Axion properties at zero and finite T from Lattice QCD

$SU(3)$ with light fermions at or close to the physical point

- Trunin, Burger, Ilgenfritz, Lombardo, Müller-Preussker

- Bonati, D’Elia, Mariti, Martinelli, Mesiti, Negro, Sanfilippo, Villadoro

- Petreczky, Schadler, Sharma

- Borsanyi, Fodor, Kampert, Katz, Kawanai, Kovacs, Mages, Pasztor, Pittler, Redondo, Ringwald, Szabo

- Burger, Ilgenfritz, Lombardo, Trunin
  arXiv:1805.06001 [hep-lat].

C. Bonati, M. D'Elia, G. Martinelli, F. Negro, F. Sanfilippo, A. Todaro
arXiv:1807.07954 [hep-lat]
At zero and non zero temperature $T$ the Axion Potential can be derived from the dependence of the vacuum free energy $F[\theta,T]$ on $\theta$

The general form of $F(\theta, T)$

$$F(\theta, T) = -\frac{1}{V_4} \log \int [\mathcal{D} A][\mathcal{D} \bar{\psi}][\mathcal{D} \psi] \exp \left( - \int_0^{1/T} dt \int d^3 x \mathcal{L}_\theta^E \right)$$

Assuming analyticity at $\theta = 0$ the free energy density can be written as:

$$F(\theta, T) - F(0, T) = \frac{1}{2} \chi(T) \theta^2 \left[ 1 + b_2(T) \theta^2 + b_4(T) \theta^4 + \cdots \right] ,$$

and it is easy to see that

$$\chi = \frac{1}{V_4} \langle Q^2 \rangle_0 \quad b_2 = -\frac{\langle Q^4 \rangle_0 - 3 \langle Q^2 \rangle_0^2}{12 \langle Q^2 \rangle_0}$$

Thus for example the axion mass is related to the topological susceptibility

$$m^2 = \frac{1}{f^2} \chi_t = \frac{1}{f^2} \int d^4 x \langle Q(x) Q(0) \rangle$$
Using the Ward ids, expanding at small quark masses and saturating the T-products with Goldston boson intermediate states (or using \( \chiPT \)) we get

\[
\chi_t = -\frac{3}{4} f_\pi^2 \left( m_\pi^2 + m_\eta^2 \right) \frac{m_u m_d m_s}{(m_u + m_d + m_s)(m_u m_d + m_d m_s + m_u m_s)}
\]

- The result depends on the number of flavours
- The topological susceptibility, and consequently the mass of the axion, vanishes whenever the mass of a quark is equal to zero

Dilute Instanton Gas Approximation

At very high $T$ ($T \gg \Lambda_{QCD}$) one can show that the $\theta$ dependence is dominated by weakly interacting objects of topological charge $\pm 1$ and the free energy is given by (Gross, Pisarski, Yaffe 1981)

$$F(\theta, T) - F(0, T) \approx \chi(T)(1 - \cos \theta)$$

so that

$$b_2 = -\frac{1}{12} \quad b_4 = \frac{1}{360} \quad b_{2n} = (-1)^n \frac{2}{(2n + 2)!}$$

and the susceptibility scales with the temperature as following

$$\chi(T) \propto m^{N_f} T^{4 - \frac{11}{3} N - \frac{1}{3} N_f}$$

when $N_f$ light flavours are present.
THE AXION AT NON ZERO TEMPERATURE
That is WHEN LQCD ENTERS THE GAME

Chiral Lagrangians allow to study the temperature dependence of the axion potential and its mass to finite temperatures below the crossover region $T_c \sim 150$ MeV.

Around $T_c$ there is no known reliable perturbative expansion under control and non-perturbative methods, such as lattice QCD are required.

A. Trunin, F. Burger, E.-M. Ilgenfritz, M. P. Lombardo and M. Muller-Preussker, arXiv:1510.02265 [hep-lat].
i) $\chi(T)$ is usually determined from the probability distribution $P(Q)$ and in particular from its variance,

$$\chi(T) = \frac{<Q^2>}{V}$$

The suppression of $\chi(T)$ at large $T$ implies that, on volumes reasonably achievable in lattice simulations, one has $<Q^2> = V \chi(T) \ll 1$. Since $Q$ is integer valued, that means that configurations with $Q \neq 0$ become very rare, leading to the need of unaffordably large statistics in order to achieve a correct sampling of the topological charge distribution.
ii) Topology is strongly correlated with the chiral properties of the theory (the $\theta$-dependence would be strictly zero in the presence of massless quarks)

The explicit chiral symmetry breaking that is present in most fermion discretizations can lead to significant lattice artifacts, thus requiring very small lattice spacings to achieve a reliable continuum extrapolation
iii) Closer to the continuum limit, because of the topological nature of the problem, standard updating algorithms fail to correctly sample the distribution of $Q$ and get trapped in path integral sectors with fixed topology.

The freezing of topological charge leads to a severe critical slowing down of numerical simulations.
iv) Finally, since finite temperature numerical simulations are usually performed with $L_s^3 \times N_t$ at fixed $L_s/N_t$, as $T = 1/(N_t a)$ becomes large also the physical spatial volume $a^3 L_s^3$ becomes small, so that the possible presence of finite volume effects should be checked.
ii) Large Cutoff (Discretization) Effects

\[ \chi^{1/4} = 73(9) \text{ MeV} \]

in reasonable agreement with

\[ \chi_{\text{ChPT}}^{1/4} = 77.8(4) \text{ MeV} \]
ii) Reducing cutoff effects

\[ \chi(T)/\chi(0) = D_0 (T/T_c)^{D_2} \]

\[ D_2^{DIGA} \sim -8 \]

\[ D_2^{our \ fit} \sim -3 \]
iii) Freezing of the topological charge at small lattice spacings

Topological charge time history and histogram for $a = 0.0824$ fm on a $32^4$ lattice, for $a = 0.0572$ fm on a $48^4$ lattice, for $a = 0.0397$ fm on a $40^4$ lattice
THE FODOR TRUMP CARDS or how these smart people obtained results at large values of $T$

Borsanyi et al.

