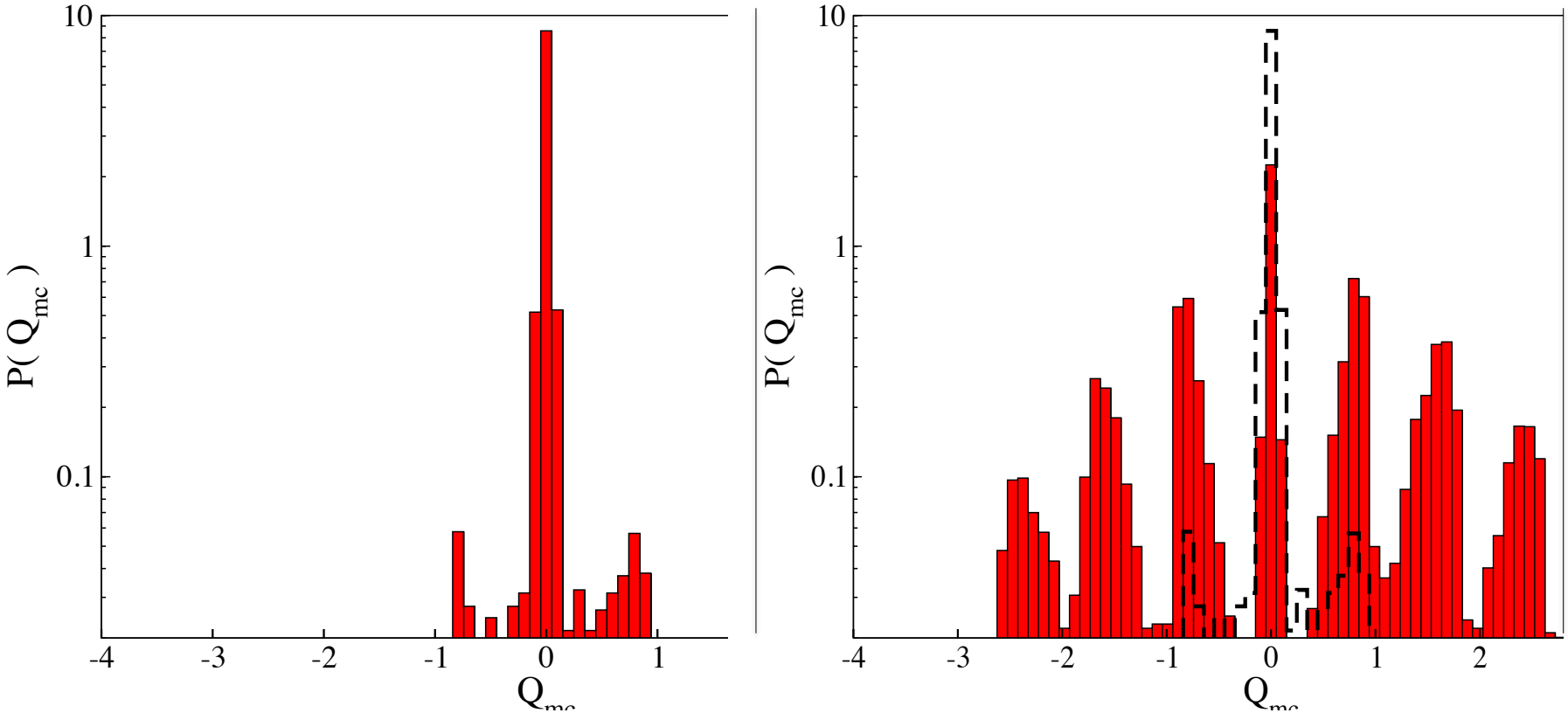
2) Systematic study of physical volume dependence (at different lattice spacings)

