Boundary CFT and topological phases of matter

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Overview

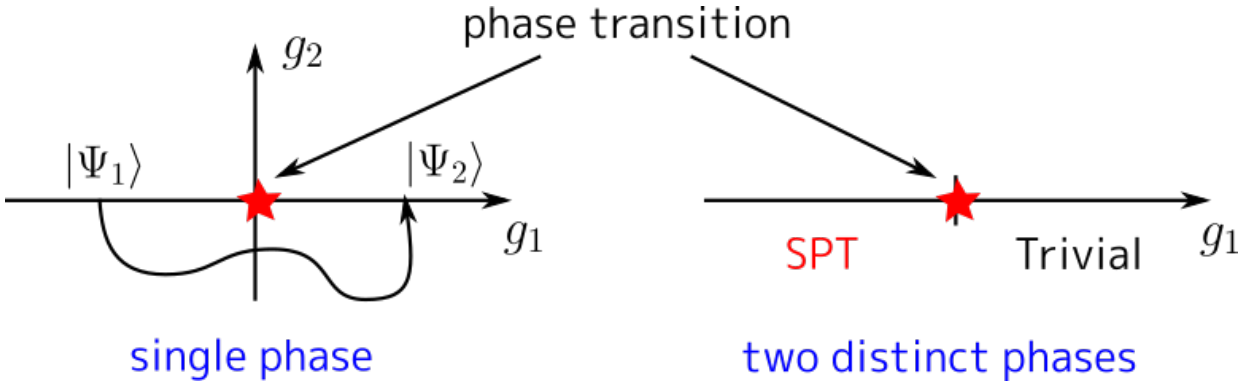
- Theme: Topological phases of matter and boundary/defect CFT
- Topological phases; fully *gapped*, can be described by topological quantum field theories.
- Conformal field theories: *gapless*
- Nevertheless, (B)CFT in (1+1)d can be used to discuss properties of topological phases of matter both in (1+1)d and (2+1)d.

Overview

- More specifically:
  - BCFT_2 and SPT_2 (or gapped phases in general)
    (topological invariants and entanglement spectrum)
  - BCFT_2 and SPT_3
    (diagnosing anomalies of boundaries of bulk topological phases)
  - BCFT_2 and topologically-ordered phases in (2+1)d
    (computation of various entanglement measures such as entanglement entropy, mutual information, negativity, etc.)
- Why (B)CFT knows something about topological phases?
  - CFT can be in proximity to topological phases either spatially (bulk-boundary correspondence) or in the phase diagram.

Symmetry-protected topological phases (SPT phases)

