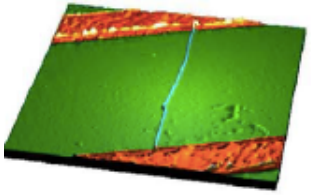
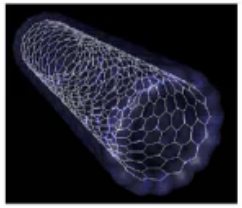
Quantum impurities in interacting environments

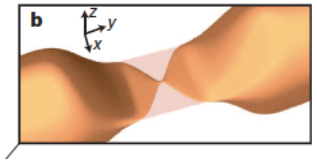
Luttinger liquids: nanotubes





Single-wall carbon nanotube
= cylindrical roll of graphene

Luttinger liquid with an impurity



Cold atoms
Esslinger '14

Natan Andrei
Rutgers university

Colin Rylands
Rutgers University

Chicheley Hall, Sept 2017

The environment - Luttinger Liquid

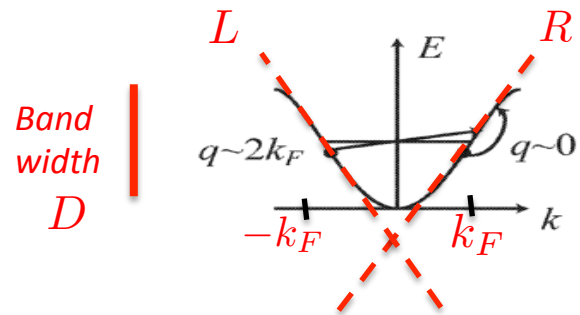
- The Luttinger Hamiltonian is the low-energy fixed point of many 1-d interacting gapless systems: XXZ Heisenberg model, Hubbard model, Lieb-Liniger model..

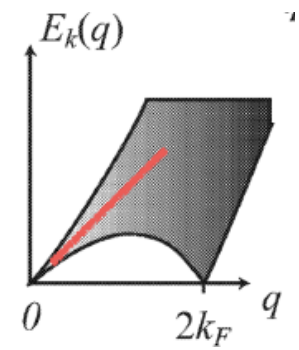
$$H_{LL} = -iv_F \int_{-\infty}^{+\infty} dx(\psi_R^\dagger \partial_x \psi_R - \psi_L^\dagger \partial_x \psi_L) + 4g \int_{-\infty}^{+\infty} dx\ \rho_R(x)\rho_L(x)$$

To obtain: *i. RG*

ii. Short cut: linearization

$$\psi(x) \approx \psi_R(x)e^{ik_F x} + \psi_L(x)e^{-ik_F x}$$



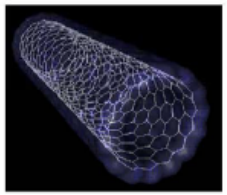


- In 1-d weak interactions change the nature of the system: fermions disappear from spectrum, collective excitations - ***bosons***

$$H_{LL} = v \int dx[K\ \Pi^2(x) + \frac{1}{K}\ (\partial\phi(x))^2]$$

$K \sim 1 - 2g/\pi$ *Luttinger parameter:* Repulsive interaction $K < 1$, attractive $K > 1$

- Realized in many experimental systems: spin chains, carbon nano-tubes, quantum wires, Quantum Hall edges



Local Scattering Center in a Luttinger Liquid

- **A local impurity** $V(x) = U\delta(x)$

$$\int V(x)\psi^\dagger(x)\psi(x)\,dx \approx U\psi^\dagger(0)\psi(0) = U(\psi_R^\dagger(0)\psi_R(0) + \psi_L^\dagger(0)\psi_L(0) + \psi_R^\dagger(0)\psi_L(0) + \psi_L^\dagger(0)\psi_R(0))$$

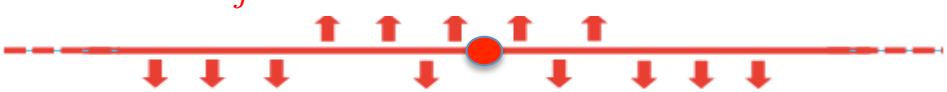
$$\psi(x) \approx \psi_R(x)e^{ik_F x} + \psi_L(x)e^{-ik_F x}$$

Forward scattering

Backward scattering

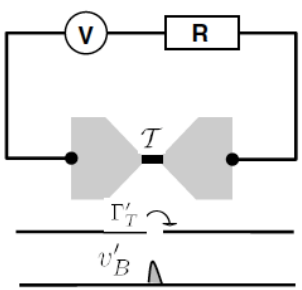
- **Examples:**

i. Spin chain $H = \sum_j J_j\ \vec{\sigma}_j \cdot \vec{\sigma}_{j+1}$ with an impurity at 0 $J_j = j + a\delta_{j,0}$



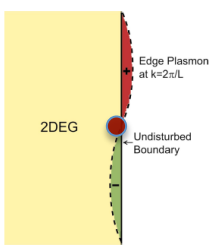
ii. 1-d coherent conductor with transmission coefficient $\mathcal{T}$ in series with impedance R

Mapped to LL with $K = 1/(1 + e^2R/h)$



Saleur and Safi '93

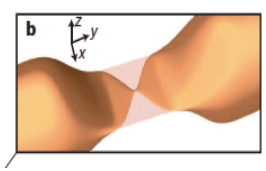
iii.



QHE edges

Wen, Chang '93..

iv.

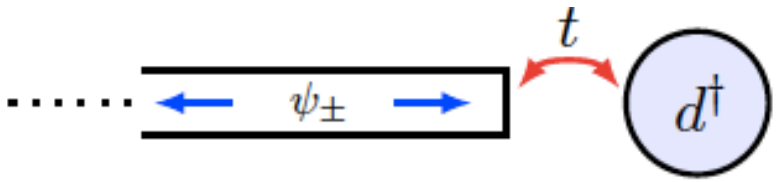


Cold atoms:

Esslinger '14

Quantum dot + Luttinger Liquid

Dot at the edge

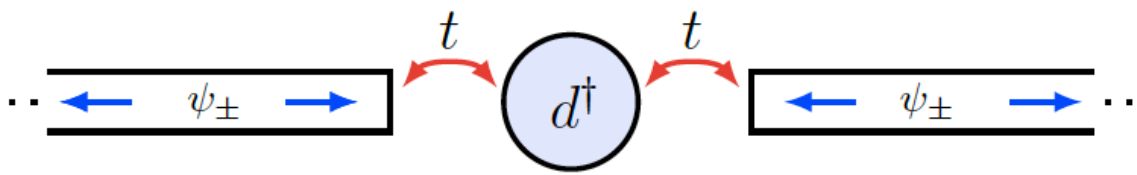


BC at the edge

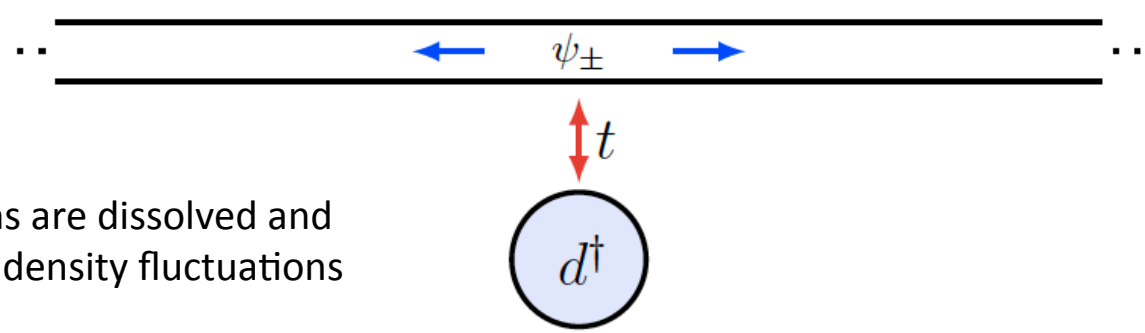
$\psi_+(0) = -\psi_-(0)$

$$H = H_{LL} + \frac{t}{2}((\psi_+^\dagger(0) - \psi_-^\dagger(0))d + \text{h.c}) + \epsilon_0 d^\dagger d + \frac{U}{2} d^\dagger d \sum_{\sigma=\pm} \psi_\sigma^\dagger(0)\psi_\sigma(0)$$

Embedded dot



Side-coupled dot



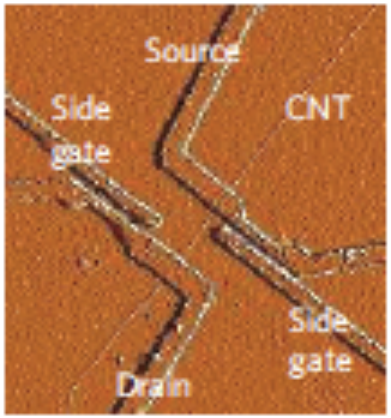
- In the wire the electrons are dissolved and excitations are bosonic density fluctuations
- On the dot the degrees of freedom are electronic

