Harmony of Defect Blocks

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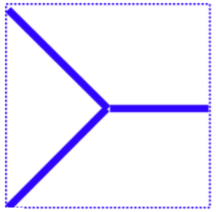
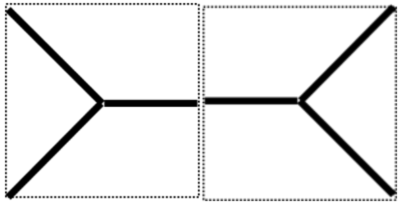
Based on work with M. Isachenkov and E. Sobko

Introduction: Conformal Bootstrap

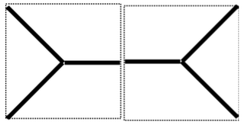
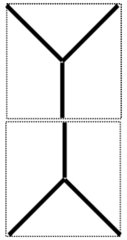
3-point functions $\langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\rangle = \gamma_{123}$  **3J symbol**

Scalar fields: $= \dfrac{\gamma_{123}}{x_{12}^{\Delta_{12}}x_{13}^{\Delta_{13}}x_{23}^{\Delta_{23}}}$ $\Delta_{12} = \Delta_1 + \Delta_2 - \Delta_3$

4-point functions $\langle \prod_{i=1}^{4} \Phi_i(x_i)\rangle = \sum_{\Phi} \gamma_{12\Phi}\gamma_{34\Phi}$  (u_1, u_2)

Conformal block

Conformal block expansion → separation of dynamics from symmetry

Crossing symmetry constraint $\sum_{\Phi} \gamma_{12\Phi}\gamma_{34\Phi}$  = $\sum_{\Psi} \gamma_{23\Psi}\gamma_{14\Psi}$ 

Introduction: Boundary/Defect Bootstrap

1-point function in presence of defect D_p along p-dim hypersurface $\mathcal{X}$

$$\langle D_p(\mathcal{X})\Phi(x)\rangle = \frac{\alpha_\Phi^{(p)}}{|x_\perp|^{\Delta_\Phi}}$$

Defect block expansion for 2-point function of defects D_p and D_q is

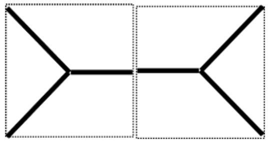
$$\langle D_p(\mathcal{X}_p)\, D_q(\mathcal{X}_q)\rangle = \sum_\Phi \alpha_\Phi^{(p)} \alpha_\Phi^{(q)} G_\Phi^{pq}(u)$$

Defect blocks

How many cross ratios u ?
What are the blocks G(u) ?
Is there crossed channel ?

[Gadde 16]

Introduction: Conformal Blocks



[Dobrev et al.]

$D_0 D_0$

In recent CFT literature: [Dolan, Osborn 03,11]
+ radial coordinates [Hogervorst,Rychkov 13]
+ recurrence relations [Penedones et al. 15]
+ spinning blocks [Costa et al 11] ....

Four-point blocks are wave functions of an integrable 2-particle

Calogero-Sutherland system [Isachenkov,VS 16] [with Sobko 16]

Boundary blocks [Liendo et al 12]; Defect blocks [Billo et al. 16]

$D_{d-1} D_0$

$D_p D_0$; *not complete*

Main Result and Plan

D_0 = pair of points

Conformal blocks for configuration of defects D_p and D_q of dim p and q are wave functions of an N-particle (spin) Calogero-Sutherland model.

$N = \min(d-p, q+2)$ for $p \geq q$ **integrable**

Solution theory exists , initiated by Heckman-Opdam ….