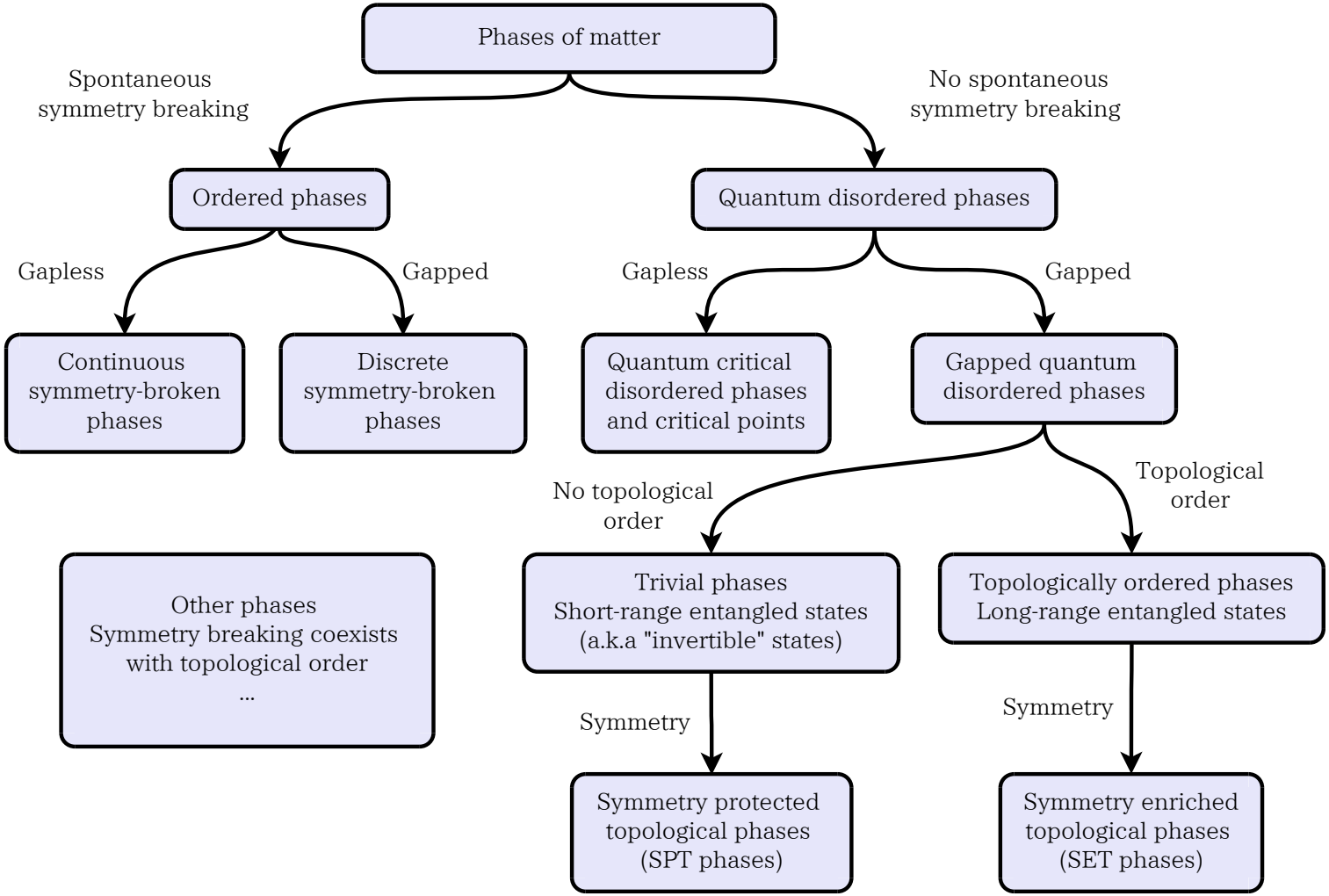
- SPT phases have a unique ground state for any spatial manifold. (I.e., no topological degeneracy, no topological order).
- *In the absence of symmetry*, SPT phases are adiabatically connected to a trivial gapped phase
- Nevertheless, *if we impose symmetry*, SPT phases are topologically distinct



- Examples:
  - time-reversal symmetric topological insulators;
  - the Haldane phase in 1d spin chains
- Bulk-boundary correspondence
- Symmetry-breaking paradigm cannot be applied: no *local* order parameter
- Need to consider topological invariants. For G-symmetric SPT phases when G does not include spacetime symmetry, (1+1)d SPT phases are classified by $H^2(G, U(1))$. [Chen-Gu-Liu-Wen (02)]

Phases of matter

ordered v.s. disordered, gappped v.s. gapless



Boundary states as gapped states

- Conformally invariant boundary states, $(L_n - \bar{L}_{-n})|B\rangle = 0$.
- Boundary states $|B\rangle$ have very short correlation length:

$$\langle B|e^{-\epsilon H}\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)e^{-\epsilon H}|B\rangle/\langle B|e^{-2\epsilon H}|B\rangle$$

  where $x_1, \cdots, x_n$ refer n different spacial positions. In the limit $\epsilon \to 0$ with $x_i \neq x_j$ the correlation function factorizes and does not depend on $x_i - x_j$. I.e., $|B\rangle$ does not have real-space correlations.
- Boundary states represent a highly excited state within the Hilbert space of a gapless conformal field theory and can be viewed as gapped ground states.

Free fermion example

- A massive free massive Dirac fermion in (1+1)d:

$$H = \int dx \left[-i\psi^\dagger \sigma_z \partial_x \psi + m\psi^\dagger \sigma_x \psi \right], \quad \psi = (\psi_L, \psi_R)^T$$

- The ground state of this Hamiltonian is given by

$$|GS\rangle = \exp\left[\sum_{k>0} \frac{m}{\sqrt{m^2+k^2}+k} \left(\psi^\dagger_{Lk}\psi_{Rk} + \psi^\dagger_{R-k}\psi_{L-k} \right) \right] |G_L\rangle \otimes |G_R\rangle$$

where $\psi_{L,Rk}$ is the Fourier component of $\psi_{L,R}(x)$, and $|G_{L,R}\rangle$ is the Fock vacuum of the left- and right-moving sector. In the limit $m \to \infty$ ($m/(v_F k) \to \infty$), $|GS\rangle$ reduces to the boundary states of the free massless fermion theory.