→ *Interplay between tunneling and interaction*

Quantum dot + Luttinger Liquid

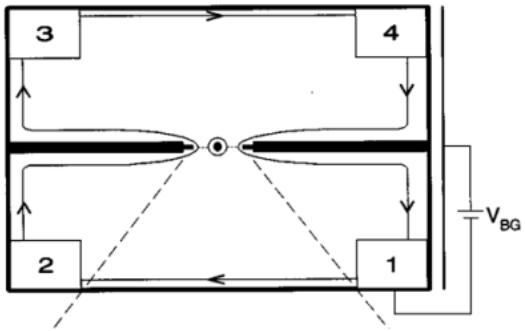
Experimental realization: Embedded dot

G. Finkelstein Group 2012



RL-Short carbon nanotube

Maasilta and Goldman 1997

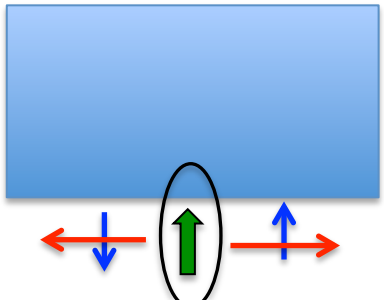


Resonant tunneling between fractional quantum Hall edges

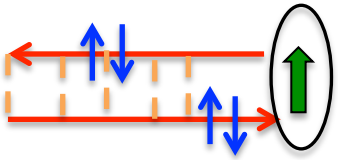
Current work:

The **Y**- problem

Spin impurity in a TI



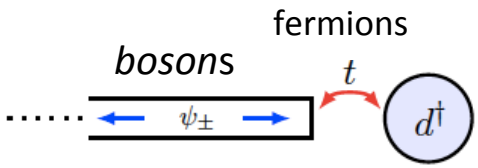
Luttinger Kondo



Outline

Why?

- The excitation spectrum of the Luttinger consists of bosons
Fermions are coherent collective excitations



- Tunneling to dot proceeds via fermions: Competition − interactions vs tunneling

How?

- Requires a new type of Bethe Ansatz, generalization of Yang-Baxter

- Derive the Bethe Ansatz equations due to imposition of Periodic Boundary Conditions

- Derive spectrum, identify the ground state, the excitations

- Calculate the dot occupation, the critical exponents, the thermodynamics, the RG flow

Local scattering in LL: The Kane-Fisher model

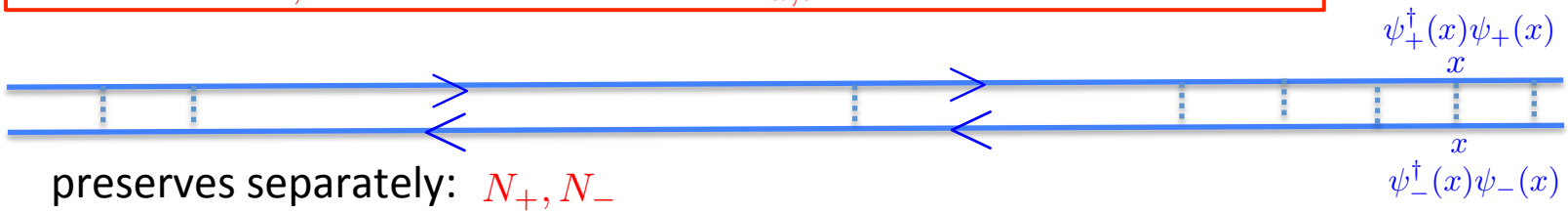
The Hamiltonian $H_{KF} = H^{bulk} + H^{imp}$ consists of:

- *Luttinger liquid* in the bulk: *left (-)* and *right (+)* moving fermions coupled via density-density

$$H^{bulk} = -i \sum_{\sigma,a} \int \psi^\dagger_{\sigma,a} \sigma \partial_x \psi_{\sigma,a}(x) + \sum_{a,b} 4g \int \psi^\dagger_{+,a} \psi^\dagger_{-,b} \psi_{-,b} \psi_{+,a}$$

$\sigma = \pm, a = \uparrow, \downarrow$

chirality spin

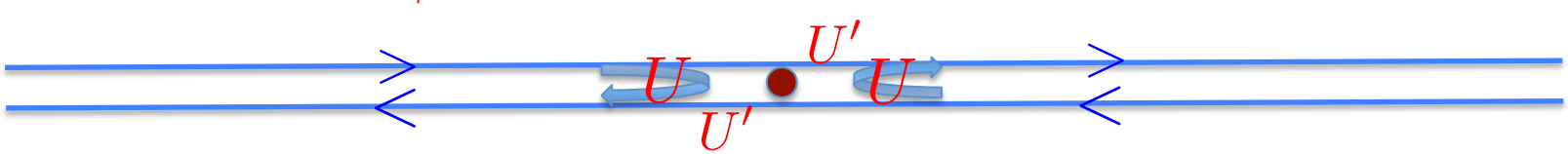


preserves separately: N_+, N_-

- localized impurity (low energy approx: $\psi(x) \approx \psi_R(x) e^{ik_F x} + \psi_L(x) e^{-ik_F x}$, then $U = U'$)

$$H^{imp} = \sum_a U \left[\psi^\dagger_{+,a}(0)\psi_{-,a}(0) + \psi^\dagger_{-,a}(0)\psi_{+,a}(0) \right] + U' [\psi^\dagger_{+,a}(0)\psi_{+,a}(0) + \psi^\dagger_{-,a}(0)\psi_{-,a}(0)]$$

preserves: $N = N_+ + N_-$

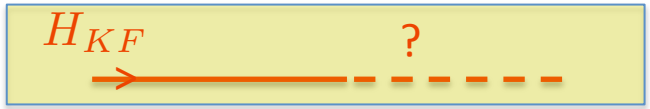


- **Related to Boundary Sine-Gordon (*Zamolodchikov '95*) by bosonization and folding (Saleur et al '97)**

The RG flow (Kane and Fisher '92)

- RG analysis (weak coupling)

$$D\frac{dU}{dD} \sim (1-K)U$$



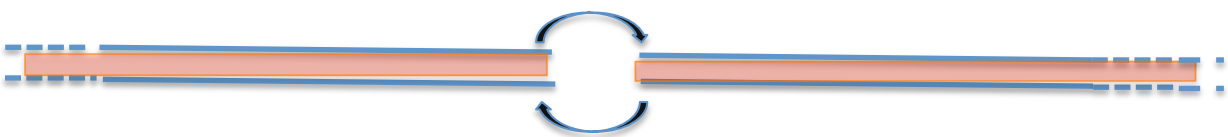
$U(D)$ grows for weak coupling, but where does it flow?

The impurity coupling is ***relevant*** for repulsive interaction in the liquid $K<1$

- Does it flow to *strong coupling*, *i.e.* cut the wire?

  - The *strong coupling regime* is described by the Weak Tunneling Hamiltonian

$$H_{WT} = H^{bulk,r} + H^{bulk,l} + H_t$$

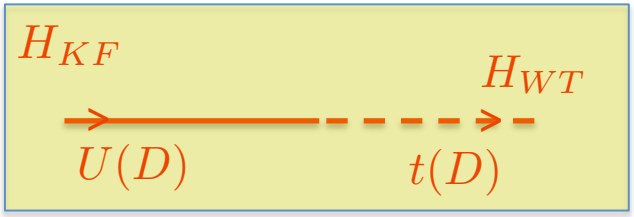


$$H^{bulk,\alpha} = -i\int_0^\infty dx(\psi^\dagger_{+,\alpha}\partial_x\psi_{+,\alpha} - \psi^\dagger_{-,\alpha}\partial_x\psi_{-,\alpha}) + 4g\int_0^\infty dx\ \rho_{+,\alpha}(x)\rho_{-,\alpha}(x)$$ with $\alpha = r, l$

$$H_t = t\big(\psi^\dagger_{+,r}(0) + \psi^\dagger_{-,r}(0)\big)\big(\psi_{+,l}(0) + \psi_{-,l}(0)\big) + \text{h.c}$$

- RG analysis of the WT Hamiltonian

$$D\frac{dt}{dD} \sim -(1-K)t$$ Tunneling is irrelevant



relevant ***irrelevant***

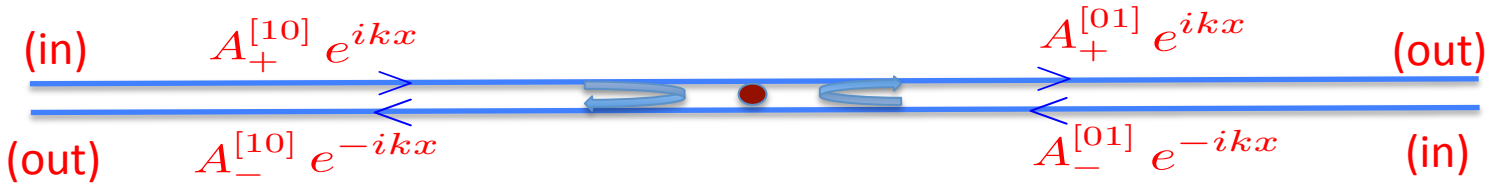
- H_{KF} - **flows from weak to strong coupling**