Modern theory of multivariate hypergeometrics

Plan

I Casimir equations and Calogero-Sutherland models

II Construction of wave functions and blocks – a sketch

Harmonic Analysis of Defects

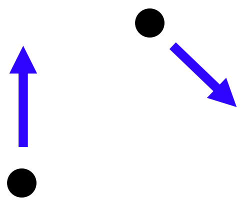
Group Theory Background

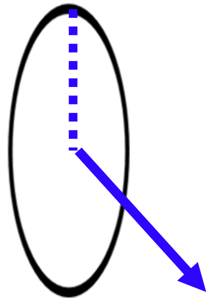
Euclidean conformal group in d dimensions is $G = SO(1, d+1)$

Isometries of p-dim defect is $G_p = SO(1, p+1) \times SO(d-p)$

p = 0: isometries of pair of points (dilations, rotations)

Conformal defect possesses $\dim G/G_p$ = (p+2)(d-p) parameters





e.g. D_0 has 2d parameters

D_1 has d+1 parameters

Space of Conformal Blocks

Let π_L and π_R be finite dimensional reps of G_p and G_q on V_L and V_R

$$\Gamma^{\pi_L \pi_R} = \{ f : G \to V_L \otimes V_R \mid f(h_L g h_R) = \pi_L(h_L)\pi_R(h_R^{-1}) f(g) \}$$

Sections of vector bundle on double coset $\mathrm{G_p \backslash G / G_q}$ with values in

$(V_L \otimes V_R)^B$ where $B = SO(p-q) \times SO(|d-p-q-2|) \subset G_{p,q}$

Space of tensor structures

$\dim G_p \backslash G / G_q = N = min(d-p, q+2)$

number *N* of ``cross ratios”

The Casimir Equation

Eigenvalue equation for Casimir elements of G on space Γ of blocks:

$$m^{1/2}(u)\mathcal{D}_2 m^{-1/2}(u) = -\frac{1}{2}\sum_{i=1}^{N}\frac{d^2}{du_i^2} + V(u)$$

m is volume of $G_p \times G_q$ orbit through $u = (u_1, \dots, u_N)$

[M. Isachenkov, VS, E. Sobko]

Scalar blocks:

$$V(u) = \sum_{i=1}^{N}\left(\frac{(a+b)^2 - 1/4}{2\sinh^2 u_i} - \frac{ab}{2\sinh^2\frac{u_i}{2}}\right)$$

$$+\sum_{i<j}\left(\frac{\epsilon(\epsilon-2)}{16\sinh^2\frac{u_i-u_j}{2}} + \frac{\epsilon(\epsilon-2)}{16\sinh^2\frac{u_i+u_j}{2}}\right) + C$$