SPT-BCFT correspondence

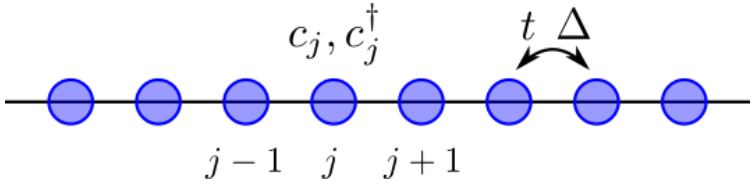
Besides the fact that boundary states look like gapped ground states, we can make a correspondence between gapped ground states and boundary states via the following arguments:

- The Jackiw-Rebbi domain wall
- Scattering from SPT phases
- The entanglement spectrum

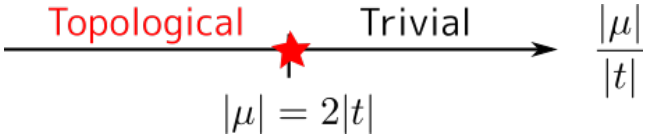
The Kitaev chain

- A simple model of (1+1)d topological superconductor: the Kitaev chain

$$H = \sum_j \Big[-tc_j^\dagger c_{j+1} + \Delta c_{j+1}^\dagger c_j^\dagger + h.c. \Big] - \mu \sum_j c_j^\dagger c_j$$



- Phase diagram: there are only two phases:



Topologically non-trivial phase is realized when $2|t| \geq |\mu|$.

- The continuum limit:

$$\mathcal{L}_* = \psi_L i(\partial_t - \partial_x)\psi_L + \psi_R i(\partial_t + \partial_x)\psi_R,$$
$$\mathcal{L}_I = -im\psi_L\psi_R,$$

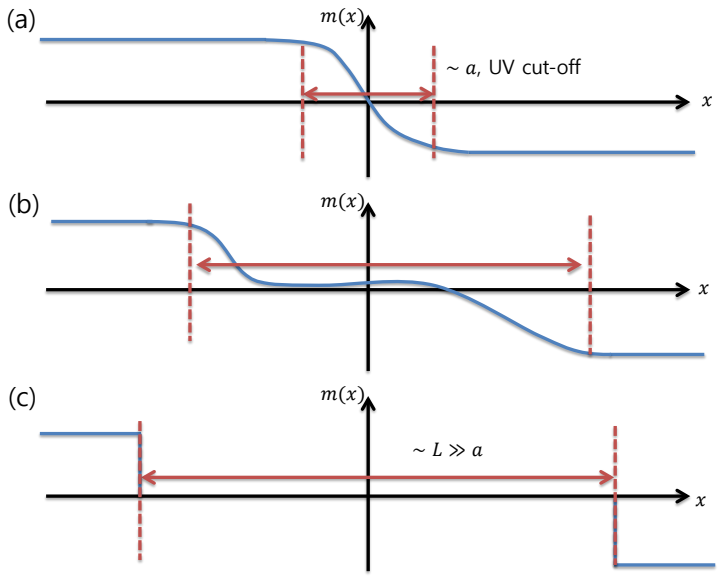
- $m < 0$ and $m > 0$ are topologically distinct

Jackiw-Rebbi domain wall

- Domain wall:

$$m(x) \to \begin{cases} +|m|, & x \to +\infty \\ -|m|, & x \to -\infty. \end{cases}$$

- A zero-energy Majorana mode is trapped at the domain wall.
- The domain wall can be "smeared" without loosing the majorana mode.