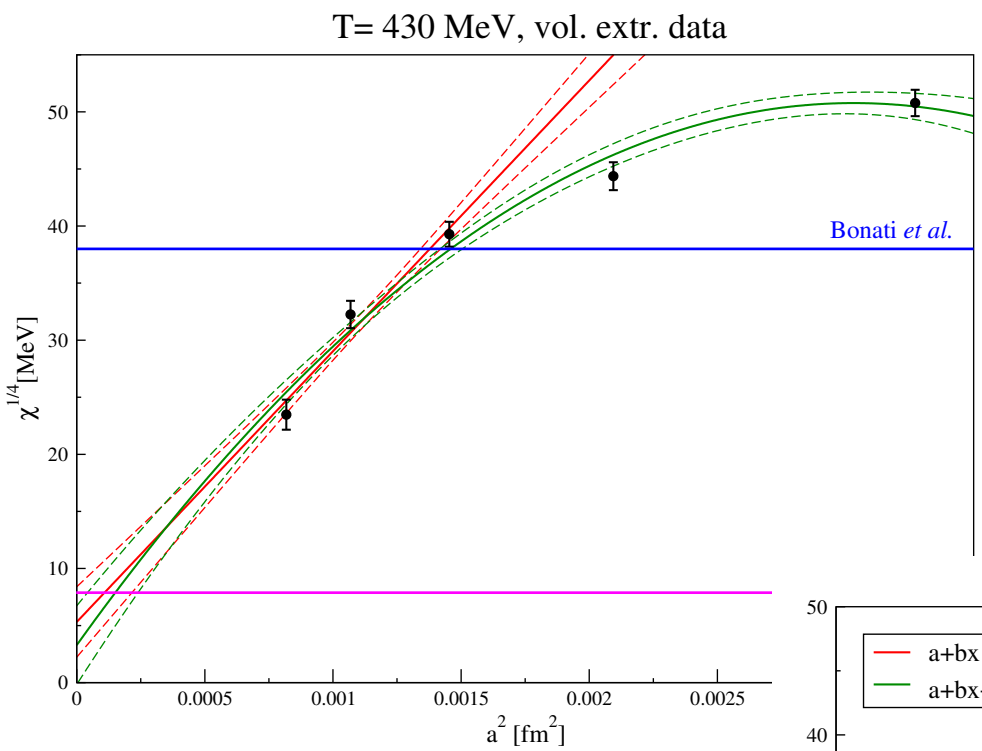
3) Systematic study of finite a dependence (not new)


T= 430 MeV
Volume Dependence at fixed lattice spacing
1e-07
1e-08
1e-09
1e-10
$a^4\chi$
0
0.2
0.4
0.6
0.8
1
1/V
32x8
40x8
24x8
24x10
40x10
32x10
28x12
48x12
36x12
48x14
32x14
40x14
36x16
48x16
64x16

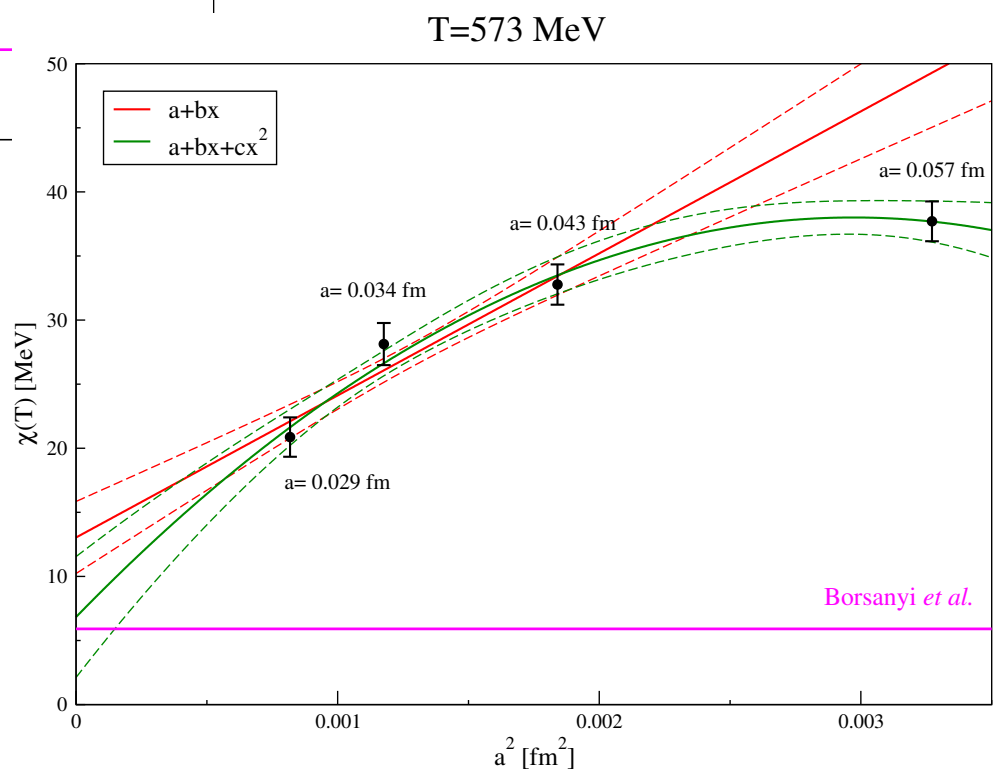
MultiCanonical Algorithm (Metadynamics, ...)