The Bethe Ansatz - I

Solve the model: $H_{KF} = H^{bulk} + H^{imp}$ **by the Bethe Ansatz**

- Eigenstate of a single particle:

$$\int dx \left[(e^{ikx} A_+^{[10]} \psi_+^\dagger(x) + e^{-ikx} A_-^{[10]} \psi_-^\dagger(x))\theta(-x) + (e^{ikx} A_+^{[01]} \psi_+^\dagger(x) + e^{-ikx} A_-^{[01]} \psi_-^\dagger(x))\theta(x) \right] |0\rangle.$$



The S-matrix relates (in)- to (out)- regions

$$\boxed{\begin{pmatrix} A_+^{[01]} \\ A_-^{[10]} \end{pmatrix}_{\text{(out)}} = S \begin{pmatrix} A_+^{[10]} \\ A_-^{[01]} \end{pmatrix}_{\text{(in)}}}$$

with $S = \dfrac{1}{1+iU} \begin{pmatrix} 1 & -iU \\ -iU & 1 \end{pmatrix}$

- Eigenstate of two fermions, L and R, crossing *away* from the impurity:

$$|F^{L,R}\rangle = \int dx\, dy\, F(x,y) \psi_+^\dagger(x) \psi_-^\dagger(y) |0\rangle$$

where $[-i(\partial_x - \partial_y) + 4g\delta(x-y)]F(x,y) = EF(x,y)$

so $F(x,y) = Ae^{ik_1 x - ik_2 y}[\theta(x-y) + e^{i\phi}\theta(y-x)]$

$$\rightarrow \boxed{S^{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\phi} & 0 & 0 \\ 0 & 0 & e^{i\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}.$$

with $e^{i\phi} = \dfrac{1-ig}{1+ig}$.

The Bethe Ansatz - II

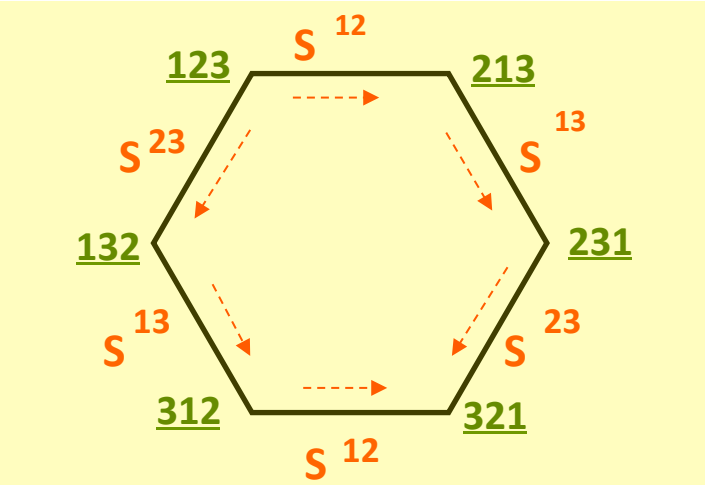
Write a Bethe Ansatz wave function:

$$|k_1, k_2\rangle = \sum_Q \sum_{\sigma_1\sigma_2} \int \theta(x_Q) A^Q_{\sigma_1\sigma_2} e^{\sigma_1 i k_1 x_1 + \sigma_2 i k_2 x_2} \psi^\dagger_{\sigma_1}(x_1)\psi^\dagger_{\sigma_1}(x_2)|0\rangle$$

Is it an eigenstate?

For consistency need Yang-Baxter relation (braiding)

$$S^{12}\, S^{10}\, S^{20} = S^{10}\, S^{20}\, S^{12}$$

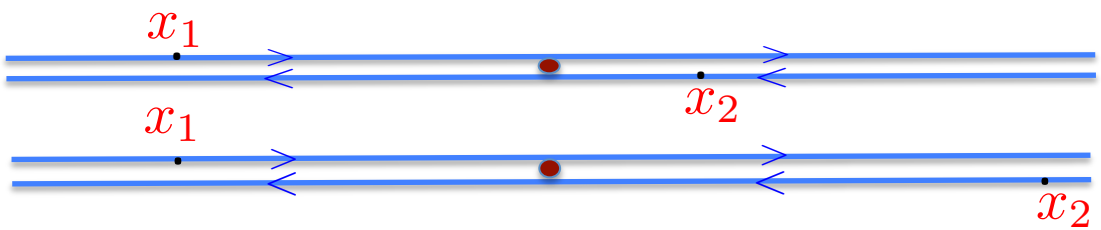


But it is violated..?

The Bethe Ansatz - III

The Hamiltonian determines the scattering of particles off the impurity and the local scattering of L-R when they cross. Other scattering relations remain undetermined: e.g. L-R on opposite sides

Example:



$A_{+-}^{[102B]}$

$A_{+-}^{[102A]}$

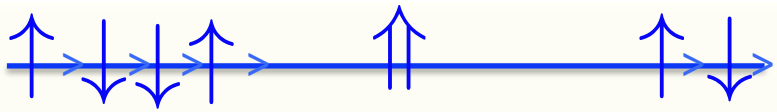
- When particles are on opposite sides of impurity the Hamiltonian acts as: $\pm i(\partial_1 - \partial_2)$ or $\pm i(\partial_1 + \partial_2)$ depending if they are of opposite or same chirality.

Allows introduction of arbitrary functions of $x_1 \pm x_2$ for opposite or same chiralities.
So we can introduce S-matrices relating regions weighted by: $\theta(\pm(x_1 - x_2))$ for $\sigma_1 = \sigma_2$ or weighted by $\theta(\pm(x_1 + x_2))$ for $\sigma_1 = -\sigma_2$.

- **Choice of S-matrix corresponds to choice of basis in degenerate subspace. It is determined by consistency.**

A simpler example: The Kondo model

$$h = -i \sum_j \partial_{x_j} + J \sum_j \delta(x_j) \vec{\sigma}_j \cdot \vec{\sigma}_0$$



Right moving spinful electrons x_j, σ_j

Localized impurity σ_0

The Hamiltonian determined the S-matrix for electron-*j* scattering off the impurity

$$S^{j0} \sim 1 + iJ(1 + \vec{\sigma}_j \cdot \vec{\sigma}_0)$$

The Bethe Ansatz - IV

- The S-matrices do not commute: $S^{i0}S^{j0} \neq S^{j0}S^{i0}$ (it is a quantum impurity)

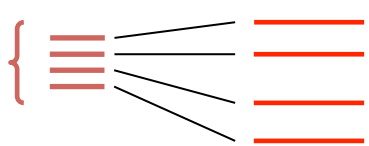
Electrons away from impurity are free so a naïve choice $S^{ij} = I$ leads to violation of YBE $S^{ij}S^{i0}S^{j0} = S^{j0}S^{i0}S^{ij}$.

- Claim: the free Hamiltonian $h = -i(\partial_{x_1} + \partial_{x_2})$ has eigenstates of the form $e^{(ik_1x_1+ik_2x_2)}F(x_1 - x_2)$ with $E = k_1 + k_2$ and $F(x_1 - x_2)$ arbitrary.

- Choose eigenstates: $e^{(ik_1x_1+ik_2x_2)}[\theta(x_1 - x_2) + S^{1,2}\theta(x_2 - x_1)]A$
where $S^{ij} = \frac{1}{2}(1 + \vec{\sigma}_i \cdot \vec{\sigma}_j)$ and then YBE is satisfied.

- What did we actually do? Chose a basis in a degenerate subspace.
The linear spectrum $E = k_1 + k_2 = (k_1 + q) + (k_2 - q)$ is infinitely degenerate.