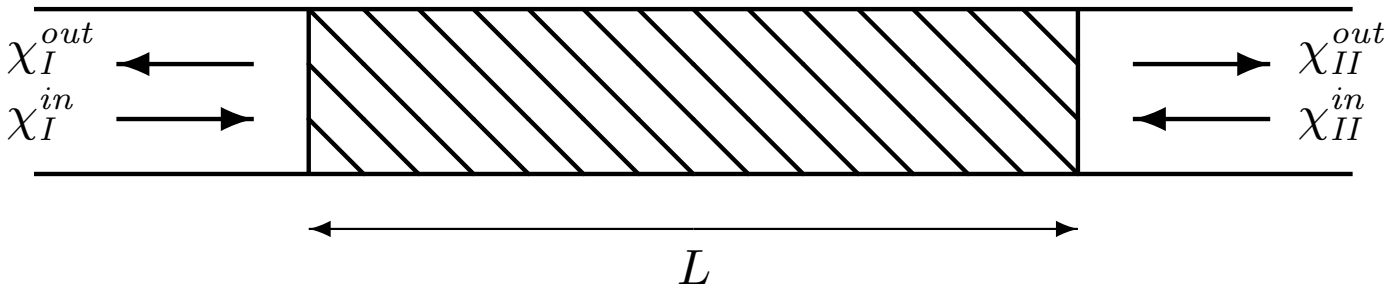


- Generalization with a relevant operator $\mathcal{O}(x)$

$$S_* \to S_* - \int dtdx\, \lambda(x)\mathcal{O}(t,x), \qquad \lambda(x) \to \begin{cases} +|\lambda|, & x \to +\infty \\ -|\lambda|, & x \to -\infty. \end{cases}$$

Scattering from SPT

- To detect an SPT phase, attach an "ideal lead"



- The modes in the leads are completely reflected from the SPT region (in $L \to \infty$).
- The topological invariant of the SPT phase can be computed from the reflection matrix [Akhmerov et al (2011)]

$$\operatorname{sgn} \det r(\varepsilon = 0) = \pm 1.$$

- The correlator inside the lead: [CardyLewellen(91),Ludwig-Affleck(91-93)]

$$\langle \psi_L^a(z) \psi_R^b(\bar{z}) \rangle = \frac{r_{ab}}{z - \bar{z}}.$$

- To the lead, the SPT region looks like an impurity

Entanglement entropy and entanglement spectrum

- Consider: The entanglement entropy S_A and the entanglement Hamiltonian H_A^E for the reduced density matrix ρ_A obtained from the ground state $|GS\rangle$ by tracing out half-space:

$$\rho_A = \mathrm{Tr}_{\bar{A}} |GS\rangle\langle GS| = \mathcal{N} e^{-H_A^E},$$
$$S_A = -\mathrm{Tr}_A \, \rho_A \ln \rho_A$$

- The entanglement hamiltonian can be obtained by a coordinate transformation $(t, x) \to (u, v)$ (Rindler coord.):

$$x = e^u \cosh v,$$
$$t = e^u \sinh v.$$

In the Rindler coord., the half of the 2d spacetime is inaccessible ("traced out").