Some approximations have been used to compute the Equation of State (EOS)
But they will not be discussed here
Trying to reduce discretization errors which for staggered fermions are rather large. At zero temperature

\[ \tilde{\chi} = \left( \frac{m_\pi}{m_\pi I} \right)^2 \chi \]

A serious study of the dependence on the quark masses is still missing however.
Reweighting of the chiral condensate to reduce discretization errors

\[ w[U] = \prod_f \prod_{n=1}^{2Q[U]} \prod_{\sigma = \pm 1} \left( \frac{2m_f}{i\sigma \lambda_n[u] + 2m_f} \right)^{n_f/4} \]

The choice of \( Q[U] \) looks rather arbitrary and has a huge effect (1-2 order of magnitude)
BUT THE REAL TRUMP CARD IS:

1) At high T calculation at fixed topological sector $Q=0,1$ only;

2) $Z_Q/Z_0(T)$ computed via average action and condensate

Compute $b_Q = \frac{d \log Z_Q/Z_0}{d \log T}$

$$\frac{d \beta}{d \log a} \langle S_g \rangle_{Q-0} + \sum_f \frac{d \log m_f}{d \log a} \langle \bar{\psi} \psi_f \rangle_{Q-0}$$

$$\frac{Z_Q}{Z_0}(T) = e^{\int_{T_0}^T d \log T' b_Q(T')} \frac{Z_Q}{Z_0}(T_0)$$
Somehow a circular argument: Freezing at high temperatures and small lattice spacing $\Rightarrow$ small $Q \Rightarrow$ only $Q=0,1 \Rightarrow$ dilute instanton gas

Freezing and or physical small volumes can mimic large discretization errors

But it is not enough yet:

$$b_Q^w = \frac{d \log Z_Q^w / Z_0^w}{d \log T} = \frac{d \beta}{d \log a} \langle S_g \rangle_{Q=0}^w + \sum_f \frac{d \log m_f}{d \log a} m_f \langle \overline{\psi}\psi_f \rangle_{Q=0}^{w+zm} ,$$

$$\langle \overline{\psi}\psi_f \rangle_{Q=0}^{w+zm} = \langle \overline{\psi}\psi_f \rangle_{Q=0}^w + \frac{|Q|}{m_f} - \left\langle \frac{1}{2m_f} \sum_{n=1}^{2|Q|} \frac{4m_f^2}{\lambda_n^2[U] + 4m_f^2} \right\rangle_Q^w .$$
But it is not enough yet: by hand treatment of Instanton-Antististanton configurations

But it is not enough yet: 3+1 versus 2+1+1 T high temperatures (indeed they approximate with $R^2$)

\[
\left. \frac{Z_1}{Z_0} \right|_{2+1+1} = \exp \left( \int_{m_{ud}^{phys}} m_{ud} \log m_{ud} \, d\bar{\psi}\psi_{ud} \right) \cdot \left. \frac{Z_1}{Z_0} \right|_{3+1}
\]
But it is not enough yet: since it is difficult to compute $<S_g>_{1-0}$

they propose to compute $<S_g>_{Q-0}$

with $Q=2,3$, etc

$<S_g>_{1-0} = <S_g>_{Q-0}/|Q|$

Also a rescaling of the topological susceptibility of a factor

$4 \mu m d / (\mu + m d)^2 = 0.88$

by hand
The results indicate that above $T_c$, cutoff effects depend dominantly on $aT = 1/N_t$ rather than $\sim a \Lambda_{QCD}$. Moreover it is impossible to obtain an acceptable fit to the data with a single exponent in the entire temperature range. Rather one could fit the data using two different values for temperatures $T > 1.5T_c$ and $T < 1.5T_c$.
The results extrapolated to the continuum confirm the results of Borsanyi et al. Although the constant does not agree with the perturbative calculation the exponent is in good agreement with DIGA.

Figure 5: Continuum extrapolated results for $\chi_t^{1/4}$ measured from gluonic definition of topological charge and $(m_t^2 \chi_{disc})^{1/4}$ fitted separately (left) and the joint continuum extrapolation of $\chi_t^{1/4}$ and $(m_t^2 \chi_{disc})^{1/4}$ (right). In the right panel, our results are compared with the continuum extrapolated results obtained in Ref. [36]. The solid orange line corresponds to a partial two-loop DIGA calculation with $\mu = \pi T$, $K = 1.9$ and $\alpha_s(\mu = 1.5$ GeV) = 0.336, while the band is obtained from the variation of $\alpha_s$ by 1σ around this central value as well as variation of the scale $\mu$ by a factor two (see text).
1) Multicanonical

2) Systematic study of physical volume dependence (at different lattice spacings)

3) Systematic study of finite $a$ dependence (not new)
Volume Dependence at fixed lattice spacing

T = 430 MeV
MultiCanonical Algorithm (Metadynamics, ...)

\[ \langle O \rangle = \frac{\langle O e^{V(Q_{mc})} \rangle_V}{\langle e^{V(Q_{mc})} \rangle_V} \]
Results at two values of $T$ agree with Borsanyi et al.
$b_2 = -\frac{\langle Q^4 \rangle_{\theta=0} - 3\langle Q^2 \rangle^2_{\theta=0}}{12\langle Q^2 \rangle_{\theta=0}}$

$b_2^{DIGA} = -1/12$