$\frac{|\langle i|H_I|j\rangle|^2}{E_i - E_j}$ Raleigh-Schrodinger perturbation theory: To perturb a degenerate level need choose a basis isn the degenerate subspace that diagonalizes perturbation



levels split by perturbation

- The choice defines the **Bethe basis**, the correct basis to turn on interaction from a degenerate level

- The Bethe basis (unlike Fock basis $S^{ij} = I^{ij}$) *separates* charge and spin since the Kondo interaction is in the spin channel only

The Bethe Ansatz - V

- The full two fermion eigenstate:

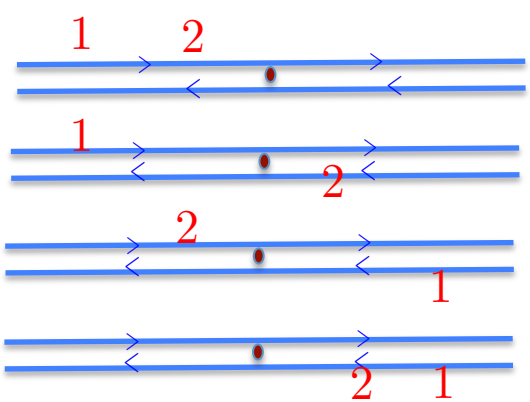
$$|k_1,k_2\rangle = \sum_Q \sum_{\sigma_1\sigma_2} \int \theta(x_Q) A^Q_{\sigma_1\sigma_2} e^{\sigma_1 i k_1 x_1 + \sigma_2 i k_2 x_2} \psi^\dagger_{\sigma_1}(x_1)\psi^\dagger_{\sigma_1}(x_2)|0\rangle$$

- Q Labels 8 regions ($N!2^N$ for N particles), depending on the ordering of the particles and impurity, and relative closeness to impurity.

$$\vec{A}_1 = \begin{pmatrix} A^{[120B]}_{++} \\ A^{[102B]}_{+-} \\ A^{[201B]}_{-+} \\ A^{[021B]}_{--} \end{pmatrix} \quad \vec{A}_2 = \begin{pmatrix} A^{[210A]}_{++} \\ A^{[102A]}_{+-} \\ A^{[201A]}_{-+} \\ A^{[012A]}_{--} \end{pmatrix} \quad \vec{A}_3 = \begin{pmatrix} A^{[201A]}_{++} \\ A^{[012A]}_{+-} \\ A^{[210A]}_{-+} \\ A^{[102A]}_{--} \end{pmatrix} \quad \vec{A}_4 = \begin{pmatrix} A^{[201B]}_{++} \\ A^{[021B]}_{+-} \\ A^{[120B]}_{-+} \\ A^{[102B]}_{--} \end{pmatrix} \quad \vec{A}_5 = \begin{pmatrix} A^{[021B]}_{++} \\ A^{[201B]}_{+-} \\ A^{[102B]}_{-+} \\ A^{[120B]}_{--} \end{pmatrix} \quad \vec{A}_6 = \begin{pmatrix} A^{[012A]}_{++} \\ A^{[201A]}_{+-} \\ A^{[102A]}_{-+} \\ A^{[210A]}_{--} \end{pmatrix} \quad \vec{A}_7 = \begin{pmatrix} A^{[102A]}_{++} \\ A^{[210A]}_{+-} \\ A^{[012A]}_{-+} \\ A^{[201A]}_{--} \end{pmatrix} \quad \vec{A}_8 = \begin{pmatrix} A^{[102B]}_{++} \\ A^{[120B]}_{+-} \\ A^{[021B]}_{-+} \\ A^{[201B]}_{--} \end{pmatrix}$$

Both particles incident on impurity but 2 is closer	Both particles incident on impurity but 1 is closer	2 moves to impurity, 1 moves away but 1 is closer	2 moves to impurity, 1 moves away but 2 is closer	Both particles move away from impurity, 2 is closer	Both particles move away from impurity, 1 is closer	1 moves to impurity, 2 moves away but 1 is closer	1 moves to impurity, 2 moves away but 2 is closer

For example: $\vec{A}_1$



The Hamiltonian determines: S^{j0} and $S^{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\phi} & 0 & 0 \\ 0 & 0 & e^{i\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\vec{A}_8 = S^{20}\vec{A}_1, \quad \vec{A}_3 = S^{10}\vec{A}_2$$

$$\vec{A}_5 = S^{20}\vec{A}_4, \quad \vec{A}_6 = S^{10}\vec{A}_7$$

$$\vec{A}_7 = S^{12}\vec{A}_8, \quad \vec{A}_4 = S^{12}\vec{A}_3$$

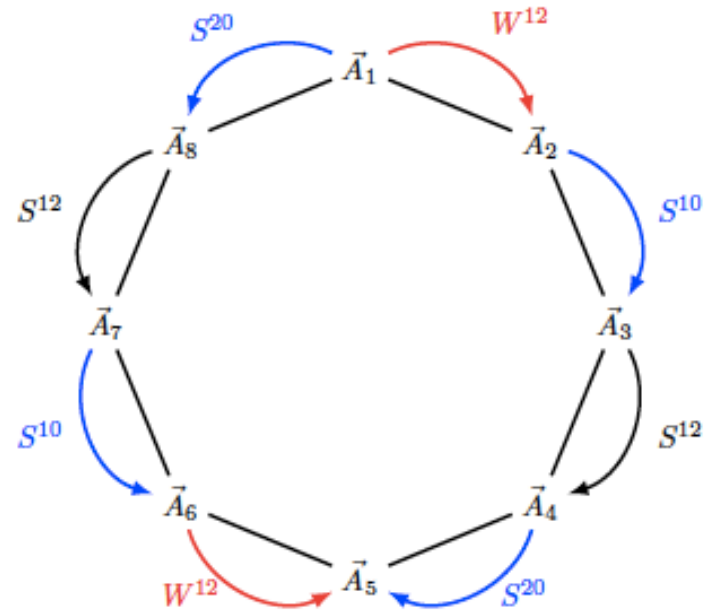
We have freedom to relate $\vec{A}_1 \leftrightarrow \vec{A}_2, \quad \vec{A}_5 \leftrightarrow \vec{A}_6$

The Bethe Ansatz - VI

We choose: $\vec{A}_2 = W^{12}\vec{A}_1, \quad \vec{A}_6 = W^{12}\vec{A}_5$ with $W^{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

Choice dictated by consistency requirement:

$$S^{20}S^{12}S^{10}W^{12} = W^{12}S^{10}S^{12}S^{20}$$



Extra regions modify consistency condition from YBE to a reflection equation

Bethe Basis: The partition to extra regions is dictated by linear derivative and the degeneracies associated with it require choice of correct basis in the degenerate subspace.

The Bethe Ansatz - VII

Generalization to N-particles:

- The **eigenstate,**

$$|\vec{k}\rangle = \sum_Q \sum_{\vec{\sigma}} \int \theta(x_Q) A^Q_{\vec{\sigma}} e^{i\sum \sigma_j k_j x_j} \prod \psi^\dagger_{\sigma_j}(x_j)|0\rangle$$

where the amplitudes $A^Q_{\vec{\sigma}}$ are related by:

$$S^{j0} = S_j \otimes_{k\neq j} I, \quad S^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\phi} & 0 & 0 \\ 0 & 0 & e^{i\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{ij} \otimes_{k\neq i,j} I, \quad W^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{ij} \otimes_{k\neq i,j} I$$

Satisfying the YBE and reflection equations

$$S^{k0}S^{jk}S^{j0}W^{jk} = W^{jk}S^{j0}S^{jk}S^{k0}$$

$$W^{jk}W^{jl}W^{kl} = W^{kl}W^{jl}W^{jk}$$

$$W^{jk}S^{jl}S^{kl} = S^{kl}S^{jl}W^{jk}$$

- **The energy eigenvalue,** $E = \sum_j k_j$

Periodic boundary conditions and spectrum

- **PBC imply:** $e^{-ik_jL}A_{\sigma_1\ldots\sigma_N} = (Z_j)^{\sigma'_1\ldots\sigma'_N}_{\sigma_1\ldots\sigma_N} A_{\sigma'_1\ldots\sigma'_N}$

with *transfer matrix* for particle - j : $Z_j = W^{j-1j}..W^{1j}S^{1j}..S^{jN}S^{j0}W^{jN}..W^{jj+1}$

- All Z_j are equivalent (no dimensionful paramenter in Hamiltonian)

- **Diagonalization of the transfer matrix** Z_1

To bring our problem into the QIS form

i. *We need to find R-matrix such that:*

$R(u \to 0) = W^{ij} \quad R(u \to \infty) = S^{ij}$

ii. *We need to find* $K^{\pm}(u)$ *(boundary) matrices such that:*

$K^-(u) \to S^{10}$

$K^+(u)$ chosen to impose BC

- Introduce XXZ R-matrix

$$\mathcal{R}(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sinh u}{\sinh(u+\eta)} & \frac{\sinh \eta}{\sinh(u+\eta)} & 0 \\ 0 & \frac{\sinh \eta}{\sinh(u+\eta)} & \frac{\sinh u}{\sinh(u+\eta)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

with crossing parameter $e^{-\eta} = e^{i\phi} = \dfrac{1-ig}{1+ig}$

Periodic boundary conditions and spectrum

- Introduce *reflection (or boundary) matrices* $K^{\pm}(u)$ H. De Vega and A. Gonzalez Ruiz '93

$$K^-(u) = \begin{pmatrix} 2i\cosh(c+\theta/2)\cosh u & \sinh(2u) \\ \sinh(2u) & 2i\cosh(c+\theta/2)\cosh u \end{pmatrix}.$$

$$K^+(u) = \begin{pmatrix} 2\sinh(-\theta)\cosh(u+\eta) & -\sinh(2u+2\eta) \\ -\sinh(2u+2\eta) & 2\sinh(-\theta)\cosh(u+\eta) \end{pmatrix}$$

with: $c = \log\left((1-U^2/4)/U\right)$
and θ an inhomogeneity parameter

then $K^-(u) \to S^{10}$ as $u = \theta/2 \to \infty$ reproduces impurity scattering