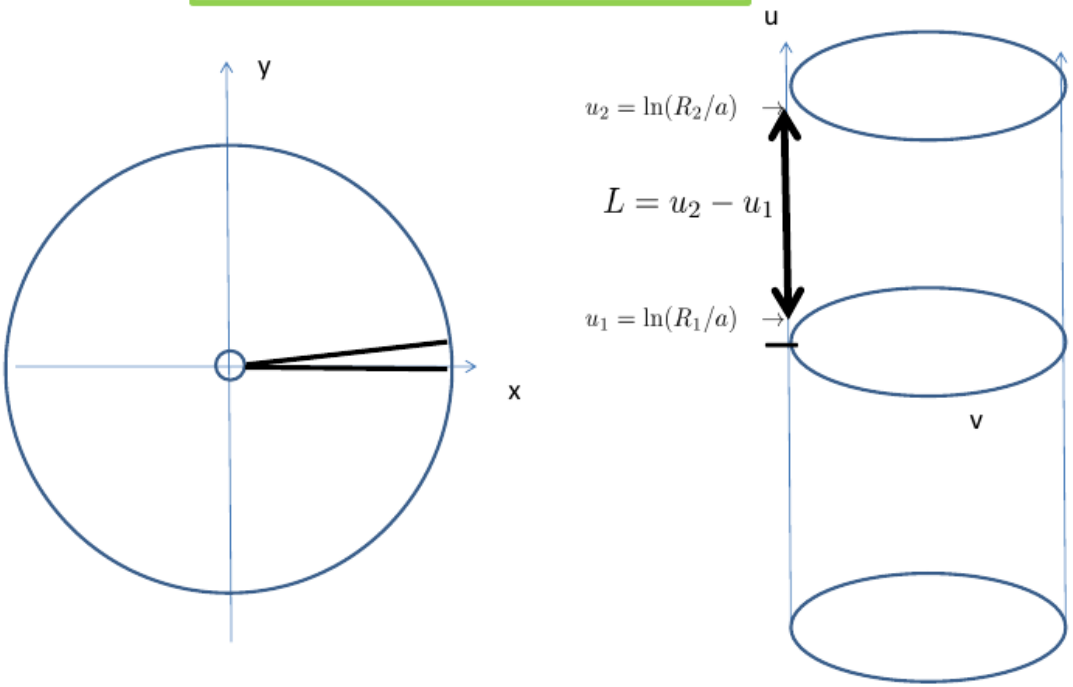
- In Euclidean signature,

$$z = x + iy = e^{w} = e^{u+iv}$$

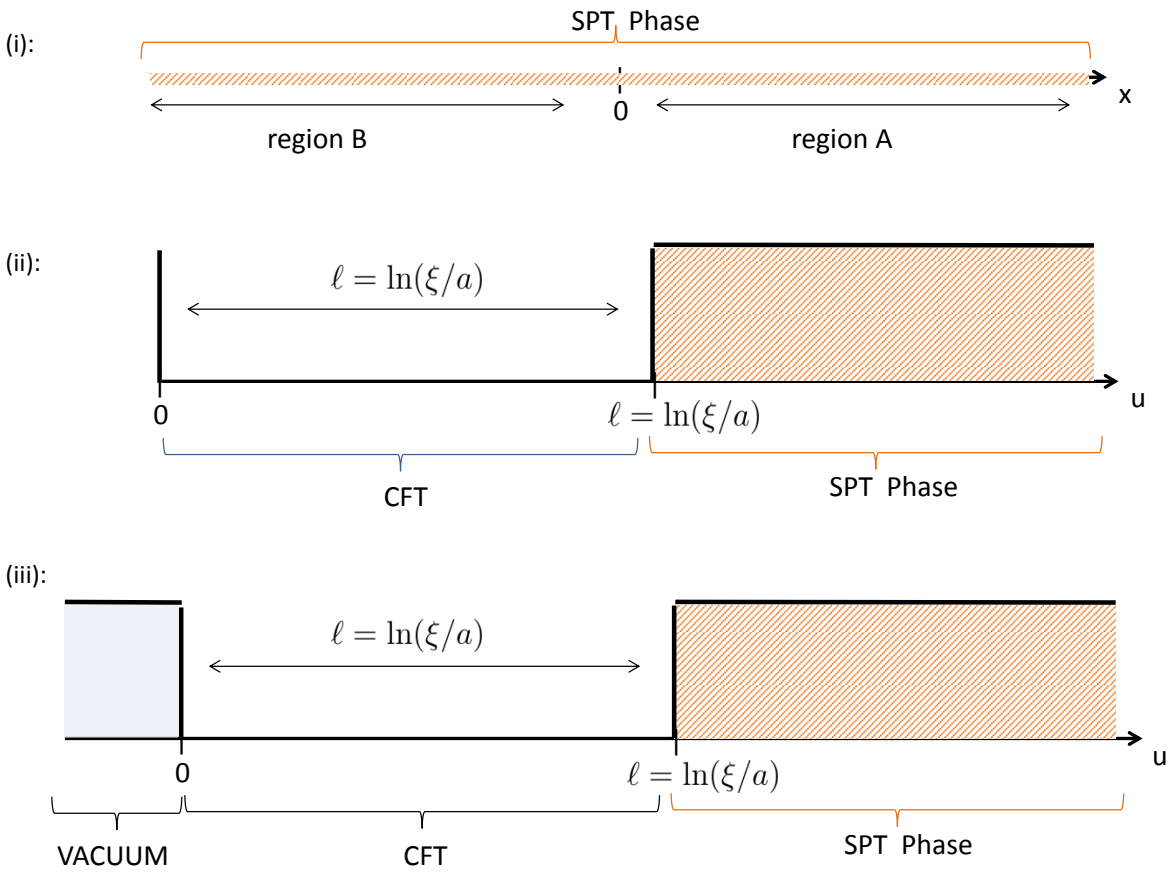
  The complex z-plane is mapped to a cylinder.

- Radial evolution in the complex z-plane = u-evolution in the cylinder
- Angular evolution in the complex z-plane = v-evolution in the cylinder = entanglement Hamiltonian
  (a.k.a corner transfer matrix, Rindler Hamiltonian)

$$\boxed{z = (x + iy) = \exp(w) = \exp(u + iv)}$$



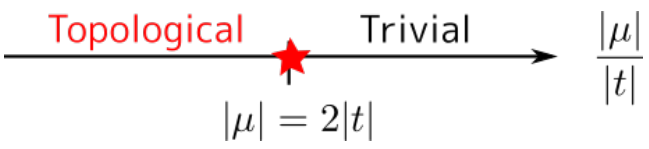
- Add a relevant perturbation and go into a massive phase $S = S_* + g \int d^2z\, \phi(z, \bar{z})$ and consider the entanglement Hamiltonian for half space.
- The above conformal map leads to an exponentially growing potential $S_* + g \int_{u_1}^{u_2} du \int_0^{2\pi} dv\, e^{yu}\, \Phi(w, \bar{w})$ with length scale $\log(\xi/a)$.