a, b, ϵ depend on d,p,q and π_L, π_R

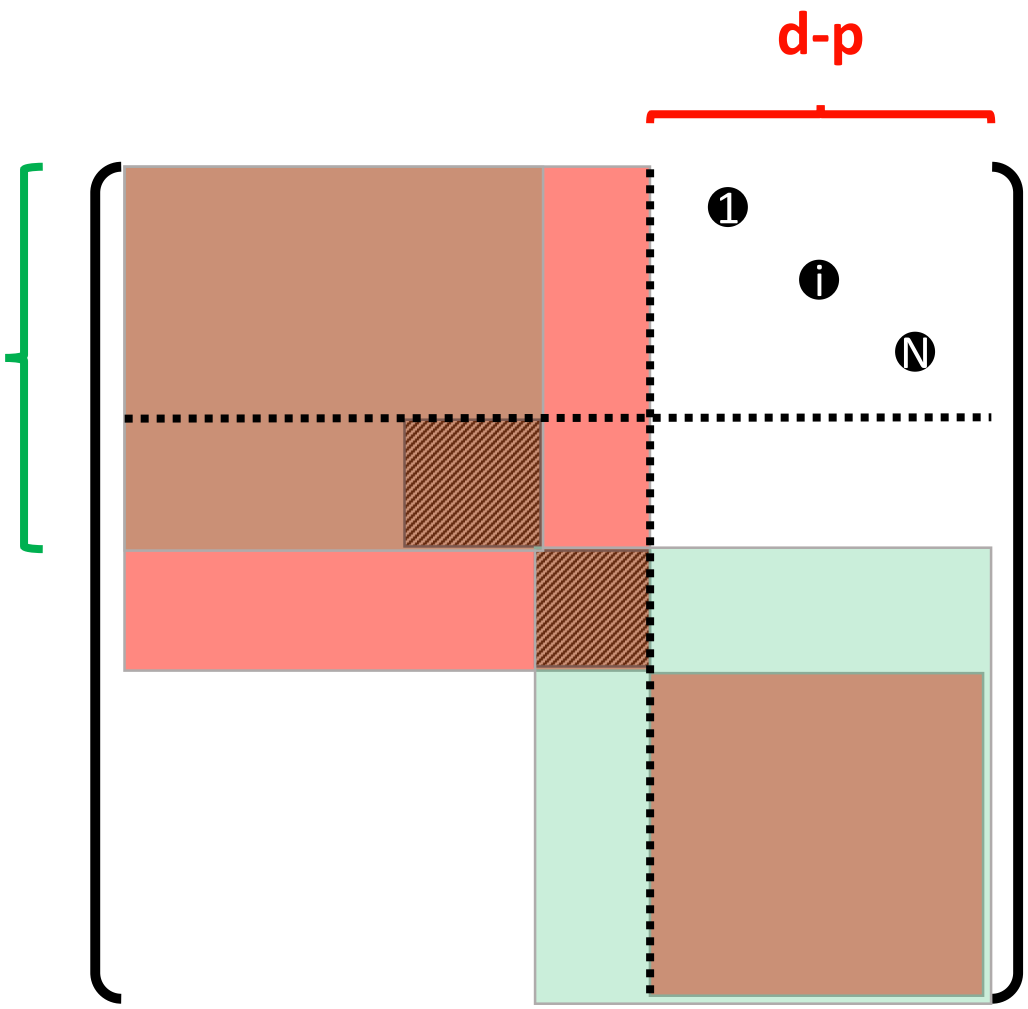
Choice of Coordinates

based on product decomposition

$$g = h_L\, a(u)\ \ h_R$$

$h_L \in G_p$ $\qquad$ $h_R \in G_q/B$

$$a(u) = \exp u_i\, \chi_i \in A$$



G_p $\qquad$ $G_p \cap G_q$

G_q $\qquad$ B

i $\qquad$ Generator χ_i of $A \subset G$

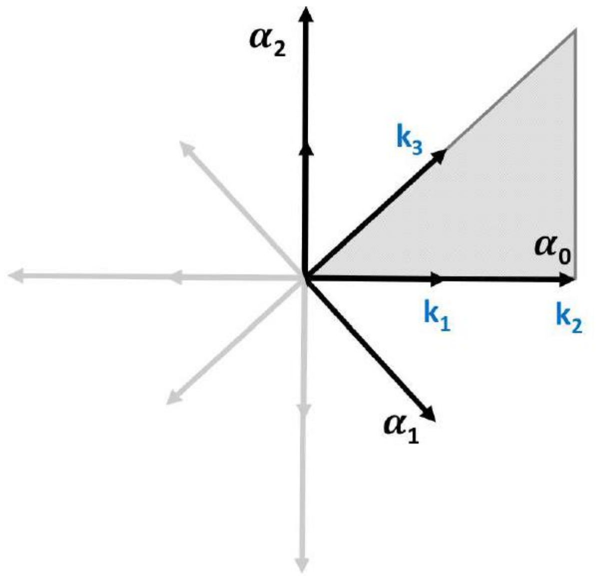
Conformal blocks for defects

Calogero-Sutherland Models

[Calogero 71]
[Sutherland 72]
[Moser 75]

Integrable multi-particle generalization of Poeschl-Teller model

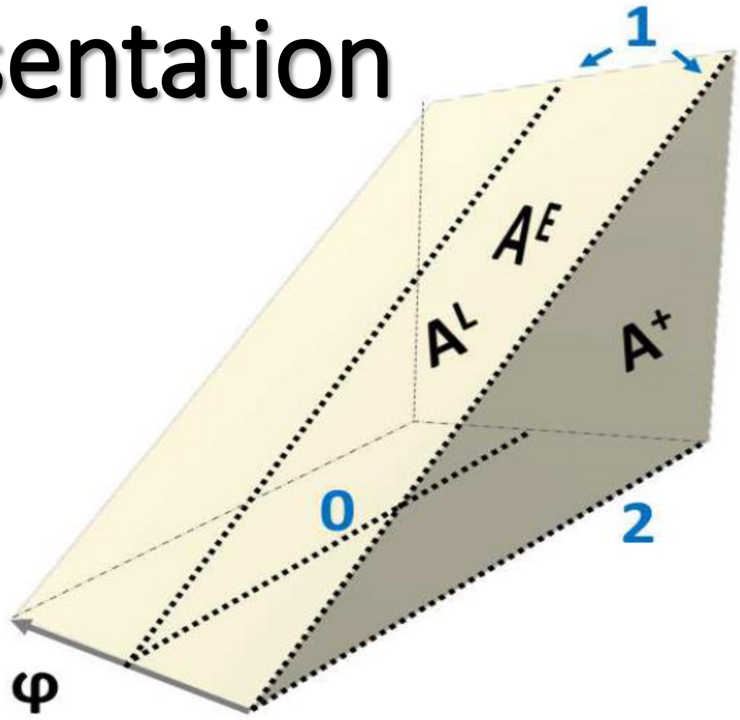
Associated with non-reduced root system – here with BC_N