$$\langle O \rangle = \frac{\left\langle O\, e^{V(Q_{mc})} \right\rangle_V}{\left\langle e^{V(Q_{mc})} \right\rangle_V}$$





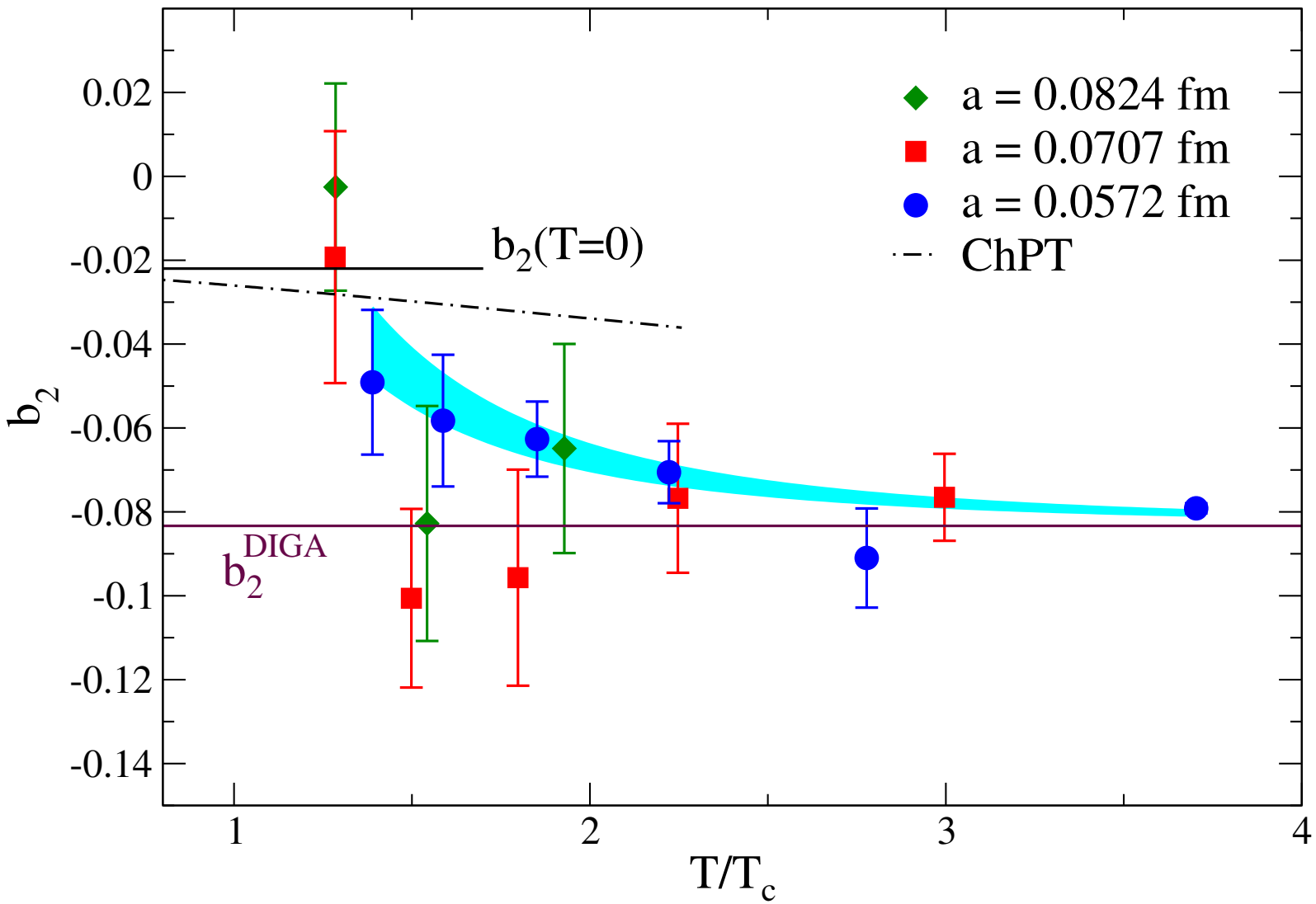
Results at two values of T agree with Borsanyi et al.