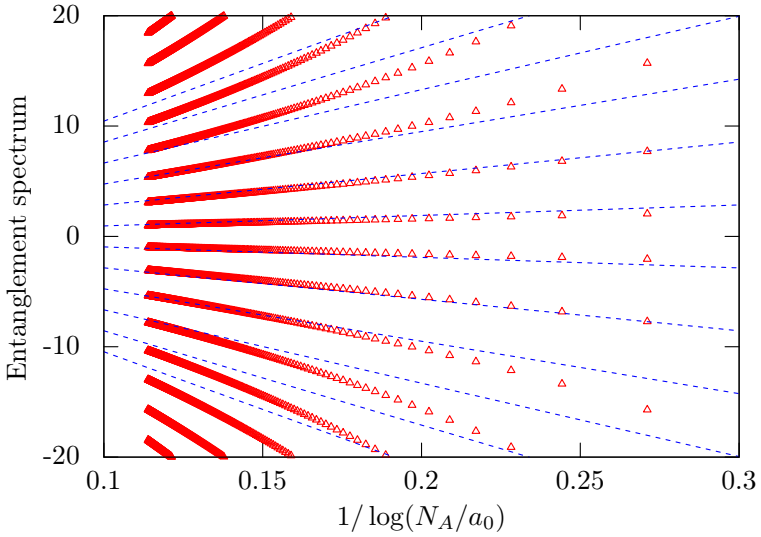


Summary (CFT case)

- Entanglement spectrum of CFT GS: $H^E = const. \frac{L_0}{\log(R/a)}$

$$H = \sum_j \Big[-tc_j^\dagger c_{j+1} + \Delta c_{j+1}^\dagger c_j^\dagger + h.c. \Big] - \mu \sum_j c_j^\dagger c_j$$

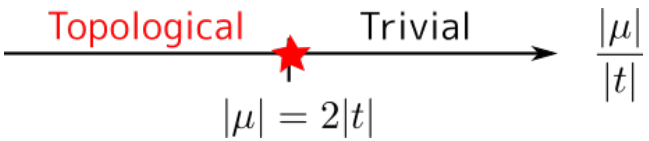


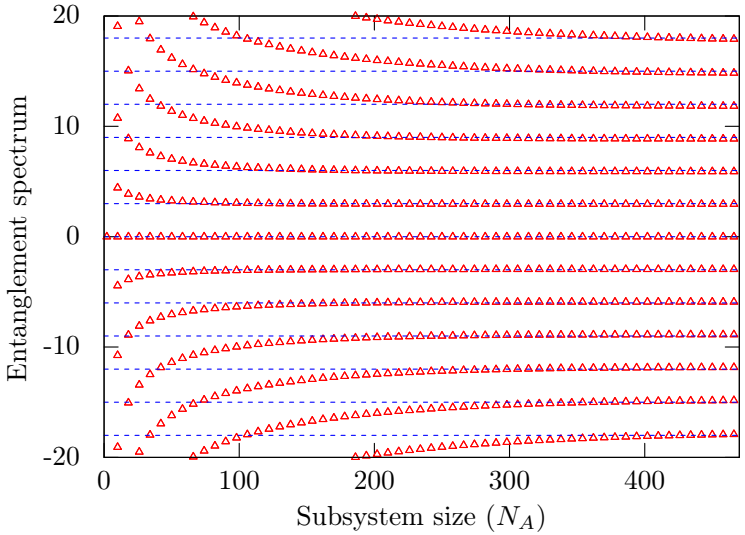


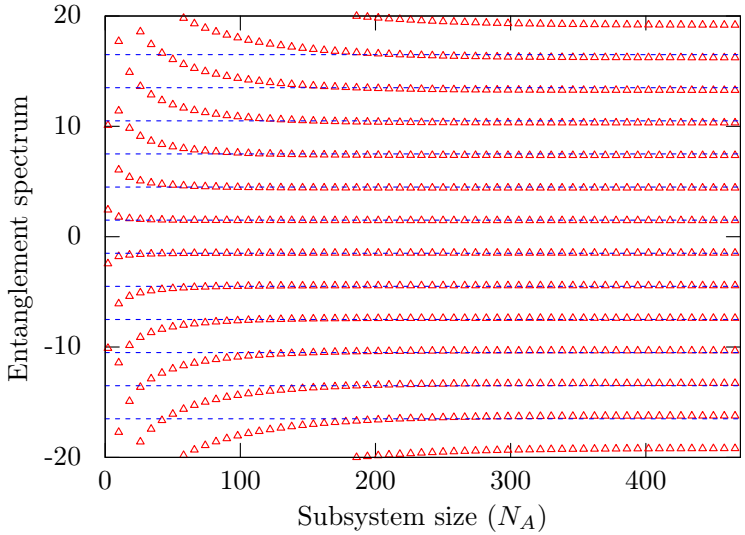
Summary (Gapped case)