$K^+(u)$ can be used to impose twisted boundary conditions

- Introduce *Monodromy matrix* (similar to *XXZ* with two boundaries)

$$\Xi_0(u) = \mathcal{C}K^+(u)\mathcal{R}_{01}(u+\theta/2)\ldots\mathcal{R}_{0N}(u+\theta/2)K^-(u)\mathcal{R}_{0N}(u-\theta/2)\ldots\mathcal{R}_{01}(u-\theta/2)$$

0- auxiliary space, and $\mathcal{C} = \frac{-\beta e^{-\eta}}{\sinh\theta\sinh\frac{3\theta}{2}}$

- Define *Transfer matrix* $t(u) = \mathrm{Tr}_0\,\Xi(u)$

Transfer matrix satisfies:
$$[t(u), t(v)] = 0$$
Sklyanin '88

- Claim: $$\boxed{Z = \lim_{\theta\to\infty} t(\theta/2)}$$

Periodic boundary conditions and spectrum

- **To diagonalize** $Z(u) = t(u = \theta/2)$ use ***ODBA*** **(off diagonal Bethe Ansatz)**

J. Cao, W.-L. Yang, K. Shi, and Y. Wang '13

Unlike QIS, *ODBA* does not require reference state

Y. Wang, W.-L. Yang, J. Cao, and K. Shi '15

We find - Eigenvalues:

$$\Lambda(\theta/2) = -4i\beta e^{i\phi} \frac{\sinh(\theta - 2i\phi)\cosh(c)\cosh^2(\theta/2)}{\sinh(\theta - i\phi)\sinh\theta} \prod_j^N \frac{\sinh(\theta/2 - \mu_j + i\phi)}{\sinh(\theta/2 + \mu_j - i\phi)}$$

with parameters $\{\mu_j, j = 1...N\}$ satisfying the BAE: for N even

$$\frac{\left[\cosh\left(i(N+1)\phi + c + i\pi/2 - \theta/2 + 2\sum_{j=1}^N \mu_j\right) - 1\right]\sinh(2\mu_j - i\phi)\sinh(2\mu_j - 2i\phi)}{2i\cosh(\mu_j + c + \theta/2 - i\phi)^2 \sinh(\mu_j - \theta - i\phi)}$$

$$= \prod_{l=1}^N \frac{\sinh(\mu_j + \mu_l - i\phi)\sinh(\mu_j + \mu_l - 2i\phi)}{\sinh(\mu_j + \theta/2 - i\phi)\sinh(\mu_j - \theta/2 - i\phi)}$$

selection rule: $\mu_j \neq \mu_k, \mu_k + i\phi$

The BAE

Recall $$e^{-ikL} = \lim_{\theta\to\infty} \Lambda(\theta/2)$$

Note: We embedded our problem in a boundary problem. **The limit restores the impurity nature.**

- To take the limit introduce two sets of Bethe parameters: $\{\lambda_j, \nu_j\}$

$$\mu_j = \begin{cases} \lambda_j + i\phi/2 + \theta/2 & \text{if } j \leq \frac{N}{2} \\ -\nu_{j-N/2} + i\phi/2 - \theta/2 & \text{if } j > \frac{N}{2}. \end{cases}$$

- We find in the limit:

$$e^{-ikL} = -\frac{1}{\alpha}\prod_j^{N/2} \frac{\sinh(\lambda_j - i\phi/2)}{\sinh(\nu_j + i\phi/2)} e^{-\lambda_j+\nu_j+i\phi}$$

with the parameters satisfying the BAE

$$\begin{cases} \sinh^N(\lambda_j - i\phi/2) = -e^{-2\lambda_j - i\phi + 2c} e^{2\sum_k(2\lambda_k - \nu_k)} \prod_k^{N/2} \sinh(\lambda_j - \nu_k)\sinh(\lambda_j - \nu_k - i\phi) \\ \sinh^N(\nu_j + i\phi/2) = \frac{2i\cosh(c - \nu_j - i\phi/2)}{e^{\nu_j - c + i\phi/2}} e^{2\sum_k \lambda_k} \prod_k^{N/2} \sinh(\nu_j - \lambda_k)\sinh(\nu_j - \lambda_k + i\phi) \end{cases}$$

The Luttinger limit (impurity decoupled)

Check: in the limit $U \to 0$ (i.e. $c \to \infty$) need to reproduce the ***Luttinger liquid*** spectrum

BAE imply: $\lambda_j = \nu_j$ or $\lambda_j = \nu_j + i\phi$

namely pairs, $\mu_{j+N/2} = -\mu_j + i\phi$ or $\mu_{j+N/2} = -\mu_j + 2i\phi$, $\mu_j,\ j \leq N/2$ undetermined

Take M pairs $\mu_{j+N/2} = -\mu_j + i\phi$ and $N/2 - M$ pairs $\mu_{j+N/2} = -\mu_j + 2i\phi$

- The latter decouple in the limit and we find:

XXZ-like equations, chirality analogous to spin - but different dynamics

$$\begin{cases} e^{-ikL} = e^{Mi\phi} \prod_{j=1}^{M} \dfrac{\sinh(\lambda_j - i\phi/2)}{\sinh(\lambda_j + i\phi/2)} \\ \dfrac{\sinh^N(\lambda_j - i\phi/2)}{\sinh^N(\lambda_j + i\phi/2)} = e^{i(N-2M)\phi} \prod_{k \neq j}^{M} \dfrac{\sinh(\lambda_j - \lambda_k - i\phi)}{\sinh(\lambda_j - \lambda_k + i\phi)}. \end{cases}$$

And we obtain the *Luttinger* spectrum

$$E = \frac{2\pi}{L} \sum_k^N n_k - \frac{2\pi}{L} \sum_j^M I_j - \frac{2M(N-M)}{L}\phi$$

with n_k and I_j being the charge and chirality quantum numbers

The Luttinger limit and the *string hypothesis*

- The $\{\lambda_j\}$ solutions in the Luttinger limit, $c \to \infty$:

  - XXZ is a well studied problem: the solutions fall into *strings*

  For simplicity choose $|\phi| = \pi/\nu$ with $\nu > 2$

Bulk excitations

i. We have positive parity *j*-string solutions (centered around zero):

$$\lambda^{(j,l)} = \lambda^j + i(j+1-2l)\phi/2 \qquad j = 1, \ldots, \nu - 1$$

ii. We have negative parity strings centered around $i\pi/2$ and for our choice only 1-strings allowed: $\lambda_\alpha^\nu + i\pi/2.$

- When the impurity is coupled, for large but finite c (i.e. $U \ll 1$), the corrections are of order $1/N$. So assume the string hypothesis,

$$\lambda - \text{real}$$
$$\mathrm{Im}\{\lambda^{(j,l)}\} = \mathrm{Im}\{\nu^{(j,l)}\} = (2j+1-l)\phi/2$$
$$\mathrm{Im}\{\lambda_j\} = \mathrm{Im}\{\nu_j\} = \pi/2$$
$$\mathrm{Im}\{\lambda_j - \nu_j\} = \phi$$

Thermodynamics I

- Constructing Free Energy $F = E - TS$ with $E\{\rho_j\} = \sum_j \int dk\, \rho_j(k) k$ and Yang-Yang entropy,

$$S = \sum_j \int \left[(\rho_j + \rho_j^h) \log(\rho_j + \rho_j^h) - \rho_j \log(\rho_j) - \rho_j^h \log(\rho_j^h) \right]$$

For convenience $|\phi| = \pi/\nu \quad \nu > 2$

Minimize w.r.t. densities solution of the BAE to obtain the TBA eqns:

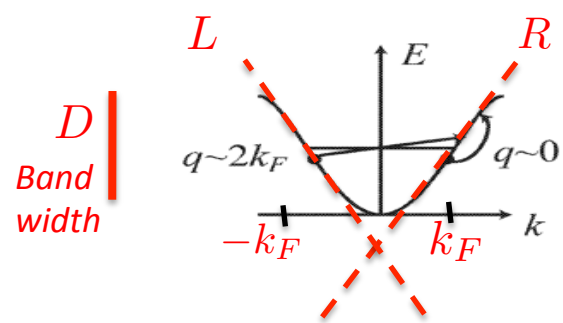
$$\log \eta_j(x) = -\delta_{j,1} \frac{2D}{T} \arctan e^{\pi x} + G * \log(1 + \eta_{j+1}(x))(1 + \eta_{j-1}(x)) + \delta_{j,\nu-2} G * \log(1 + \eta_\nu^{-1}(x))$$

$$\log \eta_{\nu-1}(x) = G * \log(1 + \eta_{\nu-2}(x)) = -\log \eta_\nu(x)$$

With $\eta_j(x) = \rho_j^h(x)/\rho_j(x)$ and $G(x) = \dfrac{1}{2\cosh \pi x}$. The density $D = \dfrac{N}{L}$ plays the role of the cut-off since we set $k_F = 0$