Eigenvalue problems → theory of multivariate hypergeometrics

[Heckman,Opdam] … Cherednik, Matsuo & many more

Monodromy Group & Representation



Symmetries of the CS potential

$$\mathcal{W}_N = W_N \ltimes \mathbb{Z}^N$$ Affine Weyl group

Monodromy group $\pi_1(\mathbb{C}^N/\mathcal{W}_N)$

$$g_i g_j = g_j g_i \quad |i-j| \geq 2$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad i = 1, \dots N-2$$

$$g_0 g_1 g_0 g_1 = g_1 g_0 g_1 g_0 \ , \ g_{N-1} g_N g_{N-1} g_N = g_N g_{N-1} g_N g_{N-1}$$

Affine braid group reflection type

Solutions of scalar CS model carry $N!\,2^N$ - dimensional representation with $(g_i - 1)(g_i - \gamma_i) = 0$ $\quad \gamma_i = \gamma_i(a, b, \epsilon) \in \mathbb{C}$ Hecke relation

Dunkl operators and dDAHA

Dunkl operators generalize role of derivatives for free particle (FP)

$$H^{FP} = \sum \partial_i^2 \qquad\qquad [\partial_i, \partial_j] = 0$$

$$\boxed{\mathcal{Y}_j = \partial_j + \sum_{\alpha \in \Sigma+} \frac{k_\alpha \langle \alpha, e_j \rangle}{1 - e^{-\langle \alpha, u \rangle}} (1 - w(\alpha)) - \langle \rho(k), e_j \rangle}$$

→ Knizhnik Zamolodchikov equations for conformal blocks in any d

Dunkl operators, coordinate functions $x_i = exp\, u_i$ and elements w of Weyl group W_N generate degenerate DAHA

Bispectral Duality & q-deformation

Generalization of (self-) duality between coordinates & momenta in FP

$$\Psi_p(u) = e^{ipu} \qquad\qquad u \leftrightarrow p$$

Dual of hyperbolic Calogero-Sutherland rational Ruijsenaars-Schneider

2^{nd} order difference equation

q-deformation to self-dual hyperbolic RS model **dDAHA → DAHA**

Outlook: Bootstrap Program

Blocks in $D_p\ D_0$ defect channel are sections in bundle over coset

Isometries of one bulk pt

Isometries of one bulk pt

$$\mathrm{SO(p+1)\ x\ SO(d\text{-}p\text{-}1)} \backslash \mathrm{SO(1,p+1)\ x\ SO(d\text{-}p)} / \mathrm{SO(p+1)\ x\ SO(d\text{-}p\text{-}1)}$$

Isometries of defect

with values in V^B where B = SO(p) x SO(d-p-2) and $V = V_1 \otimes V_2$

Space of tensor structures

Casimir equation is BC_1 (hyperbolic) x BC_1 (trigonometric)

Crossing symmetry not studied except for p = d-1

[Liendo et al 12]
[Gliozzi et al 15]

Conclusions

Conformal blocks for configuration of defects D_p and D_q of dim p and q are wave functions of an N-particle (spin) Calogero-Sutherland model.

$N = \min(d-p, q+2)$ for $p \geq q$ **integrable**

Solution theory developed in mathematics as part of modern theory of multivariate hypergeometric functions **degenerate Koornwinder-Macdonald fcts**