- Entanglement spectrum for gapped phases is given by BCFT of a nearby CFT

$$H^E = const. \frac{L_0}{\log(\xi/a)}$$







- There is *symmetry-protected degeneracy* in the topological phase.

BCFT and SPT

- So far: Connection between (1+1)d gapped phases and BCFT.
- Question: Which boundary condition? $\Longleftrightarrow$ Which gapped phases?
- Focus on the case when the massive phase is an SPT phase protected by symmetry G.
  - (i) Unique G-symmetric ground state
  - (ii) Topologically distinct in the presence of symmetry
- For a given boundary state $|B\rangle$, the corresponding SPT phase can be identified by the discrete torsion phase $\varepsilon_B(g|h) \in H^2(G, U(1))$, which can be computed as

$$g|B\rangle_h = \varepsilon_B(g|h)|B\rangle_h, \quad g, h \in G$$

where $|B\rangle_h$ is the boundary state in h-twisted sector.

- Discrete torsion phases in closed orbifold CFTs are classified by $H^2(G, U(1))$. [Vafa (86) ...]
- Open string states form a projective representation of G; fractional D-branes; [Douglas (98) ...]
- G-SPT phases in (1+1)d are classified by $H^2(G, U(1))$. [Chen-Gu-Liu-Wen (02)]
- Discrete torsion phases and entanglement spectrum (symmetry-protected degeneracy):
  Twisted partition function:

$$Z^h_{AB} = \mathrm{Tr}_{\mathcal{H}_{AB}} \left[\hat{h} e^{-\beta H^{open}_{AB}} \right]$$

  vanishes when $A \neq B$. (symmetry-enforced vanishing of partition function).
  Exchange time and space, $Z^h_{AB} = {}_h\langle A | e^{-\frac{\ell}{2} H^{closed}} | B \rangle_h$ and insert g to show

$$[\varepsilon_B(g|h) - \varepsilon_A(g|h)] Z^h_{AB} = 0$$

$\mathrm{BCFT}_2/\mathrm{SPT}_2$: Summary

- Generalized the scattering matrix approach to interacting setting.
  Clarified the relation between the discrete torsion phases in CFTs and SPTs
- Support the claim that boundary states as a gapped ground state.

$\mathrm{BCFT}_2/\mathrm{SPT}_3$: Overview

- Bulk-boundary correspondence:

$$(G\text{-}\mathrm{SPT}_3) \longleftrightarrow (\mathrm{CFT}_2 \text{ with 't Hooft anomaly})$$

-

$$\begin{aligned}(\text{'t Hooft anomaly}) &= (\text{Inability to gap } \mathrm{CFT}_2 \text{ (keeping } G)) \\ &= (\text{Modular anomaly of } G\text{-orbifold CFT}) \\ &= (\text{No solution to the Cardy condition (keeping } G))\end{aligned}$$

- For Abelian cases:

$$\begin{aligned}&(\text{Solutions to the Haldane null vector condition}) \\ &= (\text{Solutions to the Cardy condition})\end{aligned}$$

Example

- Field theory model:

$$\text{Bulk}: \quad S = \frac{1}{4\pi}\int d^2x d\tau \varepsilon_{\mu\nu\lambda} a^I_\mu K_{IJ}\partial_\nu a^J_\lambda, \quad K = \sigma_x$$

$$\text{Edge}: \quad S = \frac{1}{2\pi}\int dx d\tau\, \partial_x\phi\, \partial_\tau\theta \quad \text{with } \phi \equiv \phi + 2\pi, \theta \equiv \theta + 2\pi$$

$$\mathbb{Z}_2 \text{ symmetry}: \quad G: \phi \to \phi + \pi, \quad \theta \to \theta + \pi q \quad (q = 0, 1).$$

  Can these be SPT ?