- **Scaling limit:** remove cut-off dependence $D \to \infty$

Linearized the spectrum - valid as long as $T, p... \ll D$



Thermodynamics II

- **Scaling limit:** remove cut-off dependence $D \to \infty$ without changing the physics

Define: $\varphi_j(x) = \frac{1}{T}\log\left(\eta_j(x + \frac{1}{\pi}\log\frac{T}{D})\right)$

Approximate driving term $-\frac{2D}{T}\arctan\exp\pi(x + \frac{1}{\pi}\log\frac{T}{D}) \simeq -2e^{\pi x}$

$|\phi| = \pi/\nu$

- The universal form of the TBA:

$\nu > 2$

$$\varphi_j(x) = -\delta_{j,1}2e^{\pi x} + G * \log\left(1 + e^{\varphi_{j-1}(x)}\right)\left(1 + e^{\varphi_{j+1}(x)}\right), \quad j < \nu - 2$$

$$\varphi_{\nu-2}(x) = G * \log(1 + e^{\varphi_{\nu-1}(x)})(1 + e^{\varphi_{\nu-3}(x)})(1 + e^{-\varphi_\nu(x)})$$

$$\varphi_{\nu-1}(x) = G * \log\left(1 + e^{\varphi_{\nu-2}(x)}\right) = -\varphi_\nu(x)$$

- The free energy $F = F^{LL} + F^i$ consists of:

$$F^{LL} = E_0 - TN\int G(x)\log\left(1 + \exp\varphi_1(x)\right)$$

The bulk contribution

$$F^i = -T\int \mathrm{d}x\, G(x + \frac{1}{\pi}\log\frac{T}{T_{KF}})\log\left(1 + e^{\varphi_{\nu-1}(x)}\right)$$

The impurity contribution

- A scale emerges: $T_{KF} = De^{\pi c/\phi}$

Thermodynamics – RG and universality

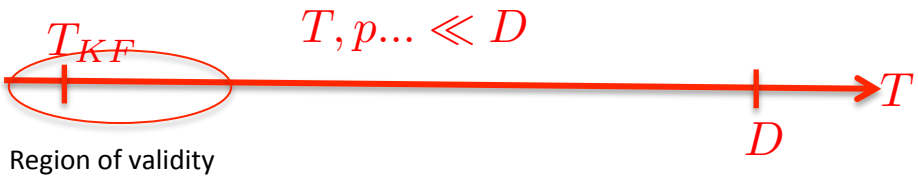
- The dynamically generated scale:

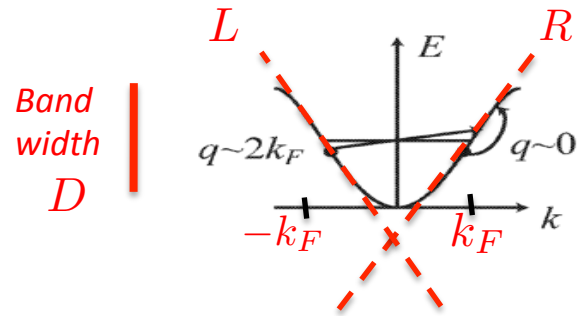
$$T_{KF} = De^{\pi c/\phi} = D\left(\frac{U}{1-U^2/4}\right)^{\frac{\pi}{2\arctan g}}$$

Recall: $c = \log\left((1-U^2/4)/U\right)$

- Two equivalent languages:

1. *Universality*: Increase band width
   -- Linearized the spectrum, valid as long as,





- High and low temperature regimes defined w.r.t T_{KF}, always small compared to D
- Equivalently send $D \to \infty$ holding T_{KF}, then all results universal.

2. *Renormalization Group*: Reduce band width while adjusting the coupling U so that the the low energy physics remains invariant, *i.e.* holding T_{KF} fixed.

for repulsive interaction RG flow: $U(D) \sim (T_{KF}/D)^{\frac{2\arctan g}{\pi}} \to \infty$ as $D \to 0$

Wire is cut in IR
Kane-Fisher '92

Thermodynamics and RG fixed points

Recall kernel:
$G(x + \frac{1}{\pi}\log\frac{T}{T_{KF}})$

- **The impurity free energy -** high and low temperature behavior

Need to solve TBA eqns for $x \to \pm\infty$ corresponding to $T \to 0, \infty$.

Denoting: $e^{\varphi_j(\pm\infty)} = \gamma^{\pm}$ we find:

$$\gamma_j^- = (j+1)^2 - 1, \qquad \gamma_{\nu-1}^- = \nu - 1 = 1/\gamma_\nu^-$$
$$\gamma_j^+ = j^2 - 1, \qquad \gamma_{\nu-1}^+ = \nu - 2 = 1/\gamma_\nu^+$$

- **The fixed points:**

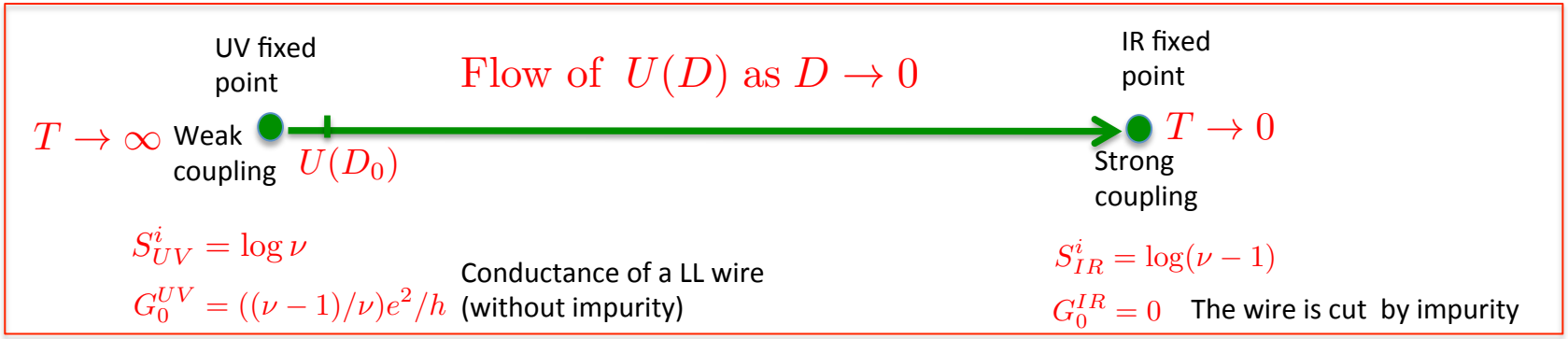
Free energy at strong (*IR* $T \to 0$) and weak (*UV* $T \to \infty$) coupling limits

$$F_{UV}^i = \frac{T}{2}\log(\nu), \qquad F_{IR}^i = \frac{T}{2}\log(\nu - 1)$$

Entropy decreases as system flows to *IR*

Zamolodchikov, Affleck Ludwig, Saleur

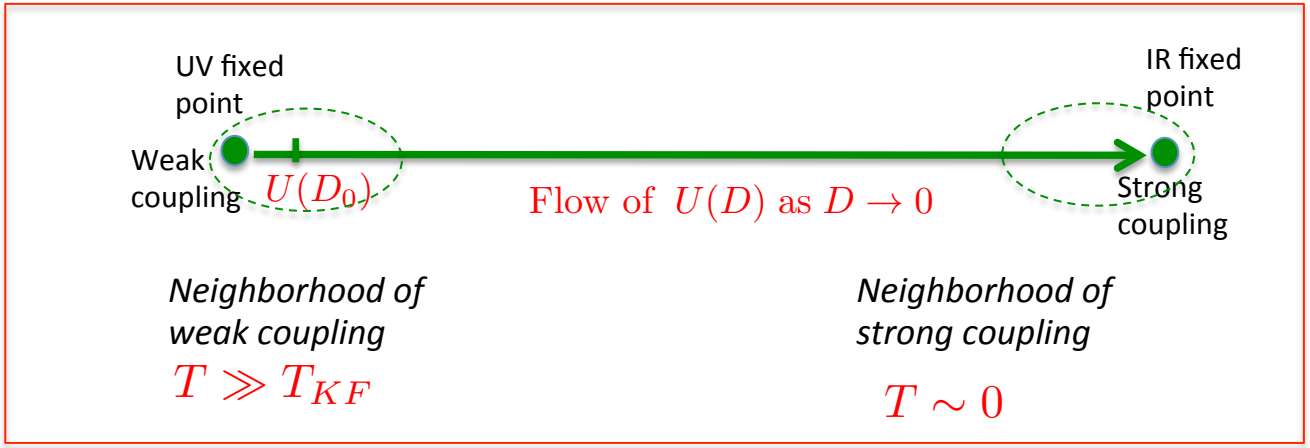
$$S_{UV}^i - S_{IR}^i = \frac{1}{2}\log\frac{\nu}{\nu - 1}$$