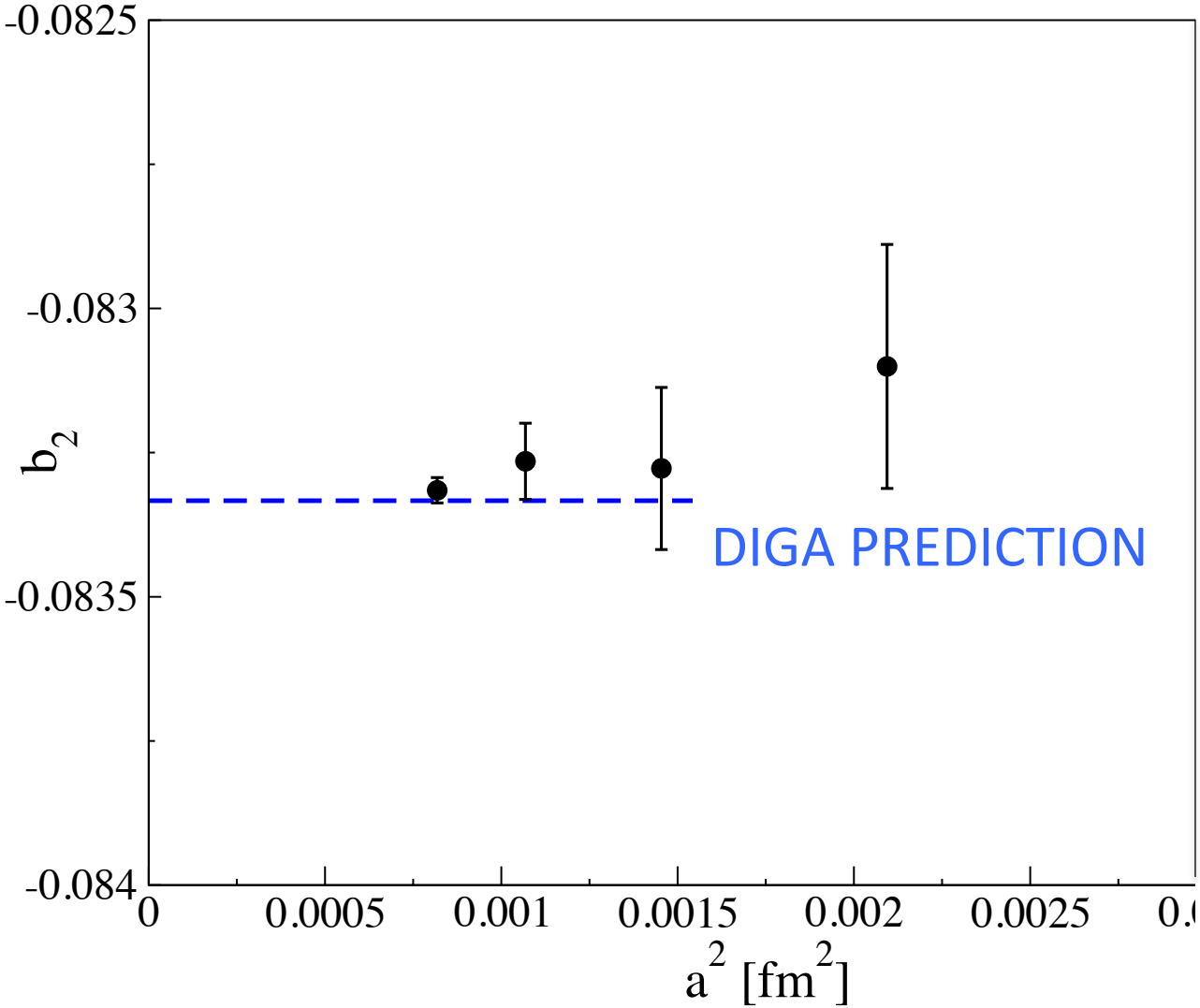


Higher Momenta

$$b_2 = -\frac{\langle Q^4\rangle_{\theta=0} - 3\langle Q^2\rangle^2_{\theta=0}}{12\langle Q^2\rangle_{\theta=0}}$$

$$b_2^{DIGA} = -1/12$$




-0.0825
-0.083
-0.0835
-0.084
b_2
0
0.0005
0.001
0.0015
0.002
0.0025
a^2 [fm^2]
DIGA PREDICTION