- When $q = 0$, $\cos\theta$ can gap the edge. "Trivial"
- When $q = 1$,

  $\cos\theta$: not allowed by the symmetry

  $\cos 2\theta$: allowed, but spontaneously breaks the symmetry

  $\cos(\theta + \phi)$: allowed, but cannot gap the edge (the Haldane condition)

  The bulk is non-deformable to a trivial state. "SPT"

- The case of $q = 1$ with two copies $N_f = 2$, can gap $\phi_1 \pm \phi_2$ and $\theta_1 \mp \theta_2$.

Gapping potential approach

- In the above example, $\cos(\theta + \phi)$ cannot gap the edge since $[\phi, \theta] \sim i$.
- <u>The Haldane null vector condition</u>: For an N-component boson theory with a given K-matrix, if there is a set of N linearly independent integer vectors $\{\mathbf{l}_i\}$ satisfying the condition

$$\mathbf{l}_i^T K^{-1} \mathbf{l}_j = 0, \quad \forall i, j = 1, \ldots, N,$$

then we can find a potential which can gap out the boson theory completely. I.e., one can find the gapping potential

$$S_{gapping} = \sum_{\mathbf{l}} c_{\mathbf{l}} \int dt dx \cos(\mathbf{l} \cdot \Phi + \alpha_{\mathbf{l}}).$$

- This "gapping potential" approach has difficulties: e.g., non-Abelian cases.

Modular anomaly approach

- Another way of diagnostic: Check modular invariance of orbifolded partition function. [SR,Zhang (12)]

$$Z^{k_1}{}_{k_2} = \mathrm{Tr}_{k_2}\left[g^{k_1} q^{H_L} \bar{q}^{H_R}\right], \quad k_{1,2} = 0, 1, \quad g : \text{generator}$$

$$Z^{orb} = \frac{1}{|G|} \sum_{g_1, g_0 \in G} \varepsilon(g_1, g_0) Z^{g_0}{}_{g_1}$$

Can the partition function be made modular invariant?

- When $q = 1$, it's not possible to have a modular invariant.

$$Z^{k_1}{}_{k_2}(\tau + 1) = e^{-\frac{\pi i q k_1^2}{2}} Z^{k_1}_{k_1 + k_2}(\tau),$$

$$Z^{k_1}{}_{k_2}(-1/\tau) = e^{\pi i k_1 k_2} Z^{k_2}_{-k_1}(\tau).$$

Boundary CFT approach

- Boundary CFT can offer an alternative way to look at this problem.
- If the CFT must exist as a boundary theory of an SPT, it should not be "Edgeable" since $\partial^2 = 0$ (if we insist G symmetry).
- One can in fact show

$$\begin{aligned}&(\text{Solutions to the Haldane null vector condition})\\&= (\text{Solutions to the Cardy condition})\end{aligned}$$

- (Aside) When G includes an orientation reversing symmetry, cross cap states can be used instead of/in addition to boundary states.

$\mathrm{BCFT}_2/\mathrm{TO}_3$

- Consider: the reduced density matrix ρ_A obtained from a ground state $|GS\rangle$ of a topologically-ordered phase by tracing out half-space.

$$\rho_A \propto \mathrm{Tr}_R\, e^{-\epsilon H} |Ishibashi\rangle\langle Ishibashi| e^{-\epsilon H}$$

- Different Ishibashi states correspond to different ground states
- With this explicit form of the reduced density matrix, various entanglement measures can be computed:
  - the entanglement entropy
  - the mutual information
  - the entanglement negativity

References

- $\mathrm{SPT}_2/\mathrm{BCFT}_2$
  - *Boundary States as Holographic Duals of Trivial Spacetimes,* [Miyaji, SR, Takayanagi, Wen (14)]
  - *Universal Entanglement Spectra of Gapped One-dimensional Field Theories,* [Cho, Ludwig, SR, PRB (16)]
  - *Relationship between Symmetry Protected Topological Phases and Boundary Conformal Field Theories via the Entanglement Spectrum,* [Cho, Shiozaki, SR, Ludwig, JPhysA Cardy Special Issue (16)]
- $\mathrm{SPT}_3/\mathrm{BCFT}_2$
  - *Boundary conformal field theory and symmetry protected topological phases in 2+1 dimensions,* [Han, Tiwari, Hsieh, SR, PRB(17)]
- $\mathrm{TO}_3/\mathrm{BCFT}_2$
  - *Edge theory approach to topological entanglement entropy, mutual information and entanglement negativity in Chern-Simons theories,* [Wen, Matsuura, SR, PRB(16)]