The *fixed points*

Thermodynamics and transport

Now study neighborhood of fixed points



Approach to *fixed points*

- Leading irrelevant operators determine **approach** to fixed points

The corrections to the asymptotic values cf. *de Sa and Tsvelik '95 for XXZ*

Specific heat: $C(T \ll T_{KF}) \sim (T/T_{KF})^{\frac{2}{\nu-1}}$, $C(T \gg T_{KF}) \sim (T_{KF}/T)^{\frac{2}{\nu}}$

- Same irrelevant operator controls conductance *Affleck and Ludwig* '92

Conductance: $G(T \ll T_{KF}) \sim (T/T_{KF})^{\frac{2}{\nu-1}}$ $G(T \gg T_{KF}) - G_0 \sim (T_{KF}/T)^{\frac{2}{\nu}}$

Vanishes in *IR K&F '92*

$G_0 = ((\nu-1)/\nu)e^2/h$ - *UV fixed point conductance*

Excitation Spectrum and Scattering

- **Ground state configuration**- solution of BAE:

$$Na_j(x) + b_j(x) = \rho_j(x) + \rho_j^h(x) + \sum_k^{\nu} A_{jk} * \rho_k(x)$$

with no holes or strings $\rho_1^h(x) = \rho_j(x) = 0, \quad j > 1$

- **Fundamental excitations-** *chiron:* a hole at x^h in the distribution ρ_1

$$\varepsilon = 2D \arctan e^{\pi x^h}$$

It scatters off the impurity, (S-matrix obtained by studying 1/L corrections to momentum)

$$S^{c,i}(\varepsilon) = e^{i\Delta^{c,i}(\frac{1}{\pi}\log(\varepsilon/T_{KF}))}$$ with $$\Delta^{c,i}(x) = \int \frac{d\omega}{8\pi i\omega} \frac{\tanh(\omega/2)}{\sinh((\pi/\phi - 1)\omega/2)} e^{i\omega x}$$

chiron-chiron scattering

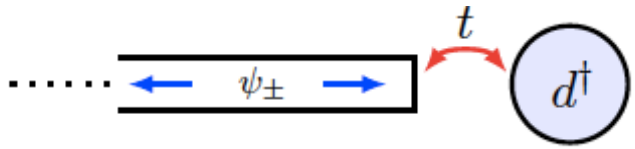
$$S^{c,c}(\varepsilon_1, \varepsilon_2) = e^{i\Delta^{c,c}(\varepsilon_1 - \varepsilon_2)}$$ with $$\Delta^{c,c}(x) = \int \frac{d\omega}{4\pi i\omega} \frac{\sinh((\pi/\phi - 2)\omega/2) e^{i\omega x}}{\cosh(\omega/2)\sinh((\pi/\phi - 1)\omega/2)}$$

- **Other approach:** Bosonization and relation to Boundary Sine-Gordon

Zamolodchikov '93 Saleur '94

Solitons, anti-solitons and breathers postulated with their on-shell S-matrices

Quantum dot at the edge – the Bethe Ansatz



Boundary condition at the edge

$$\psi_-(0) = -\psi_+(0)$$

$$H = H_{LL} + \frac{t}{2}((\psi_+^\dagger(0) - \psi_-^\dagger(0))d + \text{h.c}) + \epsilon_0 d^\dagger d + \frac{U}{2} d^\dagger d \sum_{\sigma=\pm} \psi_\sigma^\dagger(0)\psi_\sigma(0)$$

For $K = 1$ we have $H_{LL} \to H_0$ Resonance level model, Filyov and Wiegmann '81

The wave functions consist of parts with the dot occupied or unoccupied, the latter takes the Bethe Ansatz form:

$$|\vec{k}\rangle = \sum_Q \int A^Q \theta(\vec{x}_Q) e^{\sum_j^N k_j x_j} \prod_{j=1}^N \psi^\dagger(x_j)|0\rangle$$

where amplitudes are related by:

with $S^{10} = \dfrac{k_1 - \epsilon_0 - i\Gamma}{k_1 - \epsilon_0 + i\Gamma}$ and $S^{12} = \dfrac{k_1 + k_2 - 2\bar{\epsilon}_0 - i\frac{U'}{2}(k_1 - k_2)}{k_1 + k_2 - 2\bar{\epsilon}_0 + i\frac{U'}{2}(k_1 - k_2)}$

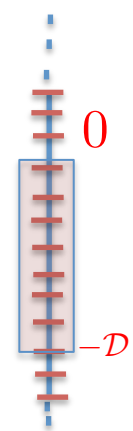
where

$\arctan(U'/2) = \arctan(U/2) - \arctan(g)$ $\bar{\epsilon}_0 = \epsilon_0 - \Gamma U'/2$ $\Gamma = t^2/2$

Imposing periodic boundary conditions leads to BAE:

$$e^{-ik_jL} = e^{i(N-1)\phi} \frac{k_j - \epsilon_0 - i\Gamma}{k_j - \epsilon_0 + i\Gamma} \prod_l^N \frac{k_j + k_l - 2\bar{\epsilon}_0 - i\frac{U'}{2}(k_j - k_l)}{k_j + k_l - 2\bar{\epsilon}_0 + i\frac{U'}{2}(k_j - k_l)}.$$

Solve for k_j and fill from cut-off $-\mathcal{D}$ upwards, as determined by minimizing E



The quantum dot – the Bethe Ansatz

It is convenient to rewrite in terms of rapidities $k_j = \mathcal{D} e^{x_j} + \bar{\epsilon}_0$

The BAE take the form

$$e^{-i\mathcal{D}e^{x_j}L} = e^{i(N-1)\phi + i\bar{\epsilon}_0 L} \frac{\cosh\frac{1}{2}\left(x_j - c + i\Delta\right)}{\cosh\frac{1}{2}\left(x_j - c - i\Delta\right)} \prod_l^N \frac{\sinh\frac{1}{2}\left(x_j - x_l - 2i\Delta\right)}{\sinh\frac{1}{2}\left(x_j - x_l + 2i\Delta\right)}$$

with

$$\Delta = \frac{\pi}{2}\left(2 - \frac{1}{K}\right) + \arctan\left(\frac{U}{2}\right)$$ *IR catastrophe index*

Recall: $K > 1/2$

$$e^c = \gamma \frac{\Gamma}{\mathcal{D}} \qquad \gamma = 1/\sqrt{1 + (U'/2)^2}$$

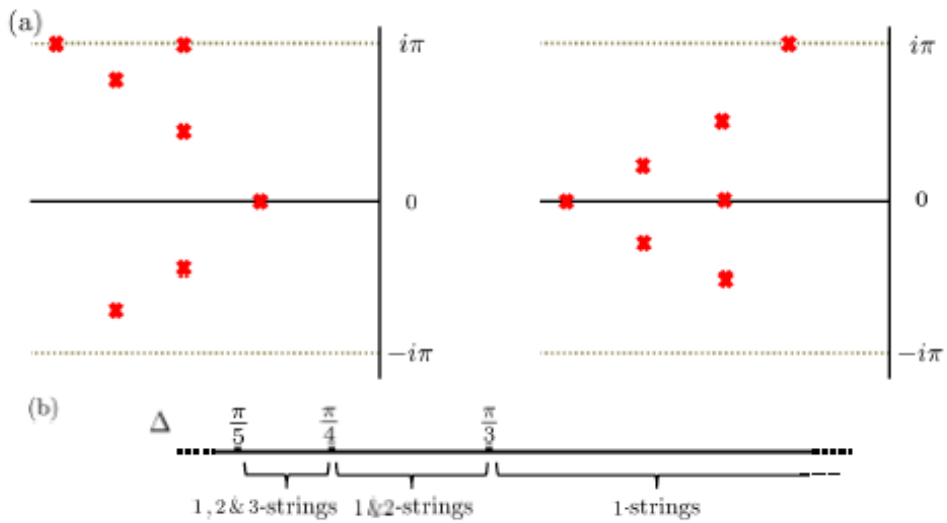
Need solve for k_j compute the energy $E = \sum_j k_j$ and identify the ground state, the excitations, compute the free energy

The quantum dot – the Bethe Ansatz

The rapidities form strings:

For $\pi\frac{\nu-1}{\nu} < \Delta \leq \pi\frac{\nu}{\nu+1}$ we have $x^l = x + i(\pi-\Delta)(n-1-2l)$ with $0 \leq n \leq \nu$

For $\frac{\pi}{\nu+1} \leq \Delta < \frac{\pi}{\nu}$ we have $x^l = x + i\pi + i\Delta(n-1-2l)$ with $0 \leq n \leq \nu$



Claim: For $\Delta > \pi/3$ the ground state consists of 1-strings

For $\pi/3 > \Delta \geq \pi/4$ the ground state consists of 1- and 2-strings

And so on

The quantum dot – the occupation

T=0 properties: The dot occupation $n_d = \langle d^\dagger d \rangle$

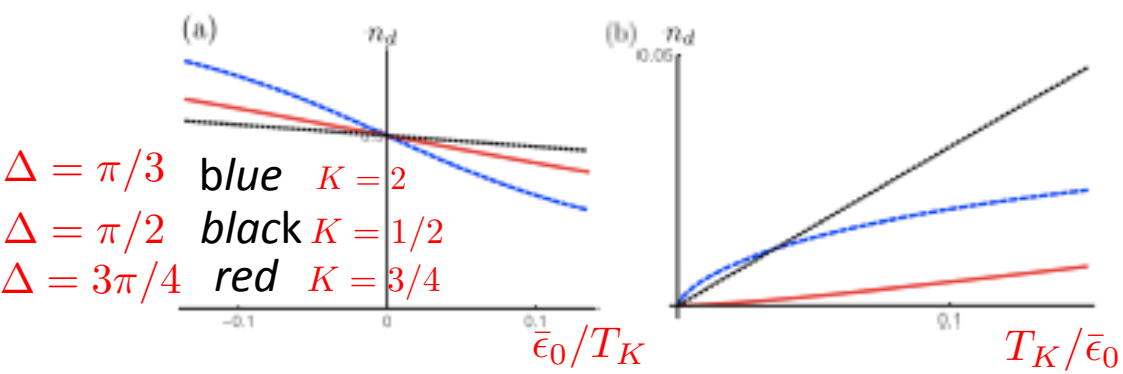
- For $\Delta > 0$ system properties are universal as $\mathcal{D} \to \infty$ while holding $T_K = \mathcal{D}\left(\gamma \frac{\Gamma}{\mathcal{D}}\right)^{\frac{\pi}{2\Delta}}$ fixed.

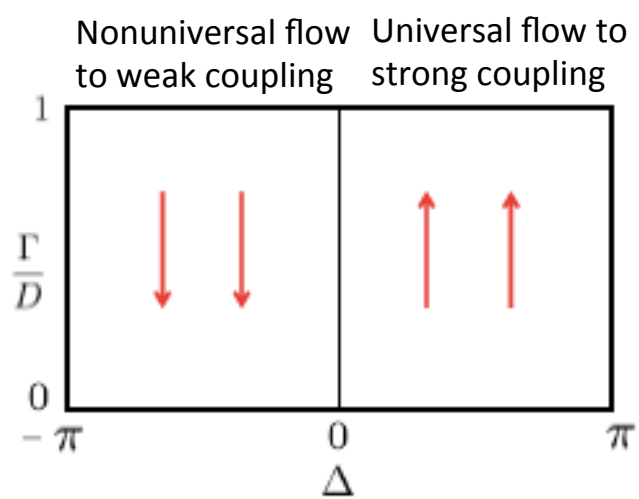
- The functional form of $n_d(\frac{\bar{\epsilon}_0}{T_K}, \Delta)$ depends on the parameter regime:

We consider $\Delta > \pi/3$

$$n_d^{<>}(\bar{\epsilon}_0, \Delta) = \frac{1}{2} - \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{e^{\frac{\pi}{2\Delta}(2n+1)a}}{2n+1} \left(\frac{\bar{\epsilon}_0}{T_K}\right)^{2n+1} \frac{\Gamma(1 + \frac{\pi}{2\Delta}(2n+1))}{\Gamma(1 + \frac{\pi - \Delta}{2\Delta}(2n+1))}$$

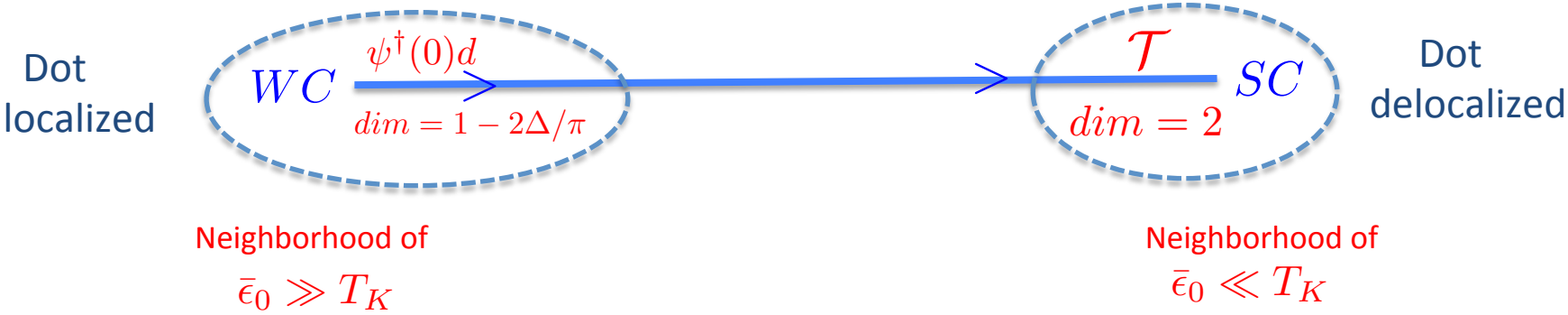
$$n_d^{>>}(\bar{\epsilon}_0, \Delta) = \frac{1}{2\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} e^{-na} \frac{\Gamma(\frac{1}{2} + \frac{\Delta}{\pi} n)}{\Gamma(1 - \frac{\pi - \Delta}{\pi} n)} \left(\frac{T_K}{\bar{\epsilon}_0}\right)^{\frac{2\Delta}{\pi} n}$$





The quantum dot – the RG flow

The RG flow for $\Delta > 0$ is universal, from weak to strong coupling



Flow of $\Gamma(\mathcal{D})$ as $\mathcal{D} \to \infty$

Similar picture at *T>0* : Compute free energy $F = T\, f(T/T_K, \bar{\epsilon}_0/T_K)$

- The entropy flow as a function of the temperature $\mathcal{S}$: $\log 2 \to 0$

  with $C_v \sim T/T_K$

- Fixed point: Fermi liquid like

Identify impurity spin and dot occupation - $S^z = n_d - 1/2$.

- Screened Kondo spin corresponds to a fully hybridized dot with occupation, $n_d = 1/2$.
- Unscreened spin corresponds to decoupled dot: either full or empty, $n_d = 0, 1$.
- The role of an external magnetic field in the Kondo model is played by the dot energy $\bar{\epsilon}_0$

Conclusions and Outlook

Conclusions:

- Solved the Kane-Fisher model exactly in all parameter regimes
- Obtained analytic for asymptotic thermodynamic and transport quantities
- Solved the weak tunneling model (dual to Kane-Fisher)
- Solved the edge coupled dot-wire, studied dot occupation and thermodynamics

To do list:

➢ *In equilibrium*

- A dot edge coupled to *n* wires
- A dot side coupled to a Luttinger wire, to n Luttinger wires
- Kondo impurity coupled to a Luttinger liquid
- Spin anisotropic systems

Conclusions and Outlook

- *Out-of- equilibrium*

i. Quench dynamics of the systems:

- Coupling the impurity to the lead (Interacting *X-ray edge problem*)
- Changing the interaction strength
- The Loschmidt echo, quantum work

ii. Nonequilibrium transport at finite voltage

iii. Persistent currents

iv. Periodically driven systems (Floquet)

Bethe Ansatz eqns in the thrmodynamic limit

- **Thermodynamic limit** $N, L \to \infty, \quad D = N/L \quad \text{fixed}$

  Introduce string and string-hole densities: $\rho_j(x), \rho_j^h(x)$ satisfying the continuous BAE:

$$N a_j(x) + b_j(x) = \rho_j(x) + \rho_j^h(x) + \sum_{k\nu}^{\nu} A_{jk} * \rho_k(x)$$

$$N a_\nu(x) + b_\nu(x) = -\rho_\nu(x) - \rho_\nu^h(x) + \sum_{k} A_{\nu k} * \rho_k(x)$$

where $a_j(x) = \frac{1}{2\pi}\frac{d}{dx}p(x, n_j, v_j)$

$$b_j(x) = -\frac{1}{4\pi}\frac{d}{dx}p(x - c/\phi, n_j, -v_j)$$

$$A_{jk}(x) = \frac{1}{2\pi}\frac{d}{dx}\Theta_{jk}(x)$$

$$p(x, n_j, v_j) = 2v_j \arctan\left((\cot n_j\phi/2)^{v_j} \tanh \phi x\right)$$

$$\Theta_{jk}(x) = p(x, |n_j - n_k|, v_j v_k) + p(x, n_j + n_k, v_j v_k) + 2\sum_q p(x, |n_j - n_k| + 2q, v_j v_k)$$

and v_j denotes the parity of string j.

Eqns similar to AKM, Tsvelik-Wiegmann '84

but impurity appears with opposite parity – It mixes L & R

- The corresponding energy is:

$$E\{\rho_j\} = -\sum_{j=1}^{\nu} D \int \rho_j(x)\left(p(x, n_j, v_j) + \theta(v_j)\pi\right)$$

Conventions

$$\phi/\pi = 1 - 1/K = 1/\nu = (2/\pi)\arctan g$$